RANDOM PROCESSES. THE FINAL TEST.

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9:00-12:00, 18 of October, 2001

Student ID:_____

∗ any supplementary material is allowed

∗ duration of the exam is 3 hours

∗ write <u>briefly</u> the main idea of your answers in the exam itself. If needed, give <u>reference</u> to your copybook, where you may elaborate other technical details.

∗ good luck !

Problem 1.

Let ξ and η be a pair of random variables, such that $\mathbf{E}\xi = \mathbf{E}\eta = 0$, $\mathbf{E}\xi^2 < \infty$, $\mathbf{E}\eta^2 < \infty$ and $\mathbf{E}\xi\eta \neq 0$. Which of the following statements are generally true:

(a) $\mathbf{E}\big(\eta | \widehat{\mathbf{E}}(\eta|\xi)\big) = \mathbf{E}(\eta|\xi)$

    TRUE/FALSE, proof/counterexample:

(b) $\widehat{\mathbf{E}}\big(\eta | \mathbf{E}(\eta|\xi)\big) = \mathbf{E}(\eta|\xi)$

    TRUE/FALSE, proof/counterexample:

(c) $\widehat{\mathbf{E}}\big(\eta | \mathbf{E}(\eta|\xi)\big) = \widehat{\mathbf{E}}(\eta|\xi)$

    TRUE/FALSE, proof/counterexample:

(d) $\widehat{\mathbf{E}}\big(\eta | \widehat{\mathbf{E}}(\eta|\xi)\big) = \widehat{\mathbf{E}}(\eta|\xi)$

    TRUE/FALSE, proof/counterexample:

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(e)
$$\left.\begin{array}{c} \widehat{\mathbf{E}}(\eta|\xi) = \xi \\ \widehat{\mathbf{E}}(\xi|\eta) = \eta \end{array}\right\} \implies \eta = \xi \quad \mathbf{P} - a.s.$$

TRUE/FALSE, proof/counterexample:

(f) $\mathbf{E}\big(\mathbf{E}(\eta|\xi)\big|\eta\big) = \eta \implies \mathbf{E}(\eta|\xi) = \eta \quad \mathbf{P} - a.s.$

TRUE/FALSE, proof/counterexample:

Problem 2.

Let $(X_n)_{n\geq 0}$ be a Markov chain, taking values in the alphabet $\mathbb{S} = \{1, ..., d\}$, with transition probabilities $\lambda_{ij} = \mathbf{P}\{X_n = j | X_{n-1} = i\}$ and initial distribution $\pi_0(i) = \mathbf{P}\{X_0 = i\}$. Define *transitions indicator* process $(\nu_n)_{n\geq 1}$:

$$\nu_n = I(X_n \neq X_{n-1})$$

(a) Which of the following filters generates $\pi_n = \mathbf{P}(X_n = i | \nu_1^n)$:

(1) $\quad \pi_n(i) = \dfrac{\lambda_{ii}\pi_{n-1}(i)}{\sum_j \lambda_{jj}\pi_{n-1}(j)} I(\nu_n = 0) + \dfrac{\pi_{n-1}(i)\sum_{k\neq i}\lambda_{ik}}{\sum_j \pi_{n-1}(j)\sum_{k\neq j}\lambda_{jk}} I(\nu_n = 1)$

(2) $\quad \pi_n(i) = \dfrac{\lambda_{ii}\pi_{n-1}(i)}{\sum_j \lambda_{jj}\pi_{n-1}(j)} I(\nu_n = 0) + \dfrac{\sum_{k\neq i}\lambda_{ki}\pi_{n-1}(k)}{\sum_j \sum_{k\neq j}\lambda_{kj}\pi_{n-1}(k)} I(\nu_n = 1)$

(3) $\quad \pi_n(i) = \pi_{n-1}(i)I(\nu_n = 0) + \dfrac{\pi_{n-1}(i)\sum_{k\neq i}\lambda_{ik}}{\sum_j \pi_{n-1}(j)\sum_{k\neq j}\lambda_{jk}} I(\nu_n = 1)$

(4) $\quad \pi_n(i) = \pi_{n-1}(i)I(\nu_n = 0) + \displaystyle\sum_k \lambda_{ki}\pi_{n-1}(k)I(\nu_n = 1)$

(b) Assume that additional observation process is available:

$$Y_n = a(X_n) + \xi_n$$

where $a(i)$ is $\mathbb{S} \mapsto \mathbb{R}$ function, $(\xi_n)_{n\geq 1}$ is a sequence of i.i.d. random variables and ξ_1 has a probability density function $f(x)$. Which of the following filters generates the estimates $\zeta_n(i) = \mathbf{P}(X_n = i | \nu_1^n, Y_1^n)$:

