RANDOM PROCESSES. THE FINAL TEST
Special Assignement

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∗ any supplementary material is allowed
∗ good luck !

Problem 1.

Consider the process $(X_n)_{n\geq 0}$ generated by a recursion:

$$X_n \;=\; \frac{X_{n-1}\xi_n}{\sqrt{X_{n-1}^2+\xi_n^2}}$$

subject to X_0 - a standard Gaussian random variable. $(\xi_n)_{n\geq 1}$ is a standard Gaussian i.i.d. sequence.

(a) Prove that X_n Gaussian random variable for any $n \geq 0$. [1]
(b) Is $(X_n)_{n\geq 0}$ Gaussian process ? Prove your answer.
(c) Find recursions for $m_n = \mathbf{E}X_n$ and $V_n = \mathbf{E}(X_n - m_n)^2$.
(d) Does X_n converge ? If yes to what limit and in what sense? [2]

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[1]To answer (a) from **Problem 1** you will need to prove the following:

Lemma 1.1. *Let α and β be a pair of independent Gaussian random variables with zero means and variances σ_α^2 and σ_β^2. Let:*

$$\gamma = \frac{\alpha\beta}{\sqrt{\alpha^2+\beta^2}}$$

then γ is Gaussian.

This lemma can be proved in several ways and you are encouraged to prove it as you wish. Though you may follow the advice:
(a) Show that in this case the distribution of γ is determined by distribution of $1/\gamma^2$.
(b) Find the characteristic function of $1/\alpha^2$ and $1/\beta^2$
(c) Use the fact that $1/\gamma^2 = 1/\alpha^2 + 1/\beta^2$ and the independence of α and β to find the distribution of $1/\gamma^2$ and hence also of γ.

[2]no need to show convergence with probability 1

Problem 2.

Consider a pair of random processes $(X_n, Y_n)_{n\geq 0}$, generated by:

$$\begin{aligned} X_n &= aX_{n-1} + b\varepsilon_n, \quad n = 1, 2, ... \\ Y_n &= AX_{n-1} + B\xi_n \end{aligned}$$

where a, b, A and B are constants and $(\varepsilon_n)_{n\geq 1}$ and $(\xi_n)_{n\geq 1}$ are independent i.i.d. standard Gaussian random sequences. The initial conditon X_0 is a Gaussian r.v. with zero mean and $P = \mathbf{E}X_0^2$.

(a) Find the recursion for the optimal estimate of the initial condition from the observations, i.e. $\pi_n = \mathbf{E}(X_0|Y_0^n)$

(b) Show [3] that the limit $V = \lim_{n\to\infty} \mathbf{E}(X_0 - \pi_n)^2$ exists and find its value.

Problem 3.

Let signal/observation model $(X_n, Y_n)_{n\geq 0}$:

$$\begin{aligned} X_n &= a_0(Y_0^{n-1}) + a_1(Y_0^{n-1})X_{n-1} + b\varepsilon_n, \quad n = 1, 2, ... \\ Y_n &= A_0(Y_0^{n-1}) + A_1(Y_0^{n-1})X_{n-1} + B\xi_n \end{aligned}$$

where b and B are constants and $A_i(Y_0^{n-1})$ and $a_i(Y_0^{n-1})$, $i = 0, 1$ are explicit bounded functionals of the vector $[Y_0, Y_1, ..., Y_{n-1}]$. $(\varepsilon_n)_{n\geq 1}$ and $(\xi_n)_{n\geq 1}$ are independent i.i.d. standard Gaussian random sequences. The initial condition (X_0, Y_0) is also a Gaussian vector.

(a) Is the pair of processes $(X_n, Y_n)_{n\geq 0}$ necessarily Gaussian ? Prove or verify your answer by example.

(b) Find the recursion for $\widehat{X}_n = \mathbf{E}(X_n|Y_0^n)$ and $P_n = \mathbf{E}\big[(X_n - \widehat{X}_n)^2|Y_0^n\big]$. Is the obtained filter linear ? time invariant ? asymptotically time invariant (i.e. time invariant as $n \to \infty$)?

(c) Verify that in case of $a_i(Y_0^{n-1}) \equiv a_i$ and $A_i(Y_0^{n-1}) \equiv A_i$, $i = 0, 1$ (a_i and A_i constants) your solution coincides with the Kalman filter.

[3] you may assume in this question that P is the positive solution of

$$P = Pa^2 + b^2 - \frac{A^2a^2P^2}{A^2P + B^2}$$