RANDOM PROCESSES. THE FINAL TEST.

P. Chigansky

9:00-12:00, 24th of October, 2003

Student ID:_______

* any supplementary material is allowed
* duration of the exam is 3 hours
* write briefly the main idea of your answers in the exam itself. If needed, give the reference to your copybook, where you may place other technical details
* note that the problems are more or less in the chronological order of the chapters in the course and **not** in any monotonic order of complexity
* the total score of the exam is 100
* good luck !

Problem 1.

Consider the simple game, where a player either doubles or loses his stake with probability $1 > p > 0$ and $(1-p)$ respectively. The player starts the game with the stake of 1 dollar and pursues the following strategy: he doubles his stake each time he loses and stops playing after he wins. Let X_n denote the total amount of money, the player wins or loses at the end of n-th game. Then X_n satisfies the recursion

$$X_n = X_{n-1} + \frac{1}{2}V_n\xi_n, \quad n \geq 1$$

subject to $X_0 = 0$, where $\xi = (\xi_n)_{n\geq 0}$ is an i.i.d. sequence with values $\{1, -1\}$ and $P(\xi_1 = 1) = p$. The *strategy* process V_n is given by $(n \geq 2)$

$$V_n = \begin{cases} 2V_{n-1} & \text{if } \xi_{n-1} = -1 \\ 0 & \text{if } \xi_{n-1} = 1 \end{cases}$$

subject to $V_1 \equiv 2$.

(a) Does $X = (X_n)_{n\geq 1}$ converge ? If yes, in what sense and for which p?

1. with probability one
2. in probability
3. in $\mathbb{L}^1$
4. in distribution
5. does not converge in any sense

Explain your answer:

(b) Describe the limit $X_\infty = \lim_{n\to\infty} X_n$

(c) Let τ be the first time at which X_n attains its limit, i.e.

$$\tau = \min\{n : X_n = X_\infty\}.$$

Find its distribution

$$P(\tau = m) =$$

(d) Check the correct answers:

(a) The player wins eventually
(b) The amount of money the player loses till he wins grows exponentially
(c) The player can win any amount of money with positive probability
(d) The player may continue the game infinitely long

Problem 2. Consider the filtering problem, where the signal $X = (X_n)_{n\ge 0}$ is a finite state Markov chain with the alphabet $\mathbb{S} = \{a_1, ..., a_d\}$, transition probabilities matrix Λ and the initial distribution p. The observations are given by

$$Y_n = X_{n-1} + \xi_n, \quad n \ge 1$$

where $\xi = (\xi_n)_{n\ge 1}$ is an i.i.d. sequence with ξ_1 having probability density $f(x)$ and $E\xi_1 = 0$, $E\xi_1^2 = 1$.

(a) The vector π_n with entries $P(X_n = a_i|Y_1^n)$, satisfies the recursion[1]

1. $$\pi_n = \frac{\Lambda^* D(Y_n)\pi_{n-1}}{\langle 1, D(Y_n)\pi_{n-1}\rangle}$$
2. $$\pi_n = \frac{\Lambda^* D(Y_n)\pi_{n-1}}{\langle 1, D(Y_n)\Lambda^*\pi_{n-1}\rangle}$$
3. $$\pi_n = \frac{D(Y_n)\pi_{n-1}}{\langle 1, D(Y_n)\Lambda^*\pi_{n-1}\rangle}$$
4. $$\pi_n = \frac{\Lambda^* D(Y_n)\pi_{n-1}}{\langle 1, \Lambda^*\pi_{n-1}\rangle}$$

$n \ge 1$, where $D(y)$ is a diagonal matrix with entries $f(y - a_j)$, $j = 1, ..., d$, $y \in \mathbb{R}$ and $\pi_0 = p$.

(b) Find the recursion for $\widehat{\pi}_n = \widehat{E}(I_n|Y_1^n)$, where I_n is the vector of indicators $I(X_n = a_i)$, $i = 1, ..., d$:

[1] $\langle x, y\rangle = \sum_{j=1}^d x_i y_i$, $x, y \in \mathbb{R}^d$

(c) Let a denote the vector with entries a_j, $j = 1, ..., d$ and suppose that is a right eigenvector of Λ, so that $\Lambda a = \gamma a$ for some real γ. Assume that the chain X_n is ergodic and stationary, i.e. $P(X_n = a_i) = \mu_i$, $i = 1, ..., d$ where vector μ solves $\Lambda^* \mu = \mu$.

Which of the recursions does X_n satisfy $(n \geq 1)$?

