RANDOM PROCESSES. THE FINAL TEST.

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9:00-12:30, 7th of April, 2000

* any supplementary material is allowed
* duration of the exam is exactly 3.5 hours - no additional time will be permitted
* total score of this exam is 110 points
* good luck !

Problem 1. (40%) *Linear Filtering*

Let $(\xi_n)_{n\in\mathbb{Z}}$ be a zero mean stationary random sequence with positive spectral density $f(\lambda) > 0$, $\lambda \in [-\pi, \pi]$.

(a) Find the optimal linear (interpolating) estimate of ξ_0 from $\{\xi_k, k \neq 0\}$, i.e. $\widetilde{\xi}_0 = \widehat{E}(\xi_0|\xi_k, k \neq 0)$.

Hint: use the orthogonality principle and express your answer in terms of spectral density $f(\lambda)$.

(b) Find the corresponding interpolation error, i.e. $\widetilde{P} = \mathbf{E}(\xi_0 - \widetilde{\xi}_0)^2$.

(c) Verify and explain your answers in (a) and (b) for the case of white noise, i.e. when ξ_n is an i.i.d. sequence.

Assume from here on, that the signal ξ_n is generated by a recursion:

$$\xi_n = a\xi_{n-1} + b\varepsilon_n,$$

where $(\varepsilon_n)_{n\in\mathbb{Z}}$ is a standard i.i.d. Gaussian sequence and a and b are constants ($|a| < 1$).

Find explicit expressions for the optimal interpolating estimate of ξ_0, given $\mathcal{F}_\xi$ and the corresponding mean square error when:

(d) $\mathcal{F}_\xi = \{\xi_k, k \neq 0\}$

Hint: prove and use the fact: $\mathbf{E}(\xi_0|\xi_k, k \neq 0) = \mathbf{E}(\xi_0|\xi_1, \xi_{-1})$

(e) $\mathcal{F}_\xi = \mathcal{F}_\xi(n) = \{\xi_1, ..., \xi_n\}$ (recursive estimate is required)

(f) $\mathcal{F}_\xi = \{\xi_k, k > 0\}$

Hint: solve (e) before (f)

Problem 2. (40%) *Nonlinear Filtering*

Let θ_n be a Markov chain with values $\{a_1, ..., a_d\}$, transition probabilities $\lambda_{ij} = \mathbf{P}\{\theta_n = a_j | \theta_{n-1} = a_i\}$ and initial distribution $p_i = \mathbf{P}\{\theta_0 = a_i\}$. It is observed in a channel with distortion, so that the observable signal is:

$$Y_n = \theta_n + \gamma\theta_{n-1} + \xi_n, \quad n \geq 1$$

where γ is a known distortion coefficient and $(\xi_n)_{n\geq 1}$ is an i.i.d. sequence with probability density function $f(x)$.

(a) Assuming that $\mathbf{E}\xi_n^2 < \infty$, derive the equations of the optimal *linear* recursive filter, i.e. find:

$$\widehat{\theta}_n = \widehat{\mathbf{E}}(\theta_n | Y_1^n), \quad P_n = \mathbf{E}(\theta_n - \widehat{\theta}_n)^2$$

(b) Derive the equation of the optimal filter, i.e. find:

$$\pi_n(i) = \mathbf{P}\{\theta_n = a_i | Y_1^n\}$$

(c) Verify that the filter obtained in (b) coincides with conventional Wonham filter for $\gamma = 0$.

(d) (Bonus+5) Assume that γ is replaced by γ_n, an i.i.d. Gaussian sequence with $\mathbf{E}\gamma_n = \gamma$ and variance σ_γ^2. Assume also that ξ_n is an i.i.d. Gaussian sequence with zero mean and variance σ_ξ^2. Find the optimal estimate $\pi_n(i)$ for this case. Does the obtained filter coincide with the filter in (b) when $\gamma = 0$?

Problem 3. (20%) *Convergence Of Random Sequences*

Define a metric between two random variables X and Y:

$$d(X,Y) = \mathbf{E}\left(\frac{|X-Y|}{1+|X-Y|}\right)$$

Show that convergence in probability is equivalent to convergence in d-metric, i.e.

(a) Show that:

$$\lim_{n\to\infty} d(X_n, X) = 0 \implies X_n \xrightarrow{\mathbf{P}} X$$

(b) (Bonus +2.5) Show that:

$$X_n \xrightarrow{\mathbf{P}} X \implies \lim_{n\to\infty} d(X_n, X) = 0$$

(c) Give a *specific* example of a metric $d'(X,Y)$, such that:

$$\lim_{n\to\infty} d'(X_n, X) = 0 \quad \begin{matrix} \Rightarrow \\ \not\Leftarrow \end{matrix} \quad X_n \xrightarrow{\mathbf{P}} X$$

(d) Give a *specific* example of a metric $d''(X,Y)$, such that:

$$\lim_{n\to\infty} d''(X_n, X) = 0 \quad \begin{matrix} \Leftarrow \\ \not\Rightarrow \end{matrix} \quad X_n \xrightarrow{\mathbf{P}} X$$

(e) (Bonus +2.5) Show that if C is a non random constant [1], then

$$X_n \xrightarrow{d} C \implies X_n \xrightarrow{\mathbf{P}} C$$

Hint: use previous results from this problem.

[1] $\xi_n \xrightarrow{d} \xi$ denotes convergence in distribution, i.e. $\mathbf{E}f(\xi_n) \to \mathbf{E}f(\xi)$ for any bounded and continuous function f