RANDOM PROCESSES. THE FINAL TEST.

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9:00-12:00, 20th, September, 2004

Student ID:_______

* any supplementary material is allowed
* duration of the exam is 3 hours
* write briefly the main idea of your answers in the exam itself. If required, give the reference to your copybook, where you may place other technical details
* note that the problems are **not** in any monotonic order of complexity
* the total score of the exam is 120 points.
* good luck !

Problem 1.

Let X be a standard Gaussian random variable (a "message") to be transmitted over noisy channel, so that the following observation sequence is available to the receiver:

$$Y_n = a_n + b_n X + \xi_n, \quad n \geq 1$$

where $\xi = (\xi_n)_{n\geq 1}$ is a standard Gaussian i.i.d. random sequence, independent of X and $(a_n, b_n)_{n\geq 1}$ are real numbers, chosen by the transmitter and known also to the receiver.

(a) Find recursive formulae for the optimal receiver $\widehat{X}_n = E(X|Y_1^n)$ and the corresponding mean square error $P_n = E(X - \widehat{X}_n)^2$.

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(b) Let $\gamma_n = E\big(a_n + b_n X\big)^2$ be the receiver output power. The optimal transmitter, which minimizes P_n and satisfies the power constraint $\gamma_n \leq \gamma$ for any $n \geq 1$ is

(1) $a_n = 0$, $b_n = 1$ and $P_n = \gamma$

(2) $a_n = 0$, $b_n = \sqrt{\gamma}$ and $P_n = 1/(1 + \gamma n)$

(3) $a_n = 0$, $b_n = \sqrt{\gamma}$ and $P_n = \gamma/(\gamma + n)$

(4) $a_n = -\gamma$, $b_n = \sqrt{\gamma}$ and $P_n = 1/(1 + n)$

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(c) If ξ_1 were a non Gaussian random variable with zero mean and unit variance, the optimal transmitter/receiver pair might attain

(1) smaller error than in the Gaussian case

(2) larger error than in the Gaussian case

(reference page ____)

(d) Assume that the transmitter gets noiseless feedback from the receiver, so that only the coefficient a_n (and not b_n) is allowed to depend on $\{Y_1, ..., Y_{n-1}\}$, the information passed to the receiver before n:

$$Y_n = a_n(Y_1^{n-1}) + b_n X + \xi_n.$$

Derive the equations for $\widehat{X}_n = E(X|Y_1^n)$ and $P_n = E(X - \widehat{X}_n)^2$, the optimal receiver in this case.

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(e) Is $Y = (Y_n)_{n\geq 1}$ a Gaussian process in **(d)** ?

(1) Yes.

(2) No

Explain:

(f) The optimal transmitter, which minimizes P_n subject to the power constraint $\gamma_n = E\big(a_n(Y_1^{n-1}) + b_n X\big)^2 \leq \gamma$ is

(1) $a_n = 0$, $b_n = \sqrt{\gamma}$ and $P_n = 1/(1+\gamma n)$

(2) $a_n = 0$, $b_n = \sqrt{\gamma}$ and $P_n = \gamma/(\gamma + n)$

(3) $a_n = -b_n \widehat{X}_{n-1}$, $b_n = \sqrt{\gamma(1+\gamma)^{n-1}}$ and $P_n = 1/(1+\gamma)^n$

(4) $a_n = -b_n \widehat{X}_{n-1}$, $b_n = \sqrt{\gamma^{n-1}(1+\gamma)}$ and $P_n = 1/(1+\gamma)^n$

Hint: Convince yourself that

$$E\big(a_n(Y_1^{n-1}) + b_n X\big)^2 = E\big(a_n(Y_1^{n-1}) + b_n \widehat{X}_{n-1}\big)^2 + b_n^2 P_{n-1}$$

and use it with the equation for P_n (without explicitly solving it).

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(g) Can the filtering error be improved, if b_n is also allowed to depend on $\{Y_1, ..., Y_{n-1}\}$

(1) Yes, by choosing

$$a_n = \_\_\_\_\_\_\_ \quad \text{and} \quad b_n = \_\_\_\_\_\_\_$$

which gives

$$P_n = \_\_\_\_\_\_\_$$

(2) No.

Explain:

Hint: If ζ_n are positive random variables then (why?)

$$E\prod_{\ell=1}^{n}\frac{1}{\zeta_\ell} \geq \exp\left\{-\sum_{\ell=1}^{n}\log E\zeta_\ell\right\}$$

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Problem 2.

Let $X = (X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables. For a fixed $n \geq 1$ let Y^n be the vector with entries

$$Y^n(i) = X_i/\sqrt{X_1^2 + ... + X_n^2}, \quad i = 1, ..., n$$

(a) Assume $EX_1^2 = 1$ and $E|X_1|^p < \infty$ for any $p \geq 1$. Does the random sequence $\sqrt{n}Y^n(1)$ converge as $n \to \infty$?

(1) Yes,

☐ P-a.s.

☐ in probability

☐ in $\mathbb{L}^2$

☐ in law

the limit is ______

(2) No.

Hint: use the law of large numbers

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(b) A random vector Z in $\mathbb{R}^n$ is said to have uniform distribution on the unit sphere in $\mathbb{R}^n$, if its Euclidian norm is unity and it's distribution is invariant under rotations, i.e for any orthogonal matrix U, such that $U^{-1} = U^*$, Z and UZ have the same distribution.

