RANDOM PROCESSES

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Remark. Any materials and lecture notes are permissible.

Problem 1.

Unobservable signal is defined as

$$X_n = X_{n-1}\varepsilon_n,$$

where $X_0 = 1$ and $(\varepsilon_n)_{n\geq 1}$ is a sequence of independent random variables such that $\Pr(\varepsilon_n = 1) = p_n$ and $\Pr(\varepsilon_n = 0) = 1 - p_n$, $0 < p_n < 1, n \geq 1$.

An information on the signal is obtained via observation process $(Y_n)_{n\geq 0}$:

$$Y_n = X_n + \xi_n,$$

where $(\xi_n)_{n\geq 0}$ is i.i.d. sequence of random variables independent of $(\varepsilon_n)_{n\geq 1}$. The distribution function of ξ_1 has a density $f_\xi(x)$.

Denote by $Y_0^n = \{Y_0, \cdots, Y_n\}$ and

$$\pi_{n|n} = \Pr(X_n = 1|Y_0^n) \quad \text{and} \quad \pi_{n|n-1} = \Pr(X_n = 1|Y_0^{n-1}).$$

1. **Non linear filtering.**
   (a) Find $\pi_{0|0}$;
   (b) For $n \geq 1$, express $\pi_{n|n}$ via $\pi_{n|n-1}$.
   (c) Derive a recursion for $\pi_{n|n}, n \geq 1$.
   (d) (*bonus +10*) Let τ the first index n such that $X_n = 0$, that is $\tau = \min\{n : X_n = 0\}$. Verify that

$$E(\tau|Y_0^n) = \sum_{k=1}^{\infty} \Pr(X_k = 1|Y_0^n).$$

2. **Kalman filter.** Assume $E\xi_1 = 0$ and $E\xi_1^2 < \infty$.
   (a) Find a model suitable for Kalman filter.
   (b) Derive a Kalman filter

3. **Degenerate case.** Assume there exists an index m such that $p_m = 0$. Show that for $n \geq m$ filtering estimates for X_n, both non linear and linear, coincide.

Problem 2.

Let θ and $X_1, \cdots, X_n, \cdots$ be random variables, $E\theta^2 < \infty$. Put $\widehat{\theta}_n = E(\theta | X_1^n)$, where $X_1^n = \{X_1, \cdots, X_n\}$. $E(\theta | X_1^n)$ is the optimal in the mean square sense estimate of θ given observations X_1^n and

$$\Delta_n = E(\theta - \widehat{\theta}_n)^2$$

is the mean square error.

1. Show that $\Delta_n \geq \Delta_{n+1}, n \geq 1$.
2. Show that $\lim_{n\to\infty} E\widehat{\theta}_n^2$ exists and is bounded from above by $E\theta^2$.

Problem 3.

Let random processes $(X_t, Y_t)_{t \geq 0}$ be defined by linear equations

$$\begin{aligned} \dot{X}(t) &= aX(t) + b\dot{W}(t) \\ \dot{Y}(t) &= AX(t) + B\dot{V}(t) \end{aligned}$$

subject to the initial conditions $X(0) = 0$, $Y(0) = 0$, where a, A, b, B are constants, and where $W(t)$ and $V(t)$ are independent Wiener processes, i.e. $\dot{W}(t)$ and $\dot{V}(t)$ are Gaussian white noises. Let t_k, $k = 0, 1, ...$, be sampling times such that $t_0 = 0$ and $t_{k+1} - t_k \equiv \Delta$. For small Δ, we have

$$\begin{aligned} X(t_{k+1}) &\approx X(t_k) + aX(t_k)\Delta + b[W(t_{k+1}) - W(t_k)] \\ Y(t_{k+1}) &\approx Y(t_k) + AX(t_k)\Delta + B[V(t_{k+1}) - V(t_k)]. \end{aligned} \tag{1}$$

1. Replacing "$\approx$" in (1) on "$=$" derive the Kalman filter for the signal $X_{t_k}, k \geq 0$ given observations $Y_{t_k}, k \geq 0$.
2. With $\Delta \to 0$ obtain the Kalman filter for the original continuous time model.