**RANDOM PROCESSES. THE FINAL TEST.**

**Prof. R. Liptser and Pavel Chigansky**

**14:00-17:00, 26 of June, 2000**

- ∗ any supplementary material is allowed
- ∗ duration of the exam is 3 hours
- ∗ note that the questions with lower relative weights are more complex
- ∗ the total score of the exam is 105 points
- ∗ please, use a separate notebook for the first question
- ∗ good luck !

**Problem 1.** *Conditional expectation and orthogonal projection*

For the random variables $X$, $Y$ and $Z$ verify the correctness of the following statements[1]. If the statement is correct prove your answer, otherwise give an *explicit* counterexample.

(a) [6] $\boldsymbol{E}\Big\{\boldsymbol{E}(X|Y)\Big|X\Big\} \stackrel{?}{=} X$

(b) [5] Independence of $X$ and $Y$ implies $\boldsymbol{E}(X|Y) = \boldsymbol{E}X$. Is the converse true, i.e.

$$\boldsymbol{E}(X|Y) = \boldsymbol{E}X \stackrel{?}{\Longrightarrow} X \text{ and } Y \text{ independent.}$$

(c) [4] $\boldsymbol{E}(X|Y) \stackrel{?}{=} \boldsymbol{E}\Big\{X\Big|\boldsymbol{E}(X|Y)\Big\}$

(d) [5]$\{X, Y, Z\}$ is Gaussian, such that $\boldsymbol{E}X = 0$ and $Z$ and $Y$ are independent.

$$\boldsymbol{E}(X|Y,Z) \stackrel{?}{=} \boldsymbol{E}(X|Y) + \boldsymbol{E}(X|Z).$$

(e) [5] Does (d) remain correct, if $\{X, Y, Z\}$ is non-Gaussian?

(f) [5] Assume $X, Y$ are non Gaussian random variables with finite second moments. Assume $\boldsymbol{E}(X|Y) = c_0 + c_1 Y$, where $c_0$ and $c_1$ are constants. Let $\widehat{\boldsymbol{E}}(X|Y)$ denote the orthogonal projection

$$\boldsymbol{E}(X|Y) \stackrel{?}{=} \widehat{\boldsymbol{E}}(X|Y)$$

(g) [4] Give an example of a pair of non Gaussian random variables $X$ and $Y$, so that $\boldsymbol{E}(X|Y) = c_0 + c_1 Y$, where $c_0$ and $c_1$ are some constants.

(h) [5] $X > Y$ implies $\mathbf{E}(X|Z) > \mathbf{E}(Y|Z)$. Show that this property is generally wrong for the orthogonal projections, i.e.

$$X > Y \not\Rightarrow \widehat{\boldsymbol{E}}(X|Z) > \widehat{\boldsymbol{E}}(Y|Z).$$

Describe one or several cases when it is nevertheless correct.

[1] all the comparisons between random variables (e.g. $X = Y$ and $X > Y$) are with probability 1

**Problem 2.** *Linear Filtering*

Let $(X_n)_{n\geq 0}$ be a scalar signal, generated by the difference equation:

$$X_n \quad = \quad a(\theta)X_{n-1} + b(\theta)\varepsilon_n, \quad n \geq 1$$

subject to $X_0$, a standard Gaussian random variable.

$(\varepsilon_n)_{n\geq 1}$ is a standard i.i.d. Gaussian sequence. $\theta$ is a random parameter, taking values from $S = \{1, ..., d\}$ with probabilities $\{p_1, ..., p_d\}$ respectively. Assume also that $\theta$, $X_0$ and $(\varepsilon_n)_{n\geq 1}$ are independent. The coefficients $a(x)$ and $b(x)$ are some given functions.

The signal $X_n$ is observed in additive white noise, so that the measurements are given by:

$$Y_n = X_{n-1} + \sigma\xi_n, \quad n \geq 1$$

where $\sigma$ is a known constant and $(\xi_n)_{n\geq 1}$ is a standard Gaussian i.i.d. sequence, independent of $(X_n)_{n\geq 1}$.

(a) [6] Is $(X_n, Y_n)_{n\geq 1}$ a conditionally Gaussian process, given $\theta$. Prove your answer.

(b) [5]Derive recursions for

$$m_n(\theta) = \boldsymbol{E}(X_n|\theta) \quad \text{and} \quad V_n(\theta) = \boldsymbol{E}\big((X_n - m_n(\theta))^2|\theta\big).$$

(c) [7] Is $(X_n, Y_n)_{n\geq 1}$ a Gaussian process ? Prove your answer.

(d) [7] Use the Kalman filter to generate $\widehat{X}_n(\theta) = \boldsymbol{E}(X_n|\theta, Y_1^n)$ and $P_n(\theta) = \boldsymbol{E}(X_n - \widehat{X}_n(\theta))^2|\theta)$.

(e) [7] Let $\widehat{X}_n = \boldsymbol{E}(X_n|Y_1^n)$ and $\pi_n(i) = \boldsymbol{P}\{\theta = i|Y_1^n\}$. Express $\widehat{X}_n = \boldsymbol{E}(X_n|Y_1^n)$ via $\widehat{X}_n(i) = \boldsymbol{E}(X_n|\theta = i, Y_1^n)$ and $\pi_n(i)$.

(f) [5] Find recursion for the filtering estimate $\pi_n(i) = \mathbf{P}\{\theta = i|Y_1^n\}$.

**Problem 3.** *Linear/Nonlinear Filtering*

Let $(\theta_n)_{n\geq 1}$ be a finite state Markov chain with states $S = \{a_1, ..., a_d\}$, transition probabilities

$$\lambda_{ij} := \boldsymbol{P}\{\theta_n = a_j | \theta_{n-1} = a_i\}, \quad n \geq 1, \quad i, j \in S$$

and initial distribution

$$p_i = \boldsymbol{P}\{\theta_0 = a_i\}, \quad i \in S$$

The observed process is given by:

$$Y_n = H(\theta_n) + \xi_n, \quad n \geq 1$$

where $H$ is known nonlinear function and $\xi_n$ is *colored* noise, generated by:

$$\xi_n = \gamma \xi_{n-1} + \varepsilon_n$$

subject to $\xi_0 \equiv 0$. The constant $\gamma$ is known and $(\varepsilon_n)_{n\geq 1}$ is an i.i.d. sequence of random variables, independent of $(\theta_n)_{n\geq 0}$. $\varepsilon_1$ is assumed to have a distribution density $f(x)$.

(a) [8] Under $\gamma = 0$, derive the optimal filter for $\theta_n$, i.e. find the recursion $\pi_n(i) = \mathbf{P}\{\theta_n = a_i | Y_1^n\}$.
(b) [8] Under $\gamma = 0$ and $\boldsymbol{E}\varepsilon_n^2 < \infty$ and $\boldsymbol{E}\varepsilon_n = 0$ use the Kalman filter for calculation of $\widehat{\theta}_n = \widehat{\boldsymbol{E}}(\theta_n | Y_1^n)$.
(c) [7] Under $\gamma \neq 0$ and $\boldsymbol{E}\varepsilon_n^2 < \infty$ and $\boldsymbol{E}\varepsilon_n = 0$ use the Kalman filter for calculation of $\widehat{\theta}_n = \widehat{\boldsymbol{E}}(\theta_n | Y_1^n)$.
(d) [6] Under $\gamma \neq 0$, derive the optimal filter for $\theta_n$, i.e. find the recursion $\pi_n(i) = \boldsymbol{P}\{\theta_n = a_i | Y_1^n\}$.