RANDOM PROCESSES, THE SOLUTION TO THE EXAM OCTOBER 24TH, 2003

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Problem 1.

(a) If $\xi_1 = 1$, then $V_1 = 2$ and $X_1 = 1$. In this case for any $n \geq 2$, $V_n \equiv 0$ and $X_n \equiv 1$. If $\xi_1 = -1$, then $X_1 = -1$ and $V_1 = 2$. At $n = 2$, $V_2 = 4$ and $X_2 = -3$ if $\xi_2 = -1$ or $X_2 = 1$ if $\xi_2 = 1$. In the latter case $X_n \equiv 1$ and $V_n \equiv 0$ for $n \geq 3$, etc. By induction arguments, it is clear that $X_n \equiv 1$ for any $n \geq \tau$ where $\tau = \min\{n : \xi_n = 1\}$. Then

$$P(\lim_{n\to\infty} X_n = 1) = P(\tau < \infty) = 1 - P(\xi_1 = -1, \xi_2 = -1, ...) = 1 - \lim_{n\to\infty}(1-p)^n = 1$$

i.e. X_n converges to 1 for any $p > 0$ with probability one and thus also in probability and in distribution. Further

$$E|X_n - 1| = 2^n P(\xi_1 = -1, ..., \xi_n = -1) = 2^n(1-p)^n.$$

So X_n converges to 1 in $\mathbb{L}^1$ if $p > 1/2$.

(b) As shown above $X_\infty = 1$.

(c) $P(\tau = m) = P(\xi_1 = -1, ..., \xi_{m-1} = -1, \xi_m = 1) = (1-p)^{m-1}p$, i.e. τ has geometrical distribution. In particular

$$E\tau = \sum_{m=1}^{\infty} mP(\tau = m) = p\sum_{m=1}^{\infty} m(1-p)^m = -p(1-p)\frac{d}{dp}\sum_{m=1}^{\infty}(1-p)^m =$$
$$-p(1-p)\frac{d}{dp}(1-p)\sum_{m=0}^{\infty}(1-p)^m = -p(1-p)\frac{d}{dp}(1-p)/p = 1/p - 1$$

(d) The player wins eventually one dollar, but the amount of money he loses till he wins grows as 2^n. Since $E\tau < \infty$ the game is finite with probability one, i.e. $P(\tau < \infty) = 1$. That is (a) and (b) are true, while (c) and (d) are false.

Problem 2.

(a) Let $G(z; y)$ be a function, such that $\pi_n(i) = G(Y_1^{n-1}, Y_n)$ for some fixed i. Then it satisfies

$$E\big(I(X_n = a_i)h(Y_n)|Y_1^{n-1}\big) = E\big(G(Y_1^{n-1}, Y_n)h(Y_n)|Y_1^{n-1}\big) \quad (1.1)$$

for any bounded function $h : \mathbb{R} \mapsto \mathbb{R}$. The left hand side reads

$$E\big(I(X_n = a_i)h(Y_n)|Y_1^{n-1}\big) = E\big(I(X_n = a_i)h(X_{n-1} + \xi_n)|Y_1^{n-1}\big) =$$
$$E\big(\sum_{j=1}^d I(X_{n-1} = a_j)I(X_n = a_i)h(a_j + \xi_n)|Y_1^{n-1}\big) =$$
$$\sum_{j=1}^d \lambda_{ji}\pi_{n-1}(j)\int_{\mathbb{R}} f(x - a_j)h(x)dx.$$

Similarly

$$E\big(G(Y_1^{n-1}, Y_n)h(Y_n)|Y_1^{n-1}\big) = E\big(\sum_{j=1}^d I(X_{n-1} = a_j)G(Y_1^{n-1}, a_j + \xi_n)h(a_j + \xi_n)|Y_1^{n-1}\big) =$$
$$E\Big(\int_{\mathbb{R}} G(Y_1^{n-1}, x)h(x)\sum_{j=1}^d \pi_{n-1}(j)f(x - a_j)dx|Y_1^{n-1}\Big)$$

By arbitrariness of h

$$\pi_n(i) = G(Y_1^n; Y_n) = \frac{\sum_{j=1}^d \lambda_{ji}\pi_{n-1}(j)f(x - a_j)}{\sum_{j=1}^d \pi_{n-1}(j)f(x - a_j)}$$

or in the vector form

$$\pi_n = \frac{\Lambda^* D(Y_n)\pi_{n-1}}{\langle 1, D(Y_n)\pi_{n-1}\rangle}, \quad n \ge 1$$

subject to p, where $D(y)$ is the diagonal matrix with entries $f(y - a_j)$, $j = 1, ..., d$.