(1) $\zeta_n(i) = \dfrac{\lambda_{ii}\zeta_{n-1}(i)f(Y_n - a_i)}{\sum_j \lambda_{jj}\zeta_{n-1}(j)f(Y_n - a_j)} I(\nu_n = 0)+$

$$+ \dfrac{\zeta_{n-1}(i)\sum_{k\neq i}\lambda_{ik}f(Y_n - a_k)}{\sum_j \zeta_{n-1}(j)\sum_{k\neq j}\lambda_{jk}f(Y_n - a_k)} I(\nu_n = 1)$$

(2) $\zeta_n(i) = \dfrac{\lambda_{ii}\zeta_{n-1}(i)f(Y_n - a_i)}{\sum_j \lambda_{jj}\zeta_{n-1}(j)f(Y_n - a_j)} I(\nu_n = 0)+$

$$+ \dfrac{f(Y_n - a_i)\sum_{k\neq i}\lambda_{ki}\zeta_{n-1}(k)}{\sum_j f(Y_n - a_j)\sum_{k\neq j}\lambda_{kj}\zeta_{n-1}(k)} I(\nu_n = 1)$$

(3) $\zeta_n(i) = \zeta_{n-1}(i)f(Y_n - a_i)I(\nu_n = 0)+$

$$+ \dfrac{\zeta_{n-1}(i)\sum_{k\neq i}\lambda_{ik}f(Y_n - a_k)}{\sum_j \zeta_{n-1}(j)\sum_{k\neq j}\lambda_{jk}f(Y_n - a_k)} I(\nu_n = 1)$$

(4) $\zeta_n(i) = \zeta_{n-1}(i)f(Y_n - a_i)I(\nu_n = 0)+$

$$+ f(Y_n - a_i)\sum_k \lambda_{ki}\zeta_{n-1}(k)I(\nu_n = 1)$$

Problem 3.

Consider process $(X_n)_{n \geq 0}$, generated by so called *bilinear* recursion:
$$X_n = \alpha X_{n-1} + \beta X_{n-1}\varepsilon_n, \quad X_0 \equiv 1$$
where α, β are real numbers and $(\varepsilon_n)_{n \geq 1}$ is a sequence of i.i.d. random variables, $\mathbf{E}\varepsilon_1^2 < \infty$.

(a) Determine in which of the following cases (X_n) converges in appropriate sense (check all correct answers and explain briefly)

(1) $\alpha = \beta = 0.75$, ε_1 is distributed uniformly on $[-1, 1]$.
in $\mathbb{L}^2$	: ☐
in $\mathbb{L}^1$	: ☐
in probability	: ☐
in distribution	: ☐

(2) $\alpha = \beta = 0.75$, ε_1 takes two values $\{-1, 1\}$ with equal probabilities.
in $\mathbb{L}^2$	: ☐
in $\mathbb{L}^1$	: ☐
in probability	: ☐
in distribution	: ☐

(3) $\alpha = 0$, $\beta = 1$, ε_1 takes two values $\{-1, 1\}$ with equal probabilities.
in $\mathbb{L}^2$	: ☐
in $\mathbb{L}^1$	: ☐
in probability	: ☐
in distribution	: ☐

(4) $\alpha = 1$, $\beta = 1/2$, ε_1 takes two values $\{-1, 1\}$ with equal probabilities.
in $\mathbb{L}^2$	: ☐
in $\mathbb{L}^1$	: ☐
in probability	: ☐
in distribution	: ☐

Note: From now on assume $\mathbf{E}\varepsilon_1 = 0$ and $\mathbf{E}\varepsilon_1^2 = 1$.

(b) Consider the observation sequence

$$Y_n = X_{n-1} + \varepsilon_n$$

Write down the recursive equations for $\widehat{X}_n = \widehat{\mathbf{E}}(X_n|Y_1^n)$ and $P_n = \mathbf{E}(X_n - \widehat{X}_n)^2$:

$\widehat{X}_n = \ldots$

$P_n = \ldots$

(c) Consider the case **(a)**-(3), i.e.

$$X_n = X_{n-1}\varepsilon_n, \quad X_0 \equiv 1$$
$$Y_n = X_{n-1} + \varepsilon_n$$

where $\varepsilon_1 = \pm 1$ with equal probabilities. In this question $\widehat{\mathbf{E}}(X_n|Y_1^n)$ and $\mathbf{E}(X_n|Y_1^n)$ are understood as 'linear' and 'nonlinear' filters respectively. A filter is 'trivial' if it does not depend on the observations and is 'precise' if it estimates the signal exactly (i.e. with zero mean square error). Choose the correct statement and prove your answer

(1) both the linear and nonlinear filters are trivial
(2) the nonlinear filter is trivial, while the linear one is precise
(3) the nonlinear filter is precise, while the linear one is trivial
(4) both the linear and nonlinear filters are precise
(5) none of the above

Proof: