RANDOM PROCESSES. THE FINAL TEST.

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9:00-12:00, 13 of June, 2003

Student ID:_______

* any supplementary material is allowed
* duration of the exam is 3 hours
* write briefly the main idea of your answers in the exam itself. If needed, give the reference to your copybook, where you may place other technical details
* note that the problems are more or less in the chronological order of the chapters in the course and **not** in any monotonic order of complexity
* the total score of the exam is 100
* good luck !

Problem 1.

Consider the following model for population growth. Suppose that the population size satisfies the equation[1]

$$X_n = \sum_{j=1}^{X_{n-1}} \xi_{n,j}, \quad X_0 = N > 0$$

where $(\xi_{n,j})_{n,j\geq 1}$ are i.i.d. random variables

$$P(\xi_{1,1} = 0) = 1 - p - q, \quad P(\xi_{1,1} = 1) = p, \quad P(\xi_{1,1} = 2) = q$$

with $p + q < 1$.

(a) The sequence X_n converges to zero in $\mathbb{L}^1$ sense if

1. $p + 2q < 1$
2. $2p + q < 1$
3. $p < 2q$
4. $2p < q$

Hint: use the smoothing property of conditional expectation

(b) The sequence X_n converges to zero in $\mathbb{L}^2$ sense if

1. $p + 2q < 1$
2. $2p + q < 1$
3. $p < 2q$
4. $2p < q$

[1]By convention $\sum_{j=1}^{0} \ldots \equiv 0$

Now consider the modified model

$$X_n = \min\left\{\widetilde{N}, \sum_{j=1}^{X_{n-1}} \xi_{n,j}\right\}$$

where $\widetilde{N}$ is some constant integer.

(b) X_n converges to zero in the sense (check all correct answers)

1. P-a.s
2. in probability
3. $\mathbb{L}^1$
4. $\mathbb{L}^2$
5. none of the above

Hint: try to verify convergence in probability first.

Problem 2.

Let $X_n = \varepsilon_n + \varepsilon_{n-1}$, $n \geq 1$ where $(\varepsilon_n)_{n\geq 0}$ is an i.i.d. standard Gaussian sequence.

(a) The estimate $\widehat{X}_n = \mathbf{E}\big(X_n | X_1^{n-1}\big)$ satisfies the equation $(n \geq 1)$

1. $\widehat{X}_{n+1} = \dfrac{n}{n+1} X_n$
2. $\widehat{X}_{n+1} = \dfrac{1}{n+1} X_n$
3. $\widehat{X}_{n+1} = \dfrac{n}{n+1}\big(X_n - \widehat{X}_n\big), \quad \widehat{X}_1 = 0$
4. $\widehat{X}_{n+1} = \dfrac{1}{n+1}\big(X_n - \widehat{X}_n\big), \quad \widehat{X}_1 = 0$

Hint: first, try to see how $\widehat{X}_n$ is related to $\mathbf{E}\big(\varepsilon_n | X_1^n\big)$.

(b) The mean square error $Q_n = \mathbf{E}(X_n - \widehat{X}_n)^2$, $n \geq 2$ equals

1. $Q_n = 1 + 1/n$
2. $Q_n = 1 - 1/n$
3. $Q_n = 1/n$
4. $Q_n \equiv 1$

(c) The estimate $\widehat{X}^\circ_n = \mathbf{E}\big(X_n | X_1^{n-1}, \varepsilon_0\big)$ satisfies the equation $(n \geq 1)$

$$\mathbf{1}.\quad \widehat{X}^\circ_{n+1} = \frac{1}{2}(\widehat{X}^\circ_n - X_n), \quad \widehat{X}^\circ_1 = \varepsilon_0$$
$$\mathbf{2}.\quad \widehat{X}^\circ_{n+1} = \frac{1}{2}(\widehat{X}^\circ_n + X_n), \quad \widehat{X}^\circ_1 = \varepsilon_0$$
$$\mathbf{3}.\quad \widehat{X}^\circ_{n+1} = X_n$$
$$\mathbf{4}.\quad \widehat{X}^\circ_{n+1} = X_n - \widehat{X}^\circ_n, \quad \widehat{X}^\circ_1 = \varepsilon_0$$

(d) The mean square error $Q^\circ_n = \mathbf{E}(X_n - \widehat{X}^\circ_n)^2$, $n \geq 2$ equals

$$\mathbf{1}.\quad Q^\circ_n = 1 + 1/n$$
$$\mathbf{2}.\quad Q^\circ_n = 1 - 1/n$$
$$\mathbf{3}.\quad Q^\circ_n = 1/n$$
$$\mathbf{4}.\quad Q^\circ_n \equiv 1$$

Problem 3.

A server provides service for a pair of identical independent clients: A and B. At any service cycle n each client requests the service with probability p, independently of its past requests. The server makes transitions between the clients or resides in the idle state (I). If the clients generate simultaneous requests, the priority is given to the client served in the last cycle. If the last cycle was idle, the clients are chosen at random with probabilities 1/2.

Let X_n denote the state of the server at time n.