(b) Recall that I_n satisfies the recursion

$$I_n = \Lambda^* I_{n-1} + \varepsilon_n, \quad n \ge 1 \tag{1.2}$$

where $E\varepsilon_n = 0$, $E\varepsilon_n\varepsilon_m^* = 0$ for $n \ne m$ and $V_n := E\varepsilon_n\varepsilon_n^* = \operatorname{diag}(p_n) - \Lambda^* \operatorname{diag}(p_{n-1})\Lambda$ with $p_n = (\Lambda^*)^n p_0$. Since the observation process satisfies $Y_n = a^* I_{n-1} + \xi_n$, $\widehat{\pi}_n$ is generated by the Kalman filter ($n \ge 1$)

$$\widehat{\pi}_n = \Lambda^*\widehat{\pi}_{n-1} + \frac{\Lambda^* P_{n-1}a}{a^* P_{n-1}a + 1}\big(Y_n - a^*\widehat{\pi}_{n-1}\big)$$
$$P_n = \Lambda^* P_{n-1}\Lambda + V_n - \frac{\Lambda^* P_{n-1}aa^* P_{n-1}\Lambda}{a^* P_{n-1}a + 1},$$

subject to $\widehat{\pi}_0 = p$ and $P_0 = \operatorname{diag}(p) - pp^*$.

(c) Note that $X_n = a^* I_n$, where I_n is the vector of indicators $I(X_n = a_i)$, $i = 1, ..., d$. Multiply recursion by a^* the equation (1.2) to obtain

$$X_n = \gamma X_{n-1} + \widetilde{\varepsilon}_n.$$

where $\widetilde{\varepsilon}_n = a^*\varepsilon_n$. Clearly $E\widetilde{\varepsilon}_n = 0$, $E\widetilde{\varepsilon}_n\widetilde{\varepsilon}_m = 0$ when $n \ne m$ and

$$E\widetilde{\varepsilon}_n^2 = a^* V_n a = a^* \operatorname{diag}(p_n)a - a^*\Lambda^* \operatorname{diag}(p_{n-1})\Lambda a =$$
$$a^* \operatorname{diag}(p_n)a - \gamma^2 a^* \operatorname{diag}(p_{n-1})a = (1 - \gamma^2)a^* \operatorname{diag}(\mu)a = (1 - \gamma^2)\langle a^2\rangle$$

where the latter equality holds, since $p_n = \mu$. Note that this suggests that $|\gamma| < 1$, which is indeed true for transition probabilities matrices (which are also called *stochastic* matrices).

(d) Since the observation process is generated by

$$Y_n = X_{n-1} + \xi_n.$$

the optimal linear estimate $\widehat{X}_n = \widehat{E}(X_n|Y_1^n)$ is generated by the Kalman filter

$$\widehat{X}_n = \gamma \widehat{X}_{n-1} + \frac{\gamma P_{n-1}}{P_{n-1}+1}(Y_n - \widehat{X}_{n-1})$$

$$P_n = \gamma^2 P_{n-1} + (1-\gamma^2)\langle a^2\rangle - \frac{\gamma^2 P_{n-1}^2}{P_{n-1}+1}$$

subject to $\widehat{X}_0 = a^*\mu$ and $P_0 = \langle a^2\rangle - (a^*\mu)^2$.

Problem 3.

(a) (W_t, W_1) is a Gaussian pair with zero mean and $EW_t^2 = t$ and $EW_tW_s = t\wedge s$, so

$$E(W_t|W_1) = \frac{\mathrm{cov}(W_t, W_1)}{\mathrm{cov}(W_1)} W_1 = tW_1$$

$$E(W_t - E(W_t|W_1))^2 = \mathrm{cov}(W_t) - \frac{\mathrm{cov}^2(W_t, W_1)}{\mathrm{cov}(W_1)} = t - t^2$$

$$E(W_s - E(W_s|W_1))(W_t - E(W_t|W_1)) = s\wedge t - ts - st + st = s\wedge t - st$$

(b) Since (W_t, W_1) is Gaussian, the conditional distribution is Gaussian as well, i.e.

$$\frac{\partial}{\partial x} P(W_t \le x|W_1) = \frac{1}{\sqrt{2\pi t(1-t)}} \exp\left\{-\frac{1}{2}\frac{(x-tW_1)^2}{t(1-t)}\right\}.$$

(c) If $W_t^x = W_t - t(W_1 - x)$, then $EW_t^x = EW_t - t(EW_1 - x) = tx$. Similarly $\mathrm{cov}(W_t^x) = E(W_t^x - xt)^2 = E(W_t - W_1t)^2 = t - t^2$ and $\mathrm{cov}(W_s^x, W_t^x) = E(W_t - W_1t)(W_s - W_1s) = s\wedge t - st$.

(d) Yes. Denote by $p(z;x)$, $z \in \mathbb{R}^n, x \in \mathbb{R}$ the probability density of the vector $(W_{t_1}^x, ..., W_{t_n}^x)$, i.e.

$$p(z;x) = \frac{\partial^n}{\partial z_1 ... \partial z_n} P\big(W_{t_1}^x \le z_1, ..., W_{t_n}^x \le z_n\big).$$

This density is Gaussian with the mean and covariance matrix, whose entries were found in (b). On the other hand, by (a) the conditional density of $(W_{t_1}, ..., W_{t_n})$ given W_1 has the same mean and covariance as $p(z;x)$ does, and hence

$$q(z;W_1) := \frac{\partial^n}{\partial z_1 ... \partial z_n} P\big(W_{t_1} \le z_1, ..., W_{t_n} \le z_n | W_1\big) = p(z;W_1), \quad P-a.s.$$

