RANDOM PROCESSES. THE FINAL TEST

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9:00-13:00, 2nd of July, 1999

* any supplementary material is allowed
* the total score of the exam is 110 points.
* duration of the exam is exactly 4 hours - no additional time will be permitted
* good luck !

Problem 1. (30%) *Convergence Of Random Sequences*

Consider a pair of random sequences $(\eta_n)_{n\geq 0}$ and $(\xi_n)_{n\geq 0}$ converging in probability to limits η and ξ respectively, i.e. for any $\varepsilon > 0$:

$$\lim_{n\to\infty} \mathbf{P}\{|\xi_n - \xi| \geq \varepsilon\} = 0, \quad \lim_{n\to\infty} \mathbf{P}\{|\eta_n - \eta| \geq \varepsilon\} = 0$$

(a) (12%) Show that the random variables ξ and η are equivalent, i.e. $\mathbf{P}\{\xi \neq \eta\} = 0$, <u>if and only if</u>

$$\lim_{n\to\infty} \mathbf{P}\{|\xi_n - \eta_n| \geq \varepsilon\} = 0$$

(prove both directions!)

(b) (6%) Verify that $a\xi_n + b\eta_n \xrightarrow{\mathbf{P}} a\xi + b\eta$

(c) (6%) Let $f(x)$ be a continuous function. Show that

$$f(\xi_n) \xrightarrow{\mathbf{P}} f(\xi)$$

Hint: use 'ε-δ' formulation of continuity

(d) (6%) Is (c) correct for discontinuous functions $f(x)$? Prove your answer or give a counterexample.

Problem 2. (40%) *Gaussian Processes*

Consider a pair of random processes $(X_n, Y_n)_{n\geq 0}$, generated by a non linear recursion:

$$\begin{aligned} X_{n+1} &= aX_n + \frac{X_n\varepsilon_{n+1} + Y_n\xi_{n+1}}{\sqrt{X_n^2 + Y_n^2}} \\ Y_{n+1} &= AX_n + \frac{X_n\xi_{n+1} - Y_n\varepsilon_{n+1}}{\sqrt{X_n^2 + Y_n^2}} \end{aligned}$$

where $(\varepsilon_n)_{n\geq 1}$ and $(\xi_n)_{n\geq 1}$ are independent i.i.d Gaussian sequences with zero mean and unit variance and a and A are constants. The initial condition $[X_0, Y_0]$ is a Gaussian vector, with zero mean and nonsingular covariance matrix Q.

(a) (7%) Prove that $(X_n, Y_n)_{n\geq 0}$ is a Gaussian process.

(b) (7%) Give an explicit formula for the probability density of the vector $[X_n, Y_n]$

(c) (7%) Give an explicit formula for the probability density of the vector

$$[X_0, X_1, ..., X_n; Y_0, ..., Y_n]$$

Hint: you may find useful the Markov property of the processes X_n and Y_n. No need to reduce the found formula to canonical form.

(d) (7%) Derive recursive equations for $\widehat{X}_n = \mathbf{E}(X_n|Y_0^n)$, where $Y_0^n = \{Y_k, k = 0, 1, ..., n\}$

(e) (7%) Define the *predicting* estimate $\widehat{X}_{n+k|n} = \mathbf{E}(X_{n+k}|Y_0^n)$, where k is some positive integer. Find $\widehat{X}_{n+k|n}$ and show that at each time point n it explicitly depends only on the *filtering* estimate $\widehat{X}_n$ and *not* on the whole information Y_0^n.

Hint: use the smoothing property [1] of the conditional expectation

(f) (**Bonus 5%**) Consider the following settings:

(i) X_0 and Y_0 are *independent*. X_0 is a Gaussian random variable whereas Y_0 has a non Gaussian probability density.

(ii) X_0 and Y_0 are *independent*. Y_0 is a Gaussian random variable whereas X_0 has non Gaussian probability density.

(iii) X_0 and Y_0 *depend*. X_0 is a Gaussian random variable whereas Y_0 has non Gaussian probability density.

(iv) X_0 and Y_0 *depend*. Y_0 is a Gaussian random variable whereas X_0 has non Gaussian probability density.

Does the estimates, found in (d) and (e) remain optimal in each case? Optimal in the class of all linear estimates?

[1] i.e. $\mathbf{E}(\alpha|\beta_1, ..., \beta_n) = \mathbf{E}(\mathbf{E}(\alpha|\beta_1, ..., \beta_n, \beta_{n+1}, ...)|\beta_1, ..., \beta_n)$, see lecture note 8, page 4, property 8

Problem 3. (40%) *Comparison between linear and non linear filters*

A random parameter θ has a binary distribution:

$$\mathbf{P}\{\theta = 1\} = \pi_0, \quad \mathbf{P}\{\theta = 0\} = 1 - \pi_0$$

It is observed in a non Gaussian noise, so that:

$$Y_n = \theta + \xi_n, \quad n \geq 1$$

where $(\xi_n)_{n\geq 0}$ is a sequece of i.i.d. random variables with probability density $f(x)$. It is required to estimate θ from the observations $Y_1^n = \{Y_k, 1 \leq k \leq n\}$.

(a) (7%)Denote by $\widehat{\theta}_n$ the optimal *linear* estimate of θ from Y_0^n. Derive recursions for the estimate $\widehat{\theta}_n$ and the filtering error $P_n = \mathbf{E}(\theta - \widehat{\theta}_n)^2$ (assume $\mathbf{E}\xi_n^2 = \sigma^2 < \infty$ and $\mathbf{E}\xi_n = 0$)

(b) (7%) Does $\widehat{\theta}$ converge? If yes, to what limit and in what sense?

(c) (7%) Let $\pi_n \stackrel{\triangle}{=} \mathbf{E}(\theta | Y_0^n) = \mathbf{P}\{\theta = 1 | Y_0^n\}$. Derive the recursion for π_n.

(d) (7%) Show that the filtering error generated by the nonlinear filter in (c) on the basis of a single observation Y_1 is given by:

$$V_1 \stackrel{\triangle}{=} \mathbf{E}(\pi_1 - \theta)^2 = \pi_0(1 - \pi_0) \int_{-\infty}^{\infty} \frac{f(x) f(x-1)}{f(x-1)\pi_0 + f(x)(1-\pi_0)} dx$$

(e) (7%) Assume uniformly distributed noise: $f(x) = \begin{cases} 1/2, & x \in [-1, 1] \\ 0, & x \notin [-1, 1] \end{cases}$

Calculate explicitly V_1 and compare it to the error obtained in (a). For which values of π_0 the equality $V_1(\pi_0) = P_1(\pi_0)$ holds ?

(f) (**Bonus 5%**) Derive a recursion for $V_n = \mathbf{E}(\pi_n - \theta)^2$ for the case of the uniform noise as in (e). Do the sequences V_n and P_n converge? If yes, specify the limit and compare the rates of convergences?

Hint: simplify the estimate π_n for this specific setting.