1. $X_n = \gamma X_{n-1} + \sqrt{(1-\gamma^2)\langle a^2 \rangle}\widetilde{\varepsilon}_n$
2. $X_n = (1-\gamma)X_{n-1} + \gamma\sqrt{\langle a^2 \rangle}\widetilde{\varepsilon}_n$
3. $X_n = \gamma X_{n-1} + \langle a^2 \rangle\sqrt{(1-\gamma^2)}\widetilde{\varepsilon}_n$
4. $X_n = (1-\gamma)X_{n-1} + \sqrt{(1-\gamma^2)\langle a^2 \rangle}\widetilde{\varepsilon}_n$

where $\langle a^2 \rangle = \sum_{i=1}^d a_i^2 \mu_i$ and $\widetilde{\varepsilon} = (\widetilde{\varepsilon}_n)_{n \geq 1}$ is a sequence of uncorrelated random variables with zero mean and unit variance.

(d) Under the assumptions of (c), derive *scalar* recursions for $\widehat{X}_n = \widehat{E}(X_n | Y_1^n)$ and $P_n = E\big(X_n - \widehat{X}_n\big)^2$:

Problem [2] **3.** *(Brownian bridge)*

Let $W = (W_t)_{0\le t\le 1}$ be the Wiener process.

(a) Find the following conditional expectations $(t \le 1)$

$$E(W_t|W_1) =$$

$$E\Big(\big(W_t - E(W_t|W_1)\big)^2\big|W_1\Big) =$$

$$E\Big(\big(W_t - E(W_t|W_1)\big)\big(W_s - E(W_s|W_1)\big)\big|W_1\Big) =$$

(b) Find the conditional density

$$\frac{\partial}{\partial x}P(W_t \le x|W_1) =$$

(c) Consider the process $W_t^x = W_t - t(W_1 - x)$, where x is a real parameter. Find its mean, variance and covariance functions:

$$EW_t^x =$$

$$E\big(W_t^x - E(W_t^x)\big)^2 =$$

$$E\big(W_t^x - E(W_t^x)\big)\big(W_s^x - E(W_s^x)\big) =$$

[2] You may encounter the integrals $(0 < t < 1)$

$$\int_0^t \frac{1}{(1-s)^2}ds = \frac{t}{1-t}$$

and

$$\int_0^t \frac{s}{(1-s)^2}ds = \frac{t}{1-t} + \ln(1-t) - 1.$$

(d) For any continuous function x_t, $t \in [0,1]$, let $\psi_n(x)$ denote a bounded real functional of the vector $(x_{t_1}, ..., x_{t_n})$ for any fixed partition $0 < t_1 < ... < t_n < 1$ of $[0,1]$. Then

$$E\big(\psi_n(W)|W_1 = x\big) = E\psi_n(W^x), \quad \forall x \in \mathbb{R}$$

Is this claim correct[3] ?

Yes □

No □

Explain your answer:

(e) Consider the Ito process

$$V_t^x = xt - (1-t)\int_0^t \frac{dW_s}{1-s}, \quad 0 \le t < 1,$$

where $x \in \mathbb{R}$. Which SDE does V^x solve

$$
\begin{aligned}
1.\quad & dV_t^x = \frac{x}{1-t}dt - \frac{x - V_t^x}{1-t}dW_t \\
2.\quad & dV_t^x = \frac{V_t^x}{1-t}dt - \frac{x}{1-t}dW_t \\
3.\quad & dV_t^x = \frac{x - V_t^x}{1-t}dt - dW_t \\
4.\quad & dV_t^x = dt - \frac{x - V_t^x}{1-t}dW_t
\end{aligned}
$$

subject to $V_0^x = 0$.

[3] More precisely $\int_{\mathbb{R}} \big|E\big(\varphi(W)|W_1 = x\big) - E\varphi(W^x)\big|dx = 0$ should be asked here and below

(f) Does V_t converge in $\mathbb{L}^2$ as $t \to 1$? If yes, describe the limit.

(g) Find the following expectations

$$EV_t^x =$$

$$E\big(V_t^x - E(V_t^x)\big)^2 =$$

$$E\big(V_t^x - E(V_t^x)\big)\big(V_s^x - E(V_s^x)\big) =$$

(h) Is V^x a Gaussian process ?

Yes ☐

No ☐

Explain your answer:

(i) For any functional $\psi_n(x)$, defined in (d),

$$E\big(\psi_n(W)|W_1 = x\big) = E\psi_n(V^x), \quad \forall x \in \mathbb{R}.$$

Is this claim correct?

Yes ☐

No ☐

Explain your answer:

(j) Are the processes V_t^x and W_t^x indistinguishable, i.e. is $P(V_t^x = W_t^x) = 1$ for any $x \in \mathbb{R}$ and for any $t \in [0, 1]$?

Yes ☐

No ☐

Explain your answer: