FINAL TEST SOLUTION
RANDOM PROCESSES 2003

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Problem 1.

(a)

$$\mathbb{E}X_n = \mathbb{E}\sum_{j=1}^{X_{n-1}} \xi_{n,j} = \mathbb{E}\sum_{\ell=0}^{\infty} I(X_{n-1}=\ell)\mathbb{E}\Big(\sum_{j=1}^{\ell}\xi_{n,j}|X_{n-1}\Big) =$$

$$\mathbb{E}\sum_{\ell=0}^{\infty} I(X_{n-1}=\ell)\ell(p+2q) = (p+2q)\mathbb{E}X_{n-1}$$

and thus $X_n \xrightarrow[n\to\infty]{\mathbb{L}^1} 0$ if $p+2q<1$.

(b) Set $\rho = p+2q$ for brevity. Clearly

$$X_n = \sum_{j=1}^{X_{n-1}} (\xi_{n,j}-\rho) + X_{n-1}\rho. \tag{1.1}$$

Moreover

$$\mathbb{E}X_{n-1}\sum_{j=1}^{X_{n-1}} (\xi_{n,j}-\rho) = \mathbb{E}X_{n-1}E\Big(\sum_{j=1}^{X_{n-1}} (\xi_{n,j}-\rho)|X_{n-1}\Big) = 0$$

and

$$\mathbb{E}\Big(\sum_{j=1}^{X_{n-1}} (\xi_{n,j}-\rho)\Big)^2 = \mathbb{E}\Big(\sum_{\ell=0}^{\infty} I(X_{n-1}=\ell)\sum_{j=1}^{\ell}(\xi_{n,j}-\rho)\Big)^2 =$$

$$\mathbb{E}\sum_{\ell=0}^{\infty} I(X_{n-1}=\ell)\mathbb{E}\Big(\sum_{j=1}^{\ell}(\xi_{n,j}-\rho)\Big)^2 =$$

$$\mathbb{E}\sum_{\ell=0}^{\infty} I(X_{n-1}=\ell)\ell\,\mathrm{Var}(\xi_{1,1}) = \text{const.}\,\mathbb{E}X_{n-1} = \text{const.}\,\rho^n$$

Squaring the eq. (1.1), obtain

$$\mathbb{E}X_n^2 = \text{const.}\,\rho^n + \rho^2\mathbb{E}X_{n-1}^2$$

that is

$$\mathbb{E}X_n^2 = N^2\rho^{2n} + \text{const.}\sum_{k=0}^{n}\rho^{n-k}\rho^{2k} = N^2\rho^{2n} + \text{const.}\,\rho^n \underbrace{\sum_{k=0}^{n}\rho^k}_{\le 1/(1-\rho)} \xrightarrow{n\to\infty} 0$$

and hence the required condition is $\rho = p + 2q < 1$.

(c) Let us verify first convergence in probability. Note that

$$\{\exists k \in [1,n] : \xi_{k,1} = 0, ..., \xi_{k,\widetilde{N}} = 0\} \subseteq \{X_n = 0\}.$$

Let $\varepsilon = (1-p-q)^{\widetilde{N}}$. Then

$$P(X_n = 0) \ge P\{\exists k \in [1,n] : \xi_{k,1} = 0, ..., \xi_{k,\widetilde{N}} = 0\} = 1 - (1-\varepsilon)^n \xrightarrow{n\to\infty} 1$$

and convergence in probability follows.

Now verify P-a.s. convergence. Note that

$$\{\exists n : \xi_{n,1} = 0, ..., \xi_{n,\widetilde{N}} = 0\} \subseteq \{\lim_{n\to\infty} X_n = 0\}.$$

For any fixed m

$$P\{\exists n \le m : \xi_{n,1} = 0, ..., \xi_{n,\widetilde{N}} = 0\} = 1 - \big(1-\varepsilon\big)^m.$$

and since

$$\{\exists n \le m : \xi_{n,1} = 0, ..., \xi_{n,\widetilde{N}} = 0\} \nearrow \{\exists n : \xi_{n,1} = 0, ..., \xi_{n,\widetilde{N}} = 0\}, \quad \text{as } m \to \infty$$

it follows

$$P(\lim_{n\to\infty} X_n = 0) \ge P\{\exists n : \xi_{n,1} = 0, ..., \xi_{n,\widetilde{N}} = 0\} = 1 - \lim_{m\to\infty} \big(1-\varepsilon\big)^m = 1.$$

$\mathbb{L}^p$, $p > 0$ convergence follows from convergence in probability, since $X_n \le \widetilde{N}$.