In other words

$$E\big(\psi_n(W)|W_1 = x\big) = \int_{\mathbb{R}^n} \psi_n(z) p(z;x) dz = E\psi_n(W^x), \quad a.s.$$

(e) By the Ito formula

$$dV_t^x = xdt + \left(\int_0^t \frac{dW_s}{1-s}\right) dt - (1-t)\frac{dW_t}{1-t} = xdt + \left(\int_0^t \frac{dW_s}{1-s}\right) dt - dW_t$$

Taking into account that

$$\int_0^t \frac{dW_s}{1-s} = \frac{xt - V_t^x}{1-t}$$

it follows

$$dV_t^x = xdt + \frac{xt - V_t^x}{1-t}dt - dW_t = \frac{x - V_t^x}{1-t}dt - dW_t,$$

i.e. V^x is the solution of the equation

$$dV_t^x = \frac{x - V_t^x}{1-t}dt - dW_t, \quad 0 \le t \le 1$$

subject to $V_0^x = 0$.

(f) $E\big(V_t^x - x\big)^2 = E\left(x(t-1) - (1-t)\int_0^t \frac{dW_s}{1-s}\right)^2 \le 2x^2(t-1)^2 + 2(1-t)^2 E\left(\int_0^t \frac{dW_s}{1-s}\right)^2.$ By the properties of the Ito integral

$$E\left(\int_0^t \frac{dW_s}{1-s}\right)^2 = \int_0^t \frac{1}{(1-s)^2}ds = \frac{t}{1-t}$$

and so

$$E\big(V_t^x - x\big)^2 \le 2x^2(t-1)^2 + 2(1-t)t \xrightarrow{t\to 1} 0,$$

i.e. V_t^x converges to x in $\mathbb{L}^2$ as $t \to 1$.

(g)

$$\begin{aligned}
&EV_t^x = xt - (1-t)E\int_0^t \frac{dW_s}{1-s} = xt \\
&\mathrm{cov}(V_t^x) = (1-t)^2 E\left(\int_0^t \frac{dW_s}{1-s}\right)^2 = (1-t)^2\frac{t}{1-t} = t - t^2 \\
&\mathrm{cov}(V_s^x, V_t^x) = (1-s)(1-t)E\left(\int_0^t \frac{dW_u}{1-u}\right)\left(\int_0^s \frac{dW_v}{1-v}\right) \\
&= (1-s)(1-t)\int_0^{s\wedge t}\frac{du}{(1-u)^2} = (1-s)(1-t)\frac{s\wedge t}{1-s\wedge t} = \\
&\begin{cases} (1-t)s & s \le t \\ (1-s)t & s > t \end{cases} = s\wedge t - st
\end{aligned}$$

(h) V^x is a Gaussian process, since it is a linear functional of W.

(i) Yes. By the same argument as in (d) - note that V^x and W^x has the same mean and covariance.

(j) If $P(V_t^x = W_t^x) = 1$, then $E(V_t^x - W_t^x)^2 = 0$. Set $x = 0$. Then

$$E(W_t^0 - V_t^0)^2 = E\left(W_t - tW_1 + (1-t)\int_0^t \frac{dW_s}{1-s}\right)^2 =$$

$$EW_t^2 + t^2 EW_1^2 + (1-t)^2 E\left(\int_0^t \frac{dW_s}{1-s}\right)^2 - 2tEW_tW_1 +$$

$$2(1-t)E(W_t - tW_1)\int_0^t \frac{dW_s}{1-s} =$$

$$t + t^2 + (1-t)^2 \frac{t}{1-t} - 2t^2 + 2(1-t)E(W_t - tW_1)\int_0^t \frac{dW_s}{1-s}$$

By the Ito formula

$$d\left(\frac{W_t}{1-t}\right) = \frac{dW_t}{1-t} + \frac{W_t}{(1-t)^2}dt$$

and hence

$$\int_0^t \frac{dW_s}{1-s} = \frac{W_t}{1-t} - \int_0^t \frac{W_s}{(1-s)^2}ds.$$

Then

$$EW_t\int_0^t \frac{dW_s}{1-s} = EW_1\int_0^t \frac{dW_s}{1-s} = \frac{t}{1-t} - \int_0^t \frac{s}{(1-s)^2}ds =$$

$$\frac{t}{1-t} - \left[\frac{1}{1-s} + \ln(1-s)\right]_{s=0}^{s=t} = \frac{t}{1-t} - \frac{1}{1-t} - \ln(1-t) + 1 = -\ln(1-t)$$

and

$$E(W_t^0 - V_t^0)^2 = t + t^2 + (1-t)^2\frac{t}{1-t} - 2t^2 - 2(1-t)^2\ln(1-t)$$

Since the "ln" term is left uncompensated, there are t's for which

$$E(W_t^0 - V_t^0)^2 > 0$$

and thus $P(W_t^0 - V_t^0 \neq 0) > 1$. In other words V_t^x and W_t^x are distinct processes, with the same distributions!