# RANDOM PROCESSES. SOLUTION OF THE FINAL TEST
**Special Assignement**

**July, 1999**

## Problem 1.

(a) First prove the auxiliary result.

**Lemma 1.1.** *if $\alpha$ and $\beta$ are independent Gaussian random variables with zero mean and variances $\sigma_\alpha^2$ and $\sigma_\beta^2$, then $\gamma = \alpha\beta/\sqrt{\alpha^2+\beta^2}$ is a Gaussian r.v. with zero mean and variance $\sigma_\alpha^2\sigma_\beta^2/(\sigma_\alpha+\sigma_\beta)^2$.*

*Proof.* (there are other elegant proves!) Note that $\gamma^{-2} = \alpha^{-2}+\beta^{-2}$. Let $\psi(s) = \mathbf{E}(e^{is/\alpha^2})$:

$$\begin{aligned}\psi_\alpha(s) &= \frac{1}{\sqrt{2\pi\sigma_\alpha^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{is}{x^2}-\frac{x^2}{2\sigma_\alpha^2}\right\}dx = \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp\left\{-\frac{is}{z^2 2\sigma_\alpha^2}-z^2\right\}dz = h\left(\sqrt{\frac{s}{2\sigma_\alpha^2}}\right)\end{aligned}$$

where

$$h(t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp\left\{-\frac{it^2}{z^2}-z^2\right\}dz$$

It easily seen that $h'(t) = -2\sqrt{i}h(t)$, so $h(t) = C\exp\{-2\sqrt{i}t\}$. Since $h(0)=1$ we finally conclude that $h(t) = \exp\{-2\sqrt{i}t\}$ Consequently $\psi_\alpha(s) = \exp\{-2\sqrt{is/2\sigma_\alpha^2}\}$ and analogously $\psi_\beta(s) = \exp\{-2\sqrt{is/2\sigma_\beta^2}\}$. Then since $\alpha$ and $\beta$ are independent, we have:

$$\begin{aligned}\psi_\gamma(s) &\stackrel{\triangle}{=} \mathbf{E}\left(e^{is/\gamma^2}\right) = \psi_\beta(s)\psi_\alpha(s) = \exp\{-\sqrt{2is}(1/\sigma_\beta+1/\sigma_\alpha)\} = \qquad (1.1)\\ &= \exp\left\{-\sqrt{2is}\left(\frac{\sigma_\beta\sigma_\alpha}{\sigma_\beta+\sigma_\alpha}\right)^{-1}\right\}\end{aligned}$$

Note that $\gamma$ has a symmetric density (Why ?), so the distribution of $\gamma$ is determined by the distribution of $1/\gamma^2$. The latter and (1.1) allows to conclude that $\gamma$ is Gaussian. □

Assume that $X_{n-1}$ is Gaussian, then clearly $X_n$ is Gaussian, since $\xi_n$ and $X_{n-1}$ are independent. Since the initial condition is Gaussian, we conclude that $X_n$ is a Gaussian r.v. for each $n$.

(b) The process $(X_n)_{n\geq 0}$ is not Gaussian. Assume that $[X_0, X_1]$ is a Gaussian vector. Then since $\mathbf{E}X_1X_0 = 0$ they are independent and hence we expect that $\mathbf{E}(X_1^2|X_0) = \mathbf{E}X_1^2$ is not a function of $X_0$.

Let's prove that the latter does not hold:

$$\mathbf{E}(X_1^2|X_0) = \mathbf{E}\left(\frac{X_0^2\xi_1^2}{X_0^2+\xi_1^2}\Big|X_0\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{X_0^2 z^2}{X_0^2+z^2}e^{-z^2/2}dz \triangleq H(X_0)$$

Obviously $H(X_0) \neq const$: $H(0)=0$ and $H(1)\neq 0$.

(c) $m_n = \mathbf{E}X_n \equiv 0$ and

$$V_n = \frac{V_{n-1}\sigma_\xi^2}{(\sqrt{V_{n-1}}+\sigma_\xi)^2}, \quad V_0 = 1$$

(d) Show that $\lim_{n\to\infty} V_n = 0$ and then $X_n \to 0$ as $n\to\infty$ in mean square sense and hence also in the mean and in probability. Let $Q_n = 1/V_n$ then

$$Q_n = (\sigma_\xi + \sqrt{Q_{n-1}})^2$$

Define an auxiliary sequence:

$$\widetilde{Q}_n = \widetilde{Q}_{n-1} + \sigma_\xi^2, \quad \widetilde{Q}_0 = Q_0$$

By induction we show that $Q_n \geq \widetilde{Q}_n$ for $n \geq 0$: assume that $Q_{n-1} \geq \widetilde{Q}_{n-1}$ then

$$Q_n = \sigma_\xi^2 + Q_{n-1} + 2\sigma_\xi\sqrt{Q_{n-1}} \geq \sigma_\xi^2 + Q_{n-1} \geq \sigma_\xi^2 + \widetilde{Q}_{n-1} = \widetilde{Q}_n$$

Clearly $\widetilde{Q}_n \to \infty$, which implies $Q_n \to \infty$ as $n\to\infty$.