Problem 2.

(a) First note that $\mathbb{E}\big(X_n|X_1^{n-1}\big) = \mathbb{E}\big(\varepsilon_n + \varepsilon_{n-1}|X_1^{n-1}\big) = \mathbb{E}\big(\varepsilon_{n-1}|X_1^{n-1}\big) := \widehat{\varepsilon}_{n-1}$. Since ε is Gaussian, $\widehat{\varepsilon}_{n-1} = \widehat{\mathbb{E}}(\varepsilon_{n-1}|X_1^{n-1})$ and can be calculated recursively:

$$\begin{aligned}
&\widehat{\varepsilon}_{n|n-1} = \widehat{\mathbb{E}}\big(\varepsilon_n|X_1^{n-1}\big) = 0 \\
&\widehat{X}_{n|n-1} = \widehat{\mathbb{E}}\big(X_n|X_1^{n-1}\big) = \widehat{\varepsilon}_{n-1} \\
&P^{\varepsilon}_{n|n-1} = \mathbb{E}\big(\varepsilon_n - \widehat{\varepsilon}_{n|n-1}\big)^2 = 1 \\
&P^{\varepsilon x}_{n|n-1} = \mathbb{E}\big(\varepsilon_n - \widehat{\varepsilon}_{n|n-1}\big)\big(X_n - \widehat{\mathbb{E}}\big(X_n|X_1^{n-1}\big)\big) = \mathbb{E}\varepsilon_n\big(\varepsilon_n + \varepsilon_{n-1} - \widehat{\varepsilon}_{n-1}\big) = 1 \\
&P^{x}_{n|n-1} = \mathbb{E}\big(X_n - \widehat{\mathbb{E}}\big(X_n|X_1^{n-1}\big)\big)^2 = \mathbb{E}\big(\varepsilon_n + \varepsilon_{n-1} - \widehat{\varepsilon}_{n-1}\big)^2 = 1 + P_{n-1}
\end{aligned}$$

and thus $n \ge 1$

$$\begin{aligned}
&\widehat{\varepsilon}_n = 1/(1+P_{n-1})\big(X_n - \widehat{\varepsilon}_{n-1}\big) \\
&P_n = 1 - 1/(1+P_{n-1})
\end{aligned}$$

subject to $\widehat{\varepsilon}_0 = 0$ and $P_0 = 1$.

The sequence $R_n = 1/P_n$ satisfies

$$R_n = 1 + R_{n-1}, \quad R_0 = 1$$

and thus $R_n = n+1$, i.e. $P_n = 1/(n+1)$, $n \ge 0$. This leads to

$$\widehat{\varepsilon}_n = \frac{n}{n+1}\big(X_n - \widehat{\varepsilon}_{n-1}\big), \quad \widehat{\varepsilon}_0 = 0, \quad n \ge 1$$

and in turn

$$\widehat{X}_{n+1} = \frac{n}{n+1}\big(X_n - \widehat{X}_n\big), \quad \widehat{X}_1 = 0, \quad n \ge 1.$$

(b) $Q_n = P^x_{n|n-1} = 1 + P_{n-1} = 1 + 1/n$, $n \geq 2$.

(c) Note that given ε_0 and $X_1, ..., X_n$, the output of the recursion

$$\widehat{\varepsilon}'_n = X_n - \widehat{\varepsilon}'_{n-1}, \quad \widehat{\varepsilon}'_0 = \varepsilon_0$$

gives ε_n exactly (just try to unroll this recursion to see this), i.e. $\widehat{\varepsilon}_n \equiv \varepsilon_n$ and thus it is the conditional expectation. Since $\widehat{X}^\circ_{n+1} = \widehat{\varepsilon}'_n$, it follows that

$$\widehat{X}^\circ_{n+1} = X_n - \widehat{X}^\circ_n, \quad \widehat{X}^\circ_1 = \varepsilon_0$$

(d) Since $\widehat{X}^\circ_{n+1} = \widehat{\varepsilon}'_n \equiv \varepsilon_n$, $Q^\circ_{n+1} = \mathbb{E}(X_{n+1} - \widehat{X}^\circ_{n+1})^2 = \mathbb{E}(\varepsilon_{n+1} + \varepsilon_n - \varepsilon_n)^2 \equiv 1$.

Problem 3.

