**NONLINEAR FILTERING**[1]

**(Home exam, Fall, 2005)**

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**Problem 1**

Consider the two dimensional[2] system of linear SDE's ($t \in [0,T]$)

$$dX_t = aX_t dt + bdW_t$$
$$dY_t = AX_t dt + dV_t$$

subject to $Y_0 \equiv 0$ and a random initial condition $X_0$ with probability distribution $F(x)$, $\int_{\mathbb{R}} x^2 dF(x) < \infty$. As usual, the coefficients are real numbers and $W$ and $V$ are independent Wiener processes, independent of $X_0$. If $X_0$ were a Gaussian random variable, then the conditional distribution of $X_t$, given $\mathscr{F}_t^Y$ would be Gaussian, completely specified by its mean and covariance, generated by the Kalman-Bucy equations.

This problem deals with derivation of the finite dimensional filter, when $X_0$ is non Gaussian. Throughout $f$ is a measurable function, such that $\mathrm{E}f^2(X_t) < \infty$, $t \in [0,T]$.

**(a)** Let $X_t' = X_0 e^{at}$ and define

$$V_t' = V_t + \int_0^t AX_s' ds, \quad t \in [0,T].$$

Define a probability $\widetilde{\mathrm{P}}$ (via appropriate Radon-Nikodym derivative), equivalent to $\mathrm{P}$ and such that $V'$ is a Wiener process under $\widetilde{\mathrm{P}}$, independent of $X_0$ and $W$.

**(b)** Express $\mathrm{E}\big(f(X_t)|\mathscr{F}_t^Y\big)$ by means of the conditional expectations under $\widetilde{\mathrm{P}}$.

**(c)** Calculate the conditional $\widetilde{\mathrm{P}}$-expectations by means of an appropriate two-dimensional Kalman-Bucy filter

**(d)** Using the result of **(b)** and **(c)**, write an explicit[3] expression for $\mathrm{E}\big(f(X_t)|\mathscr{F}_t^Y\big)$.

**(e)** Verify your answer in **(d)** in the case of Gaussian $F(x)$.

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[2] a complete probability space $(\Omega, \mathscr{F}, \mathrm{P})$ is assumed to support all the mentioned random objects

[3] up to integrations with respect to $F(x)$ and Gaussian densities and dependence on the solutions of concrete SDE's

**Problem 2.**

Consider the linear diffusion

$$dX_t = aX_t dt + dW_t, \quad X_0 = 0,$$

where $W$ is a Wiener process and $a$ is an unknown random parameter, to be estimated from $\mathscr{F}_t^X$. Below $a$ and $W$ are assume to be independent.

**(a)** Assume that $a$ takes a finite number of values $\{\alpha_1, ..., \alpha_d\}$ with positive probabilities $\{p_1, ..., p_d\}$. Find the recursive formulae ($d$-dimensional system of SDE's) for $\pi_t(i) = \mathrm{P}(a = \alpha_i | \mathscr{F}_t^X)$.

**(b)** Find the explicit solutions to the SDE's in **(a)**.

**(c)** Does $\pi_t(i)$ converges to $\mathbf{1}_{\{a=\alpha_i\}}$, $i = 1, ..., d$ ? If yes, in what sense ?

**(d)** Assume that $\mathrm{E}a^2 < \infty$ and find an explicit expression for the orthogonal projection $\widehat{\mathrm{E}}(a|X_{[0,t]})$ and the corresponding mean square error.

**(e)** Assume that $a$ is a standard Gaussian random variable. Is the process $X$ Gaussian ? Is the pair $(a, X)$ Gaussian ? Is $X$ conditionally Gaussian, given $a$ ?

**(f)** Is the optimal nonlinear filter in this case finite dimensional ? If yes, find the recursive equations for the sufficient statistics.

**(g)** Does the mean square error $P_t = \mathrm{E}\big(a - \mathrm{E}(a|\mathscr{F}_t^X)\big)^2$ converges to zero as $t \to \infty$?