RANDOM PROCESSES. THE FINAL TEST.

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9:00-12:00, 28 of June, 2002

Student ID:_______

* any supplementary material is allowed
* duration of the exam is 3 hours
* write briefly the main idea of your answers in the exam itself. If needed, give reference to your copybook, where you may place other technical details.
* the total score of the exam is 110
* good luck !

Problem 1.

In this problem X, Y, Z, V are random variables with zero means. Verify the following statements:

(a) If X and Y are uncorrelated and each one is Gaussian, then the pair (X, Y) is Gaussian.

true/**false** here is the proof/counterexample:

(b) If $\widehat{\mathbf{E}}(X|Y) = \mathbf{E}(X|Y)$ and $\mathbf{E}(X|Y)$ is Gaussian random variable with positive variance, then Y is Gaussian

true/**false** here is the proof/counterexample:

(c) In **(b)** the pair (X, Y) is Gaussian

true/**false** here is the proof/counterexample:

(d) If X and $\widehat{\mathbf{E}}(X|Y)$ are uncorrelated, then X and Y are uncorrelated.

true/**false** here is the proof/counterexample:

(e) If X and $\mathbf{E}(X|Y)$ are independent, then X and Y are independent.

true/**false** here is the proof/counterexample:

(f) If X and V are independent and $Y = \alpha X + V$ with some constant α, then for Z independent of V

$$\widehat{\mathbf{E}}\Big(\widehat{\mathbf{E}}(Z|X)\big|Y\Big) = \widehat{\mathbf{E}}(Z|Y)$$

true/**false** here is the proof/counterexample:

(g) In **(f)**

$$\mathbf{E}\Big(\mathbf{E}(Z|X)\big|Y\Big) = \mathbf{E}(Z|Y)$$

true/**false** here is the proof/counterexample:

Problem 2.

Let $(X_n)_{n\geq 1}$ be an i.i.d. sequence of binary random variables $\mathbf{P}(X_1 = \pm 1) = 1/2$. Define two observations processes

$$Y_n = X_n - X_{n-1}, \quad n \geq 2$$

and

$$Z_n = X_n / X_{n-1}, \quad n \geq 2$$

Consider the following recursions, which are driven by an arbitrary input sequence of numbers $x_2, x_3, ...$

$$(1) \quad \psi_n \equiv 1/2$$

$$(2) \quad \begin{cases} \psi_n = (x_n + \psi_{n-1})/(1 + P_{n-1}) \\ P_n = P_{n-1}/(1 + P_{n-1}) \end{cases}$$

$$(3) \quad \psi_n = \frac{1 - 2\psi_{n-1}}{8} x_n^2 + \frac{1}{4} x_n + \psi_{n-1}$$

$$(4) \quad \psi_n = \frac{1 - \psi_{n-1}}{2} x_n + \frac{1 + \psi_{n-1}}{2}$$

$$(5) \quad \psi_n \equiv 0$$

$$(6) \quad \begin{cases} \psi_n = P_{n-1}(x_n + \psi_{n-1})/(1 + P_{n-1}) \\ P_{n-1} \text{ as in (2)} \end{cases}$$

If either Y_n or Z_n is substituted into one of these recursions [1], the obtained equation can be regarded as a filtering estimate of X_n.

Hints:

* *Note that $Y_n \in \{-2, 0, 2\}$ and $Z_n \in \{-1, 1\}$, so that some recursions may look simpler*

* *Solve (c) before (d)*

[1] i.e. e.g. $x_2 := Y_2, x_3 := Y_3, ...$

(a) $\pi_n = \mathbf{P}(X_n = 1|Y_2^n)$ satisfies the recursion #________

This recursion is to be initialized by ________

For my proof see page #________ of the copybook

(b) $\widehat{X}_n = \widehat{\mathbf{E}}(X_n|Y_2^n)$ satisfies the recursion #________

This recursion is to be initialized by ________

For my proof see page #________ of the copybook

(c) $\rho_n = \mathbf{P}(X_n = 1|Z_2^n)$ satisfies the recursion #________

This recursion is to be initialized by ________

For my proof see page #________ of the copybook

(d) $\widetilde{X}_n = \widehat{\mathbf{E}}(X_n|Z_2^n)$ satisfies the recursion #________

This recursion is to be initialized by ________

For my proof see page #________ of the copybook

(e) Check the correct answer

(1) $\mathbf{E}(X_1|Y_2^n)$ converges to X_1, while $\mathbf{E}(X_1|Z_2^n)$ converges to zero
(2) $\mathbf{E}(X_1|Y_2^n)$ converges to zero, while $\mathbf{E}(X_1|Z_2^n)$ converges to X_1
(3) Both $\mathbf{E}(X_1|Y_2^n)$ and $\mathbf{E}(X_1|Z_2^n)$ converge to X_1
(4) Both $\mathbf{E}(X_1|Y_2^n)$ and $\mathbf{E}(X_1|Z_2^n)$ converge to zero

Hint: *estimation error of X_1 and X_n given e.g. Y_2^n is the same (why ?)*

Explain briefly your answer:

(f) Check the correct answer(s). The convergence in **(e)** is

(1) in $\mathbb{L}_1$
(2) in $\mathbb{L}_2$
(3) in probability
(4) with probability one *(bonus+3)*
(5) in distribution

Problem 3.

Let θ be a binary random variable, with values $\{0, 1\}$ and $\mathbf{P}(\theta = 0) = P(\theta = 1) = 1/2$. It is to be estimated from the observation of

$$Y_t = \int_0^t \theta ds + W_t = \theta t + W_t$$

where W_t is a Wiener process independent of θ.

Linear estimate

Consider the estimate $\widehat{\theta}_t = 1/t \int_0^t dY_s = Y_t/t$.

(a) Calculate the estimate error mean and variance

$$\mathbf{E}(\theta - \widehat{\theta}_t) = ... \qquad \mathbf{E}(\theta - \widehat{\theta}_t)^2 = ...$$

(b) $\widehat{\theta}_t$ satisfies the equation

(1) $d\widehat{\theta}_t = 1/t dY_t$
(2) $d\widehat{\theta}_t = 1/t\big(dY_t - \widehat{\theta}_t dt\big)$
(3) $d\widehat{\theta}_t = \big(dY_t - \widehat{\theta}_t dt\big)$
(4) $d\widehat{\theta}_t = \widehat{\theta}_t dt + 1/t dY_t$

Here is the proof:

<u>Nonlinear estimate</u>

Consider the nonlinear estimate of θ

$$\pi_t = \frac{1}{1+\exp\{t/2 - Y_t\}}$$

(c) Calculate the estimate error mean

$$\mathbf{E}(\theta - \pi_t) = ...$$

(d) π_t satisfies the equation

(1) $d\pi_t = \pi_t dY_t$
(2) $d\pi_t = \pi_t\big(dY_t - \pi_t dt\big)$
(3) $d\pi_t = \pi_t(1-\pi_t)\big(dY_t - \pi_t dt\big)$
(4) $d\pi_t = \pi_t(1-\pi_t)dY_t$

Hint: *first derive an equation for* $\xi_t = \exp\{t/2 - Y_t\}$ *and then proceed to calculation of* $d\pi_t$

Here is my proof: