RANDOM PROCESSES. THE FINAL TEST.

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9:00-12:00, July, 2004

Student ID:________

* any supplementary material is allowed
* duration of the exam is 3 hours
* write briefly the main idea of your answers in the exam itself. If required, give the reference to your copybook, where you may place other technical details
* note that the problems are **not** in any monotonic order of complexity
* the total score of the exam is 105 (incl. bonus question)
* good luck !

Problem 1.

Let $X = (X_n)_{n\geq 0}$ be the solution of the random recursion

$$X_n = aX_{n-1} + \varepsilon_n, \quad n \geq 1,$$

where $\varepsilon = (\varepsilon_n)_{n\geq 1}$ is a standard i.i.d. Gaussian sequence, independent of X_0, which is a standard Gaussian random variable as well. The parameter a is unknown.

Assume that a is a random variable with values in $\mathbb{S} = \{r_1, ..., r_d\}$ and $p_i = P(a = r_i)$, independent of X_0 and ε.

(a) Is X a Gaussian process ?

Yes. ☐
No. ☐

Explain:

(b) Is X_n a *conditionally* Gaussian[1]random variable for each fixed $n \geq 0$, given a?

Yes. ☐
No. ☐

Explain:

[1]Recall that the random vector θ is *conditionally Gaussian* given ξ if the conditional distribution has Gaussian density with mean and variance, possibly depending on ξ.

(c) Is $X = (X_n)_{n \geq 0}$ a *conditionally* Gaussian random process, given a?

Yes. ☐
No. ☐

Explain:

(d) Is a a *conditionally* Gaussian random variable, given X_0^n?

Yes. ☐
No. ☐

Explain:

(e) Derive the filtering equations for the optimal estimates $\pi_n(i) = P\big(a = r_i | X_0^n\big)$.

$$\pi_n(i) = \ldots$$

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From now on assume that a is a standard Gaussian random variable, independent of X_0 and ε.

(f) Is X a Gaussian process ?

Yes. ☐
No. ☐

Explain:

(g) Is X_n a *conditionally* Gaussian random variable for each fixed $n \geq 0$, given a?

Yes. ☐
No. ☐

Explain:

(h) Is X_n a *conditionally* Gaussian random process, given a?

Yes. ☐
No. ☐

Explain:

(i) Is a a *conditionally* Gaussian random variable, given X_0^n?

Yes. ☐
No. ☐

Explain:

(j) Does X_n converge to zero?

P-a.s.	☐
in probability	☐
$\mathbb{L}^1$	☐
$\mathbb{L}^2$	☐
in law	☐

(ref. page ______)

(k) Derive the filtering equations for the optimal *linear* estimate $\widehat{a}_n = \widehat{E}\big(a|X_0^n\big)$ and the corresponding mean square error $\widehat{P}_n = (a - \widehat{a}_n)^2$.

$$\widehat{a}_n = ...$$

$$\widehat{P}_n = ...$$

(ref. page ______)

(l) Derive the filtering equations for the optimal estimate $\bar{a}_n = E\big(a|X_0^n\big)$ and the corresponding *conditional* mean square error $\bar{P}_n = E\big((a-\bar{a}_n)^2|X_0^n\big)$.

$$\bar{a}_n = ...$$

$$\bar{P}_n = ...$$

(ref. page ______)

Hint: use **(i)**.

(m) Does $\bar{P}_n$ converge to zero, i.e. perfect estimation is possible ?

P-a.s.	☐
in probability	☐
$\mathbb{L}^1$	☐
$\mathbb{L}^2$	☐
in law	☐

(ref. page ______)

Hint: you may need **(j)**

Problem 2.

The equation

$$\ddot{x}_t + 2\beta\dot{x}_t + x_t = 0$$

describes the position of damped pendulum. Obviously for any $\beta > 0$, the $\lim_{t\to\infty} x_t = 0$, i.e. the pendulum is stable. Let $y_t = \dot{x}_t$, then

$$\begin{aligned}\dot{x}_t &= y_t \\ \dot{y}_t &= -x_t - 2\beta y_t.\end{aligned}$$

Suppose that the damping coefficient is perturbed by Gaussian white noise of intensity σ (i.e. the pendulum operates in a random media)

$$\begin{aligned}\dot{x}_t &= y_t \\ \dot{y}_t &= -x_t - 2(\beta + \text{"white noise"})y_t.\end{aligned} \tag{*}$$

What is the critical value of σ, such that noise destabilizes the system ?

Of course the answer depends on the model of "white noise" and the meaning of "stability" in the stochastic setting. If Ito formalism is assumed the system (*) becomes

$$\begin{aligned}dx_t &= y_t dt \\ dy_t &= -x_t dt - 2y_t(\beta dt + \sigma dW_t).\end{aligned}$$

and is considered hereafter. Assume for simplicity that the initial conditions x_0 and y_0 are standard Gaussian random variables.

(a) Find the averages of x_t and y_t

$$Ex_t = ...$$

$$Ey_t = ...$$

(b) Find the Ito equations for $q_t = x_t^2$, $r_t = y_t^2$ and $u_t = x_t y_t$:

$$dq_t = ...$$

$$dr_t = ...$$

$$du_t = ...$$

(ref. page ______)

(c) Find the equations for $\bar{q}_t = Eq_t$, $\bar{r}_t = Er_t$ and $\bar{u}_t = Eu_t$:

$$\dot{\bar{q}}_t = ...$$

$$\dot{\bar{r}}_t = ...$$

$$\dot{\bar{u}}_t = ...$$

(ref. page ______)

(d) *(bonus+5)* The system is stable (in $\mathbb{L}^2$ sense) if $\bar{q}_t + \bar{r}_t \to 0$ as $t \to \infty$. Choose the correct answers:

the system is unstable for any $\sigma > 0$	☐
the system is unstable for $\sigma = \sqrt{\beta}$	☐
the system is unstable for any $\sigma \geq \sqrt{\beta}$	☐
there is a $\sigma > 0$ such that the system is stable	☐
the system is stable for all sufficiently small $\sigma > 0$	☐
the system is stable for any $0 \leq \sigma < \sqrt{\beta}$	☐

Explain:

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Hint: some of the answers are not hard to check, others may require some eigenvalue analysis of a cubic equation.