(a) Let ξ^a_n (ξ^b_n) be the sequence of requests (say, taking value 1 when service is requested and 0 otherwise) from client A (B). Clearly ξ^a_n and ξ^b_n are independent i.i.d. sequences with $P(\xi^a_n = 1) = P(\xi^b_n = 1) = p$. Introduce an i.i.d. sequence η_n (independent of ξ^a_n and ξ^b_n) with $P(\eta_n = A) = P(\eta_n = B) = 1/2$.

Then X_n satisfies the following recursion[1]

$$\begin{aligned} X_n = \xi^a_n \xi^b_n \big[& AI(X_{n-1} = A) + BI(X_{n-1} = B) + \eta_n I(X_{n-1} = I)\big] \\ & + A\xi^a_n(1 - \xi^b_n) + B(1 - \xi^a_n)\xi^b_n + I(1 - \xi^a_n)(1 - \xi^b_n) \end{aligned} \tag{1.2}$$

Due to independence of $(\xi^a_n, \xi^b_n, \eta_n)$ and X^{n-1}_0, X_n is a Markov chain regardless of distribution of $(\xi^a_n, \xi^b_n, \eta_n)$ (i.e. none of the conditions ruins the Markov property)

(b) From (1.2)

$$\begin{aligned}
&P(X_n = A|X_{n-1} = A) = \mathbb{E}\{\xi^a_n\xi^b_n + \xi^a_n(1 - \xi^b_n)\} = p^2 + p(1 - p) = p \\
&P(X_n = I|X_{n-1} = A) = \mathbb{E}(1 - \xi^a_n)(1 - \xi^b_n) = (1 - p)^2 \\
&P(X_n = B|X_{n-1} = A) = \mathbb{E}(1 - \xi^a_n)\xi^b_n = (1 - p)p \\
&P(X_n = A|X_{n-1} = I) = 1/2\mathbb{E}\xi^a_n\xi^b_n + \mathbb{E}\xi^a_n(1 - \xi^b_n) = 1/2p^2 + p(1 - p) = p - p^2/2 \\
&P(X_n = I|X_{n-1} = I) = \mathbb{E}(1 - \xi^a_n)(1 - \xi^b_n) = (1 - p)^2 \\
&P(X_n = B|X_{n-1} = I) = 1/2\mathbb{E}\xi^a_n\xi^b_n + \mathbb{E}\xi^b_n(1 - \xi^a_n) = 1/2p^2 + p(1 - p) = p - p^2/2 \\
&P(X_n = A|X_{n-1} = B) = \mathbb{E}\xi^a_n(1 - \xi^b_n) = p(1 - p) \\
&P(X_n = I|X_{n-1} = B) = ... = (1 - p)^2 \\
&P(X_n = B|X_{n-1} = B) = ... = p
\end{aligned}$$

i.e. $\Lambda = \begin{pmatrix} p & (1-p)^2 & (1-p)p \\ p - p^2/2 & (1-p)^2 & p - p^2/2 \\ p(1-p) & (1-p)^2 & p \end{pmatrix}$

(c) Let $f_\lambda(t) = \lambda \exp\{-\lambda t\}$ and $\mathcal{F}_{n-1} = \{\alpha^{n-1}_1, \beta^{n-1}_1\}$ for brevity.

[1]The multiplication for symbols A, B, I is symbolic, e.g. $A1 = A$, $A0 = 0$, $A + 0 = A$ ($A + B$ is of course not defined and never happens!)

Let $\pi_t(I) = G(\alpha_n, \beta_n; \mathcal{F}_{n-1})$ and fix a bounded function $h(s,t)$. Then G should satisfy

$$\mathbb{E}\big(I(X_n = I)h(\alpha_n, \beta_n)|\mathcal{F}_{n-1}\big) = \mathbb{E}\big(G(\alpha_n, \beta_n; \mathcal{F}_{n-1})h(\alpha_n, \beta_n)|\mathcal{F}_{n-1}\big)$$

The left hand side becomes

$$\mathbb{E}\big(I(X_n = I)\int_0^\infty \int_0^\infty h(t,s)f_1(s)f_1(t)dtds|\mathcal{F}_{n-1}\big) = \\ \pi_{n|n-1}(I)\int_0^\infty \int_0^\infty h(t,s)f_1(s)f_1(t)dtds$$

whereas the right hand side is equal to

$$\mathbb{E}\Big(\big[I(X_n = A) + I(X_n = I) + I(X_n = B)\big]G(\alpha_n, \beta_n; \mathcal{F}_{n-1})h(\alpha_n, \beta_n)|\mathcal{F}_{n-1}\Big) = \\ \pi_{n|n-1}(A)\int_0^t \int_0^t G(s,t;\mathcal{F}_{n-1})h(s,t)f_\lambda(s)f_1(t)dsdt + \\ \pi_{n|n-1}(I)\int_0^t \int_0^t G(s,t;\mathcal{F}_{n-1})h(s,t)f_1(s)f_1(t)dsdt + \\ \pi_{n|n-1}(B)\int_0^t \int_0^t G(s,t;\mathcal{F}_{n-1})h(s,t)f_1(s)f_\lambda(t)dsdt.$$