**Problem 2.**

(a) Define an additional process:

$$Z_n = Z_{n-1}, \quad Z_0 = X_0$$

Clearly $\mathbf{E}(X_0|Y_0^n) = \mathbf{E}(Z_n|Y_0^n)$. Now consider a filtering problem of a vector random process $\theta_n = (X_n, Z_n)$ from $Y_0^n = \{Y_0, ..., Y_n\}$:

$$\begin{aligned}\theta_n &= \underbrace{\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}}_{\widetilde{a}}\theta_{n-1} + \underbrace{\begin{pmatrix} b \\ 0\end{pmatrix}}_{\widetilde{b}}\varepsilon_n \\ Y_n &= \underbrace{\begin{pmatrix} A \\ 0\end{pmatrix}^\top}_{\widetilde{A}}\theta_{n-1} + B\xi_n\end{aligned}$$

Since all the processes are jointly Gaussian the Kalman filter generates the optimal estimate $\widehat{\theta}_n = \mathbf{E}(\theta_n|Y_0^n)$:

$$\begin{aligned}\widehat{\theta}_n &= \widetilde{a}\widehat{\theta}_{n-1} + \frac{\widetilde{a}P_{n-1}\widetilde{A}}{\widetilde{A}^\top P_{n-1}\widetilde{A}+B^2}\Big(Y_n - \widetilde{A}^\top\widehat{\theta}_{n-1}\Big) \\ P_n &= \widetilde{a}P_{n-1}\widetilde{a}^\top + \widetilde{b}^\top\widetilde{b} - \frac{\widetilde{a}P_{n-1}\widetilde{A}\widetilde{A}^\top P_{n-1}\widetilde{a}}{\widetilde{A}^\top P_{n-1}\widetilde{A}+B^2} \qquad (1.2)\end{aligned}$$

subject to $\widehat{\theta}_0 = 0$ and $P_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. And the desired result is $\pi_n = [0,1] \cdot \widehat{\theta}_n$

(b) Note that $P_n$ is a symmetric matrix:

$$P_n \triangleq \begin{pmatrix} \gamma_x^1 & \gamma_n^{xz} \\ \gamma_n^{xz} & \gamma_n^z \end{pmatrix}$$

From the equation (1.2) it follows that:

$$\gamma_n^x = a^2 \gamma_{n-1}^x + b^2 - \frac{\left(Aa\gamma_{n-1}^x\right)^2}{\left(A\gamma_{n-1}^x\right)^2 + B^2}, \quad \gamma_0^x = P \tag{1.3}$$

$$\gamma_n^{xz} = a\gamma_{n-1}^{xz} - \frac{aA^2\gamma_{n-1}^x\gamma_{n-1}^{xz}}{\left(A\gamma_{n-1}^x\right)^2 + B^2}, \quad \gamma_0^{xz} = P \tag{1.4}$$

$$\gamma_n^z = \gamma_{n-1}^z - \frac{\left(A\gamma_{n-1}^{xz}\right)^2}{\left(A\gamma_{n-1}^x\right)^2 + B^2}, \quad \gamma_0^z = P \tag{1.5}$$

Assuming that $P$ is the positive solution of

$$P = a^2P + b^2 - A^2a^2P^2/(A^2P + B^2)$$

from (1.3) we conclude that $\gamma_n^x \equiv P$. Then the equation for $\gamma_n^{xz}$ (1.4) is merely a geometrical sequence ($\gamma_0^{xz} = P$):

$$\gamma_n^{xz} = \gamma_{n-1}^{xz} \underbrace{a\left(1 - \frac{A^2P}{A^2P^2 + B^2}\right)}_{\triangleq \beta} = \beta^n P \tag{1.6}$$

and equation for $\gamma_n^z$ (1.5) turns to be geometric series ($\gamma_0^z = P$)

$$\gamma_n^z = \gamma_{n-1}^z - \frac{A^2P^2\beta^{2(n-1)}}{A^2P^2 + B^2} = P - \frac{A^2P^2}{A^2P^2 + B^2} \sum_{k=0}^{n-1} \beta^k$$

so that

$$\lim_{n\to\infty} \gamma_n^z = P - \frac{A^2P^2}{A^2P^2 + B^2}\left(\frac{1}{1-\beta}\right)$$

where $\beta$ is defined in (1.6)