(a) The process X_n would not be Markov, if

1. under simultaneous requests, the clients were chosen with probability, other than 1/2
2. under simultaneous requests, the clients were not chosen at random
3. the clients were not identical (i.e. had unequal request probabilities)
4. the clients were not independent (but still independent of the past!)
5. none of the above

(b) The transition matrix of X_n is (the rows correspond to A, I and B respectively)

$$\mathbf{1}.\quad \Lambda = \begin{pmatrix} 1-p & p^2 & (1-p)p \\ 1/2-p^2/2 & p^2 & 1/2-p^2/2 \\ p(1-p) & p^2 & 1-p \end{pmatrix}$$

$$\mathbf{2}.\quad \Lambda = \begin{pmatrix} 1-p & p^2 & (1-p)p \\ p-p^2/2 & (1-p)^2 & p-p^2/2 \\ p(1-p) & p^2 & 1-p \end{pmatrix}$$

$$\mathbf{3}.\quad \Lambda = \begin{pmatrix} p & (1-p)^2 & (1-p)p \\ 1/2-p^2/2 & p^2 & 1/2-p^2/2 \\ p(1-p) & (1-p)^2 & p \end{pmatrix}$$

$$\mathbf{4}.\quad \Lambda = \begin{pmatrix} p & (1-p)^2 & (1-p)p \\ p-p^2/2 & (1-p)^2 & p-p^2/2 \\ p(1-p) & (1-p)^2 & p \end{pmatrix}$$

(c) Let α_n and β_n denote CPU load of the clients A and B at cycle n respectively. Suppose that when a client is not getting the service, its load has standard exponential distribution, while under service the load is higher and has exponential distribution with parameter $\lambda < 1$, i.e.

$$\frac{d}{dt}P(\beta_n \le t|X_n \in \{I, A\}) = \frac{d}{dt}P(\alpha_n \le t|X_n \in \{I, B\}) = e^{-t}I(t \ge 0)$$
$$\frac{d}{dt}P(\beta_n \le t|X_n = B) = \frac{d}{dt}P(\alpha_n \le t|X_n = A) = \lambda e^{-\lambda t}I(t \ge 0)$$

Moreover the CPU loads are conditionally independent, given the state of the server X_n. The estimate $\pi_n(I) = P\big(X_n = I|\alpha_1^n, \beta_1^n\big)$ satisfies

1. $\pi_n(I) = \dfrac{\pi_{n|n-1}(I)}{\lambda\pi_{n|n-1}(A)e^{(1+\lambda)\alpha_n} + \pi_{n|n-1}(I) + \lambda\pi_{n|n-1}(B)e^{(1+\lambda)\beta_n}}$
2. $\pi_n(I) = \dfrac{\pi_{n|n-1}(I)}{\lambda\pi_{n|n-1}(A)e^{\lambda\alpha_n} + \pi_{n|n-1}(I) + \lambda\pi_{n|n-1}(B)e^{\lambda\beta_n}}$
3. $\pi_n(I) = \dfrac{\pi_{n|n-1}(I)}{\lambda\pi_{n|n-1}(A)e^{(1-\lambda)\alpha_n} + \pi_{n|n-1}(I) + \lambda\pi_{n|n-1}(B)e^{(1-\lambda)\beta_n}}$
4. $\pi_n(I) = \dfrac{\pi_{n|n-1}(I)}{\lambda\pi_{n|n-1}(A)e^{-\lambda\alpha_n} + \pi_{n|n-1}(I) + \lambda\pi_{n|n-1}(B)e^{-\lambda\beta_n}}$

where the usual notations are used (e.g. $\pi_{n|n-1}(A) = P(X_n = A|\alpha_1^{n-1}, \beta_1^{n-1})$, etc.).

Problem 4.

Let S_t be the solution of the Ito equation

$$dS_t = -rS_t dt + \sigma S_t dW_t, \quad S_0 = 1$$

where $r, \sigma > 0$ and $(W_t)_{t \geq 0}$ is a Wiener process.

(a) The mean $m_t = \mathbf{E}S_t$ satisfies the equation $(m_0 = 1)$

1. $\dot{m}_t = -rm_t$
2. $\dot{m}_t = (-r + \sigma^2/2)m_t$
3. $\dot{m}_t = (-r - \sigma^2/2)m_t$
4. $\dot{m}_t = \sigma^2/2m_t$

(b) The second moment $Q_t = \mathbf{E}S_t^2$ satisfies the equation $(Q_0 = 1)$

1. $\dot{Q}_t = (-2r - \sigma^2)Q_t$
2. $\dot{Q}_t = (-2r + \sigma^2)Q_t$
3. $\dot{Q}_t = \sigma^2 Q_t$
4. $\dot{Q}_t = -2rQ_t$

(c) S_t is nonnegative for any $t \geq 0$

1. True.
2. False.

(d) S_t is a Gaussian process for any $\sigma > 0$

1. True.
2. False.

(e) S_t converges to zero in probability for any $\sigma > 0$

1. True.
2. False.

(f) For any integer $p \geq 1$, σ can be chosen so that S_t converges to zero in $\mathbb{L}^p$.

1. True. σ should satisfy ______
2. False.