So[2]

$$G(s,t;\mathcal{F}_{n-1}) = \frac{\pi_{n|n-1}f_1(s)f_1(t)}{\pi_{n|n-1}(A)f_\lambda(s)f_1(t) + \pi_{n|n-1}(I)f_1(s)f_1(t) + \pi_{n|n-1}(B)f_1(s)f_\lambda(t)}$$

and thus

$$\pi_n(I) = \frac{\pi_{n|n-1}(I)f_1(\alpha_n)f_1(\beta_n)}{\pi_{n|n-1}(A)f_\lambda(\alpha_n)f_1(\beta_n) + \pi_{n|n-1}(I)f_1(\alpha_n)f_1(\beta_n) + \pi_{n|n-1}(B)f_1(\alpha_n)f_\lambda(\beta_n)} = \\ \frac{\pi_{n|n-1}(I)\exp\{-\alpha_n - \beta_n\}}{\lambda\pi_{n|n-1}(A)\exp\{-\lambda\alpha_n - \beta_n\} + \pi_{n|n-1}(I)\exp\{-\alpha_n - \beta_n\} + \lambda\pi_{n|n-1}(B)\exp\{-\alpha_n - \lambda\beta_n\}} = \\ \frac{\pi_{n|n-1}(I)}{\lambda\pi_{n|n-1}(A)\exp\{(1-\lambda)\alpha_n\} + \pi_{n|n-1}(I) + \lambda\pi_{n|n-1}(B)\exp\{(1-\lambda)\beta_n\}}$$

Problem 4.

(a) Since $\mathbb{E}\int_0^t S_u dW_u = 0$, $m_t = \mathbb{E}S_t = 1 - \int_0^t r\mathbb{E}S_u du$ and hence $\dot{m}_t = -rm_t$, $m_0 = 1$.

(b) Apply the Ito formula to S_t^2

$$dS_t^2 = 2S_t dS_t + \frac{1}{2}2S_t^2\sigma^2 dt$$

that is

$$S_t^2 = S_0^2 - 2\int_0^t rS_u^2 du + 2\int_0^t \sigma S_u^2 dW_u + \int_0^t S_t^2\sigma^2 dt$$

[2]this answer may be guessed - it should be the similar to the scalar observation case

Taking $\mathbb{E}(\cdot)$ from both sides obtain equation for $Q_t = \mathbb{E}S_t^2$

$$\dot{Q}_t = (-2r + \sigma^2)Q_t$$

(c) True.The solution of this equation is[3],

$$S_t = \exp\big\{\sigma W_t - (r + \sigma^2/2)t\big\} > 0$$

Indeed, $S_0 = 1$ and by Ito formula

$$dS_t = S_t\sigma dW_t - rS_t dt - 1/2\sigma^2 S_t dt + 1/2 S_t\sigma^2 dt = -rS_t dt + \sigma S_t dW_t.$$

(d) False. The process can not be Gaussian since e.g. $S_t \ge 0$ for all t.

(e) True. From **(a)** we know that S_t converges in $\mathbb{L}^1$ and hence in probability.

(f) True. If $p = 1$, the claim holds by **(a)**.
With integer $p > 1$, apply the Ito formula to S_t^p

$$dS_t^p = pS^{p-1}dS_t + \frac{1}{2}p(p-1)S^{p-2}\sigma^2 S_t^2 dt = -rpS_t^p dt + p\sigma S_t^p dW_t + \frac{1}{2}p(p-1)S_t^p\sigma^2 dt$$

Set $Q_t^p = \mathbb{E}S_t^p$ and take $\mathbb{E}(\cdot)$ from both sides to obtain

$$\dot{Q}_t^p = \big[-pr + \frac{1}{2}p(p-1)\sigma^2\big]Q_t^p.$$

Clearly this equation is stable if $pr > 1/2p(p-1)\sigma^2$ or $\sigma^2 < 2r/(p-1)$.

[3] See exercise 8.7