**Problem 3.**

(a) The process $(X_n, Y_n)_{n\geq 1}$ is not necessarily Gaussian. E.g. $a_0 = 0$, $A_0 = 0$, $a_1 = 0$ and $A_1 = e^{-iY_{n-1}}$. Then:

$$Y_1 = e^{-iY_0}X_0 + B\xi_1$$

Assume that $(Y_1, X_0)$ is Gaussian. Clearly $\mathbf{E}Y_1 = 0$ and (assuming that $X_0$ and $Y_0$ independent and $\mathbf{E}Y_0^2 = 1$)

$$\begin{aligned}\mathbf{Var}(Y_1|X_0) &= \mathbf{E}\big(e^{-iY_0}X_0 + B\xi_1\big)^2 = X_0^2\mathbf{E}e^{-i2Y_0} + B^2 = \\ &= X_0^2 e^{-2s^2} + B^2 = \text{funct}(X_0)\end{aligned}$$

The latter contradicts the assumption.

(b) Though the process $(X_n, Y_n)$ is not generally Gaussian, it is *conditionally* Gaussian (the dependencies of $a_i, A_i$ on $Y_0^{n-1}$ are omitted for brevity)

$$\begin{aligned}\varphi_n(\lambda,\mu) &= \mathbf{E}\big(e^{-i\lambda X_n - i\mu Y_n}|Y_0^{n-1}\big) = \mathbf{E}\big(e^{-i\lambda X_n - i\mu Y_n}|Y_0^{n-1}\big) = \\ &= \mathbf{E}\big(\exp\{-i\lambda(a_0 + a_1X_{n-1} + b\varepsilon_n) - \\ &\quad -i\mu(A_0 + A_1X_{n-1} + B\xi_n)\}|Y_0^{n-1}\big) = \\ &= \mathbf{E}\Big(\mathbf{E}\big[\exp\{-i\lambda(a_0 + a_1X_{n-1}) - 1/2b^2\lambda^2 \\ &\quad -i\mu(A_0 + A_1X_{n-1}) - 1/2B^2\mu^2\}|X_{n-1}, Y_0^{n-1}\big]|Y_0^{n-1}\Big)\end{aligned}$$

The latter suggests that, given $Y_0^{n-1}$ and $X_{n-1}$, the pair $(X_n, Y_n)$ is Gaussian. We proceed by induction: assume that the conditional density of $X_{n-1}$, given $Y_0^{n-1}$ is Gaussian with $\mathbf{E}(X_{n-1}|Y_0^{n-1}) \triangleq m_{n-1}(Y_0^{n-1})$ and $\mathbf{E}\Big(\big[X_{n-1} - m_{n-1}\big]^2|Y_0^{n-1}\Big) = P_{n-1}(Y_0^{n-1})$. Then

$$\begin{aligned}\varphi_n(\lambda,\mu) &= \mathbf{E}\Big(\mathbf{E}\big[\exp\{-i\lambda(a_0 + a_1X_{n-1}) - 1/2b^2\lambda^2 \\ &\quad -i\mu(A_0 + A_1X_{n-1}) - 1/2B^2\mu^2\}\big]|Y_0^{n-1}\Big) = \\ &= \exp\big\{ - i\lambda(a_0 + a_1m_{n-1}) - i\mu(A_0 + A_1m_{n-1}) \\ &\quad -1/2(a_1^2P_{n-1} + b^2)\lambda^2 - 1/2\lambda\mu A_1a_1P_{n-1} \\ &\quad -1/2(A_1^2P_{n-1} + B^2)\mu^2\big\}\end{aligned}$$

which implies that the density of $X_n$ given $Y_0^n$ is Gaussian. Hence the optimal filter is given by:

$$m_n = a_0 + a_1m_{n-1} + \frac{A_1a_1P_{n-1}}{A_1^2P_{n-1} + B^2}(Y_n - A_0 - A_1m_{n-1}) \tag{1.7}$$

$$P_n = a_1^2P_{n-1} + b^2 - \frac{A_1^2a_1^2P_{n-1}^2}{A_1^2P_{n-1} + B^2} \tag{1.8}$$

**Note 1:** the essential difference between Kalman filter and so called *conditionally Gaussian filter* given by (1.7) is that the Riccati equation (1.8) in the latter depends on the observation process $(Y_n)_{n\geq 1}$ and hence can not be computed *off line*. In certain sense, it is an *adaptive* filter, since its parameters vary with the recorded data. This filter is extremely useful in control theory.

**Note 2:** the equations (1.7) and (1.8) can be derived directly from the conditional density recursion, derived in Problem 7.1.

(c) If all the functionals are constant the filter gets the form of conventional Kalman filter - the Riccati equation becomes decoupled from the observation process.