STOCHASTIC PROCESSES

1. Basics of mathematical probability

Problem 1.1

Let $I_A(\omega)$ denote the indicator function of a set (event) A, i.e.:

$$I_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$

Verify the following properties of indicators

(a) $\mathbb{P}\{A\} = \mathbb{E}I_A$
(b) $I_\emptyset = 0$ and $I_\Omega = 1$
(c) $I_A + I_{\overline{A}} = 1$
(d) $I_{A\cap B} = I_A \cdot I_B$
(e) $I_{A\cup B} = I_A + I_B - I_{A\cap B}$
(f) $I_{\bigcup_{i=1}^n A_i} = 1 - \prod_{i=1}^n (1 - I_{A_i})$
(g) For nonintersecting sets A_i the union $\cup_i A_i$ is denoted by $\sum A_i$. Show $I_{\sum_{i=1}^n A_i} = \sum_{i=1}^n I_{A_i}$
(h) $I_{A\triangle B} = (I_A - I_B)^2$, where $A\triangle B$ denotes symmetric difference of sets, i.e. $(A\backslash B) \cup (B\backslash A)$

Problem 1.2

On the probability space $([0,1], \mathcal{B}, \lambda)$, consider the random variables $X(\omega) = I(\omega \leq 1/2)$ and $Y(\omega) = \omega^2$

(1) Find the expectations of X and Y by integration on the given probability space with respect to λ
(2) Find the expectations of X and Y by integration with respect to their distribution functions.

Problem 1.3

Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. binary random variables such that $P(X_i = 0) = P(X_i = 1) = \frac{1}{2}$. Given two constants, a and b, $(a \neq b)$, and $Y_0 = b$, define a new sequence:

$$Y_n = \begin{cases} a & \text{if} \quad X_n = 0 \\ Y_{n-1} & \text{if} \quad X_n = 1 \end{cases}$$

Verify the convergence of Y_n with probability one (P-a.s.), in probability, in the mean square and in the mean.

Problem 1.4

Let X_n, Y_n and V_n, $n \geq 1$ be sequences of random variables, converging in $\mathbb{L}^2$ to X, Y and V respectively. Verify the following properties:

(1) "Linearity" of the $\mathbb{L}^2$ limit:

$$aX_n + bY_n \xrightarrow{\mathbb{L}^2} aX + bY$$

where a and b are deterministic constants.

(2) Commutativity of the expectation and $\mathbb{L}^2$ limit:

$$\mathbb{E}X = \lim \mathbb{E}X_n$$

(3) Continuity of the scalar product:

$$\mathbb{E}XY = \lim \mathbb{E}X_n Y_n$$

$$\mathbb{E}X^2 = \lim \mathbb{E}X_n^2$$

(4) Verify that $\mathbb{E}X_n Y_n = \mathbb{E}V_n$ implies $\mathbb{E}XY = \mathbb{E}V$.

Problem 1.5

Let U be a r.v., distributed uniformly on $[0, 1]$. Define a sequence:

$$Z_n = U^n \qquad n \geq 1$$

Does the sequence of sums $S_n = \sum_{i=1}^{n} Z_i$ converge with probability one ? In probability ?

Problem 1.6

Given the deterministic sequence $(a_n)_{n\geq 1}$, such that $\lim_{n\to\infty} a_n = a$, and a sequence of random variables $(X_n)_{n\geq 1}$, such that $\lim_{n\to\infty} \mathbb{E}(X_n - a_n)^2 = 0$, prove that X_n converges in $\mathbb{L}^2$ and determine the limit.

Problem 1.7

Let $\{\xi_i\}$ be a sequence of i.i.d. normal random variables, namely $\xi_i \sim \mathcal{N}(0, \sigma^2)$. Define a pair of sequences:

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n} \xi_i; \qquad S_n = \frac{1}{n-1}\sum_{i=1}^{n} (\xi_i - \mu_n)^2$$

(1) Show that μ_n converges in the mean square sense to a limit.

(2) Show that S_n converges in m.s. sense to a limit.

Hint: You may need the following fact: for a Gaussian vector $X = [X_1 X_2 X_3 X_4]$ with zero mean

$$\mathbb{E}X_1X_2X_3X_4 = \mathbb{E}X_1X_2\mathbb{E}X_3X_4 + \mathbb{E}X_1X_3\mathbb{E}X_2X_4 + \mathbb{E}X_1X_4\mathbb{E}X_2X_3.$$

(3) Show, that for any fixed n, S_n and μ_n are independent.

Hint: Recall that two Gaussian r.v. (X, Y) are independent if they are orthogonal, i.e. if $E(X - EX)(Y - EY) = 0$.

Problem 1.8

Show that:

$$\begin{matrix} \xi_n \xrightarrow{P} \xi \\ \eta_n \xrightarrow{P} \eta \end{matrix} \quad \Longrightarrow \xi_n \eta_n \xrightarrow{P} \xi\eta$$

Problem 1.9 (*)

(Large Deviations primitives) The law of large numbers states that if $\xi_1, \xi_2, ...$ is a sequence of (zero mean) i.i.d. random variables with $E|\xi_1| < \infty$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \xi_k = 0$$

in probability. It turns out that under certain conditions, the convergence rate is exponential.

(1) Derive Chernoff inequality. Assume that ξ is a r.v. such that for $\lambda \in \Lambda \subset \mathbb{R}^+$, the log-characteristic function is well defined, i.e. $\psi(\lambda) = \log \mathbb{E} e^{\lambda \xi_1} < \infty$.

$$\mathbb{P}\big(\xi \geq a\big) \leq e^{-I(a)}$$

where $I(a) = \sup_{\lambda\in\Lambda} \big\{\lambda a - \psi(\lambda)\big\}$.
Hint: use Chebyshev inequality

(2) Show that

$$\mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n} \xi_k \geq a\right) \leq e^{-nI(a)}$$

(3) Show that the probability of *large deviations* for the weak LLN decays at least exponentially, i.e.

$$\mathbb{P}\left(\Big|\frac{1}{n}\sum_{k=1}^{n} \xi_k\Big| \geq a\right) \leq Ce^{-nI(a)}$$

and find the explicit expressions for the rate function $I(a)$, when ξ_1 is

(i) Gaussian $\mathcal{N}(0,1)$

(ii) Bernoulli with values $\{-1, 1\}$ and probabilities $1/2$. Assume that $0 < a < 1$.