Y^n has uniform distribution on the unit sphere in $\mathbb{R}^n$ for any $n > 1$ if

(1) X_1 is Gaussian with zero mean

(2) X_1 is Bernulli, i.e. $P(X_1 = \pm 1) = 1/2$

(3) X_1 takes values in $\{\pm 1, \pm 2, ...\}$ and $P(X_1 = \ell) = P(X_1 = -\ell)$

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(c) It is known that the uniform distribution on the unit sphere in $\mathbb{R}^n$, $n > 1$ is unique, i.e. there is only one distribution which is invariant under rotations. Let Z^n be a random vector with this distribution. Then

(1) $\sqrt{n}Z^n(1)$ converges weakly to a uniform random variable on $[-1, 1]$

(2) $\sqrt{n}Z^n(1)$ converges P-a.s. to a uniform random variable on $[-1, 1]$

(3) $\sqrt{n}Z^n(1)$ converges weakly to a standard Gaussian random variable

(4) $\sqrt{n}Z^n(1)$ converges P-a.s. to a standard Gaussian random variable

Explain:

Problem 3.

Let $X = (X_n)_{n\geq 0}$ be a binary Markov chain , switching between 0 and 1 with transition probabilities

$$\lambda_0 = P(X_n = 0|X_{n-1} = 0), \quad \lambda_1 = P(X_n = 1|X_{n-1} = 1)$$

and equiprobable initial distribution. The observation process is given by

$$Y_n = X_n + \varepsilon_n, \quad n \geq 1$$

where $\varepsilon = (\varepsilon_n)_{n\geq 1}$ is an i.i.d. sequence, independent of X and ε_1 has probability density $f(x)$.

Introduce the process $Z = (Z_n)_{n\geq 0}$

$$Z_n = \prod_{k=0}^{n} X_k.$$

(a) Does Z_n converge ?

(1) Yes,

☐ P-a.s.

☐ in probability

☐ in $\mathbb{L}^2$

☐ in law

the limit is ______

(2) No.

Explain:

(b) Does the sequence $\widehat{Z}_n = P(Z_n = 1|Y_1^n)$ converge ?

(1) Yes,

☐ in probability

☐ in $\mathbb{L}^2$

☐ in law

the limit is ______

(2) No.

Explain:

(c) Is Z a Markov process ?

(1) Yes, the transition probabilities are

$$P(Z_n = 0|Z_{n-1} = 0) = \_\_\_\_\_\_$$
$$P(Z_n = 1|Z_{n-1} = 1) = \_\_\_\_\_\_$$

(2) No.

Explain:

(d) The conditional probability $\pi_n = P(X_n = 1|Y_1^n)$ satisfies the recursion:

$$\pi_n =$$

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(e) The conditional probability $\widehat{Z}_n = P(Z_n = 1|Y_1^n)$ satisfies

1) $\widehat{Z}_n = \widehat{Z}_{n-1}\pi_n$

2) $\widehat{Z}_n = \dfrac{\widehat{Z}_{n-1}}{\pi_{n-1}}\pi_n$

3) $\widehat{Z}_n = \dfrac{\lambda_1\widehat{Z}_{n-1} + (1-\lambda_0)(1-\widehat{Z}_{n-1})}{\lambda_1\pi_{n-1} + (1-\lambda_0)(1-\pi_{n-1})}\pi_n$

4) $\widehat{Z}_n = \dfrac{\lambda_1(1-\widehat{Z}_{n-1}) + (1-\lambda_0)\widehat{Z}_{n-1}}{\lambda_1(1-\pi_{n-1}) + (1-\lambda_0)\pi_{n-1}}\pi_n$

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Problem 4.

Let $B = (B_t)_{t\geq 0}$ be a Wiener process. It turns out that any random variable X with $EX^2 < \infty$, generated[1] by a trajectory of B on the interval $[0, 1]$, obeys the representation via Itô integral:

$$X = EX + \int_0^1 Z_s dB_s$$

for some random process $Z = (Z_t)_{0\leq t\leq 1}$. Use the Itô formula to find Z_t for each of the following random variables:

(a)

$$B_1 = \_\_\_\_\_ + \int_0^1 \_\_\_\_\_ dB_t$$

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(b)

$$B_1^2 = \_\_\_\_\_ + \int_0^1 \_\_\_\_\_ dB_t$$

(reference page _____)

[1]or more precisely X is measurable w.r.t. $\mathcal{F}_1^B = \sigma\{B_s, 0 \leq s \leq 1\}$

(c)

$$\int_0^1 B_s ds = \_\_\_\_\_ + \int_0^1 \_\_\_\_\_ dB_t$$

Hint: Apply the Itô formula to tB_t

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(d)

$$B_1^3 = \_\_\_\_\_ + \int_0^1 \_\_\_\_\_ dB_t$$

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(e)

$$\exp(B_1) = \_\_\_\_\_ + \int_0^1 \_\_\_\_\_ dB_t$$

Hint: Apply the Itô formula to $\exp\{B_t - t/2\}$.

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(f)

$$\sin(B_1) = \underline{\qquad} + \int_0^1 \underline{\qquad} dB_t$$

Hint: Apply the Itô formula to $\exp\{t/2\}\sin(B_t)$.

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