

RANDOM PROCESSES. THE FINAL TEST.

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9:00-12:00, 29 of June, 2001

Student ID:_____

- * any supplementary material is allowed
- * duration of the exam is 3 hours
- * write briefly the main idea of your answers in the exam itself. If needed, give reference to your copybook, where you may place other technical details.
- * good luck !

Problem 1.

Let $\mathbf{E}(\xi|\eta)$ and $\widehat{\mathbf{E}}(\xi|\eta)$ denote the conditional expectation and orthogonal projection, where ξ and η are random variables. Which of the following statements are generally true:

(a) $\mathbf{E}(\widehat{\mathbf{E}}(\xi|\eta)|\eta) = \mathbf{E}(\xi|\eta)$

TRUE/FALSE, proof/counterexample:

(b) $\mathbf{E}(\widehat{\mathbf{E}}(\xi|\eta)|\eta) = \widehat{\mathbf{E}}(\xi|\eta)$

TRUE/FALSE, proof/counterexample:

(c) $\widehat{\mathbf{E}}(\mathbf{E}(\xi|\eta)|\eta) = \mathbf{E}(\xi|\eta)$

TRUE/FALSE, proof/counterexample:

(d) $\widehat{\mathbf{E}}(\mathbf{E}(\xi|\eta)|\eta) = \widehat{\mathbf{E}}(\xi|\eta)$

TRUE/FALSE, proof/counterexample:

Problem 2.

Let $(X_n)_{n \geq 0}$ be a Markov chain with values in $\{-1, 0, 1\}$, transition matrix $\Lambda_{ij} = \mathbf{P}(X_n = j | X_{n-1} = i)$:

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1/4 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

and initial distribution $p_0 = [\alpha, \beta, \gamma]$.

(a) The sequence X_n

i. **converges/diverges** in probability. Proof:

ii. **converges/diverges** in \mathbb{L}^2 . Proof:

iii. **converges/diverges** in distribution. Proof:

iv. (bonus +5). **converges/diverges** with probability one. Proof:

(b) the limit has the following distribution ¹:

(c) the limit is deterministic if α , β and γ satisfy the relation:

¹pass to **Problem 3** if you think that X_n diverges in all senses

Problem 3.

Let $(X_n)_{n \geq 0}$ be a Markov chain with finite alphabet $\{a_1, \dots, a_m\}$, transition probabilities $\lambda_{ij} = \mathbf{P}(X_n = a_j | X_n = a_i)$ and initial distribution $p_0(i) = \mathbf{P}(X_0 = a_i)$. Let Y_n be defined by

$$Y_n = I(X_n \in \mathcal{J})$$

where \mathcal{J} is some subset of the alphabet (Y_n is called *aggregation* of X_n). As usual denote $\pi_{n|n-1}(i) = \mathbf{P}(X_n = a_i | Y_0^{n-1})$ and $\pi_n(i) = \mathbf{P}(X_n = a_i | Y_0^n)$.

(a) Choose the correct answer :

- (i) $\pi_n(i) = \frac{I(a_i \in \mathcal{J})}{\sum_{j \in \mathcal{J}} \pi_{n|n-1}(j)} I(Y_n = 1) + \frac{I(a_i \notin \mathcal{J})}{\sum_{j \notin \mathcal{J}} \pi_{n|n-1}(j)} I(Y_n = 0)$
- (ii) $\pi_n(i) = \frac{\pi_{n|n-1}(i) I(a_i \in \mathcal{J})}{\sum_{j \in \mathcal{J}} \pi_{n|n-1}(j)} I(Y_n = 1) + \frac{\pi_{n|n-1}(i) I(a_i \notin \mathcal{J})}{\sum_{j \notin \mathcal{J}} \pi_{n|n-1}(j)} I(Y_n = 0)$
- (iii) $\pi_n(i) = \frac{\pi_{n|n-1}(i)}{\sum_{j \in \mathcal{J}} \pi_{n|n-1}(j)} I(Y_n = 1) + \frac{\pi_{n|n-1}(i)}{\sum_{j \notin \mathcal{J}} \pi_{n|n-1}(j)} I(Y_n = 0)$
- (iv) $\pi_n(i) = \frac{\pi_{n|n-1}(i) I(a_i \in \mathcal{J})}{\sum_{j \notin \mathcal{J}} \pi_{n|n-1}(j)} I(Y_n = 1) + \frac{\pi_{n|n-1}(i) I(a_i \notin \mathcal{J})}{\sum_{j \in \mathcal{J}} \pi_{n|n-1}(j)} I(Y_n = 0)$

(b) Let $\hat{\pi}_n(i) = \hat{\mathbf{E}}(I(X_n = a_i) | Y_0^n)$. Write down the Kalman filter equations for $\hat{\pi}_n(i)$:

Problem 4. (mouse & cat)

A mouse tries to escape a cat and is awaiting for it (probably before deciding where to run), at a fixed position, which is standard Gaussian r.v. θ . The cat measures θ with an error, so that

$$Y_n = \theta + \xi_n$$

where $(\xi_n)_{n \geq 0}$ is standard i.i.d. Gaussian sequence.

(a) Write down the recursions for $m_n = \mathbf{E}(\theta|Y_0^n)$ and $P_n^m = \mathbf{E}(\theta - m_n)^2$ (cat's optimal tracking policy):

(b) The mouse assumes that the cat moves according to the optimal rule from (a) and wants to estimate cat's position with minimal mean square error. The information available to the mouse is θ , i.e. its own position. Write down the recursion for the optimal estimate of the cat's position, i.e. $c_n = \mathbf{E}(m_n|\theta)$ and $P_n^c = \mathbf{E}(c_n - m_n)^2$

(c) Find the limits for the filtering errors for the mouse and the cat:

$$\lim_{n \rightarrow \infty} P_n^m =$$

$$\lim_{n \rightarrow \infty} P_n^c =$$

(d) Choose the correct answer:

- i. $P_n^c \geq P_n^m$ for all $n \geq 0$.
- ii. $P_n^c \leq P_n^m$ for all $n \geq 0$.
- iii. $P_n^c \equiv P_n^m$ for all $n \geq 0$.
- iv. None of the above is true

(e) Does your answer in (d) hold for general mouse/cat models (e.g. non-constant, non Gaussian, nonlinear cat/mouse positions, etc.)

i. yes, here is the proof:

ii. no, here is a counterexample: