

STOCHASTIC PROCESSES

2. STATIONARY RANDOM PROCESSES

Problem 2.1

Consider a continuous time random process $\xi_t = \alpha \sin(\beta t + \gamma)$, where α , β and γ are random variables. γ is independent of α and β and has uniform distribution $\gamma \sim \mathcal{U}(0, 2\pi)$. Show that the n -dimensional distribution

$$F_{t_1, t_2, \dots, t_n}(x_1, x_2, \dots, x_n) \triangleq \mathbb{P}\{\lambda : \xi_{t_1}(\lambda) \leq x_1, \xi_{t_2}(\lambda) \leq x_2, \dots, \xi_{t_n}(\lambda) \leq x_n\}$$

is invariant under time shift, namely

$$F_{t_1, t_2, \dots, t_n}(x_1, x_2, \dots, x_n) = F_{t_1+h, t_2+h, \dots, t_n+h}(x_1, x_2, \dots, x_n).$$

Problem 2.2

Verify the following properties of correlation sequence $R(k)$ of a stationary processes

- (a) $R(k)$ is non negative definite, i.e. $\sum_{k,m} a_k R(k-m) \bar{a}_m \geq 0$, *if and only if* the corresponding spectral density function is a non negative function, namely:

$$S(\lambda) = \sum_{k=-\infty}^{\infty} R(k) e^{-i\lambda k} \geq 0 \quad \forall \lambda \in \mathbb{R}$$

- (b) if the correlation function can be decomposed as

$$R(n) = h(n) * \bar{h}(-n) := \sum_{k=-\infty}^{\infty} h(k) \bar{h}(k-n)$$

where \bar{z} stands for the complex conjugate of z , then $R(n)$ is a non-negative definite sequence.

- (c) if $R'(k, m)$ and $R''(k, m)$ are correlation functions, then

$$R'(k, m)R''(k, m), \quad R'(k, m) + R''(k, m)$$

are correlation functions as well.

Problem 2.3

Verify whether or not each of the following sequences can be correlation sequence of some stationary process.

- a) $R(n) = e^{-n^2}$
 b) $R(n) = \begin{cases} 1 & n = 0 \\ 0.7 & |n| = 1 \\ 0 & \text{otherwise} \end{cases}$

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$$c) R(n) = \begin{cases} N - |n| & |n| \leq N \\ 0 & \text{otherwise} \end{cases}$$

Problem 2.4 (*)

Let X be a column vector, formed by N subsequent samples of a stationary in the wide sense zero mean random sequence $\{X_n\}_{n \in \mathbb{Z}}$. Let $R \equiv EXX^T$ and let λ_j , $j = 1, \dots, N$ be the eigenvalues of R .

- (a) Show that $\min_{\lambda} S_x(\lambda) \leq \lambda_j \leq \max_{\lambda} S_x(\lambda)$, where $S_x(\lambda)$ is the spectral density of the process X_n .
- (b) ("*Eigenfilter*") Consider the observation sequence

$$Y_n = X_n + \xi_n$$

where $X = (X_n)_{n \in \mathbb{Z}}$ is zero mean stationary random sequence, with correlation function $EX_k X_l = r_x(k-l)$ and $\xi = (\xi_n)_{n \in \mathbb{Z}}$ is a zero mean white noise sequence, i.e. $E\xi_k \xi_l = \sigma^2 \delta_{k-l}$.

$Y = (Y_n)_{n \in \mathbb{Z}}$ is passed through the linear filter:

$$\tilde{X}_n = \sum_{k=0}^{N-1} Y_{n-k} a_k = \sum_{k=0}^{N-1} X_{n-k} a_k + \sum_{k=0}^{N-1} \xi_{n-k} a_k$$

where a_k are the filter coefficients. Find the filter coefficients, so that the signal-to-noise ratio at the output is maximal, i.e.

$$a' = \operatorname{argmax}_{a \in \mathbb{R}^N} \frac{E \left(\sum_{k=0}^{N-1} X_{n-k} a_k \right)^2}{E \left(\sum_{k=0}^{N-1} \xi_{n-k} a_k \right)^2}$$

3. LINEAR ESTIMATION

Problem 3.1

This problem deals with the Kolmogorov-Wiener approach to linear estimation. Consider a stationary random process $(X_n, Y_n)_{n \geq 0}$.

(a) *Smoothing (interpolation) problem*

Assume the following spectral densities exist:

$$S_x(\lambda) = \sum_{k=-\infty}^{\infty} EX_k X_0 e^{-i\lambda k}$$

$$S_y(\lambda) = \sum_{k=-\infty}^{\infty} EY_k Y_0 e^{-i\lambda k}$$

$$S_{xy}(\lambda) = \sum_{k=-\infty}^{\infty} EX_k Y_0 e^{-i\lambda k}$$

Find a sequence of coefficients (numbers) $(\tilde{a}_n)_{n \in \mathbb{Z}}$, so that

$$E\left(X_n - \sum_{k=-\infty}^{\infty} Y_k \tilde{a}_{n-k}\right)^2 \leq E\left(X_n - \sum_{k=-\infty}^{\infty} Y_k a_{n-k}\right)^2$$

for any other real sequence $(a_n)_{n \in \mathbb{Z}}$. Find an expression for the corresponding mean square smoothing error.

(b) (*) *Filtering problem*

In addition to assumptions in (a), assume that the complex spectral density of Y can be factored, i.e. there exists a function $B(z)$ such that

$$S_y(z) = B(z)B(z^{-1})$$

where $S_y(z) = \sum_{n=-\infty}^{\infty} R_y(n)z^{-n}$ for a complex number z , $|z| < 1$ (e.g. $z = e^{i\lambda}$) and, moreover, the power expansion of $B(z)$ has only non-positive powers of z (i.e. $B(z)$ corresponds to a "casual" sequence)

Find a sequence of coefficients $(\tilde{a}_n)_{n \geq 0}$, such that

$$E\left[X_n - \sum_{k=0}^{\infty} Y_{n-k} \tilde{a}_k\right]^2 \leq E\left[X_n - \sum_{k=0}^{\infty} Y_{n-k} a_k\right]^2$$

for any other sequence $(a_n)_{n \geq 0}$. Find an expression for the corresponding mean square filtering error.

(c) *Filter with finite memory*

Find a vector $\tilde{a} = (\tilde{a}_0, \dots, \tilde{a}_p)^*$, such that:

$$E\left(X_n - \sum_{k=0}^p Y_{n-k} \tilde{a}_k\right)^2 \leq E\left(X_n - \sum_{k=0}^p Y_{n-k} a_k\right)^2$$

for any vector $a \in \mathbb{R}^{p+1}$. Find the corresponding mean square error.

Problem 3.2

- (a) Show that there exists a stationary (in the wide sense) process $X = (X_n)_{n \in \mathbb{Z}}$, satisfying the equation

$$X_n = aX_{n-1} + \varepsilon_n, \quad n \in \mathbb{Z}, \quad |a| < 1 \quad (3.1)$$

where $\varepsilon = (\varepsilon_n)_{n \in \mathbb{Z}}$ is a "white noise" sequence (i.e. sequence of orthonormal random variables with zero means).

- (b) Let $Y = (Y_n)_{n \in \mathbb{Z}}$ be a random sequence generated by

$$Y_n = X_n + \xi_n, \quad n \in \mathbb{Z},$$

where $\xi = (\xi_n)_{n \in \mathbb{Z}}$ is another "white noise" sequence, independent of ε . Show that the pair (X, Y) is a stationary random process.

- (c) Find the optimal smoothing (interpolation) coefficients and the corresponding error (see (a) of the previous problem).
 (d) (*) Find the optimal filtering coefficients and the corresponding filtering error (see (b) of the previous problem).
 (e) Find the optimal coefficients for the filter of order 2 and the corresponding filtering error (see (c) of the previous problem).
 (f) Derive the Kalman filter equations for $\hat{X}_n = \hat{E}(X_n | Y_1^n)$, $n \geq 0$.
 (g) Specify the conditions under which the Kalman filter is stable, i.e. for which the limit

$$P_\infty := \lim_{n \rightarrow \infty} E(X_n - \hat{X}_n)^2$$

exists and is independent of the initial condition P_0 . Is the filtering equation stable for "unstable" signal model (i.e. $|a| > 1$)? Find the steady state filtering error P_∞ and the formula for the stabilized filter.

- (h) Sort all the filters in decreasing order of the filtering error.
 (i) Assume that X_n is an unknown random parameter, i.e. $X_n = X_0$ for all n , which can be modelled by setting $a = 1$ and $E\varepsilon_k^2 = 0$. Is X still a stationary process? Show that $P_\infty = 0$ in (e) and i.e. the parameter is estimated perfectly as $n \rightarrow \infty$.
 (j) Simulate your results in MATLAB. Find empirically the variance of the filtering error and compare it to the theoretical results. Try to modify the optimal gain in the Kalman filter and verify that the filtering error increases.

Problem 3.3

Let $(X_n)_{n \geq 0}$ be a discrete time telegraph signal:

$$X_n = \begin{cases} X_{n-1} & \text{with prob. } p \\ -X_{n-1} & \text{with prob. } 1-p \end{cases}$$

subject to $\mathbb{P}\{X_0 = \pm \ell\} = 1/2$. Assume it is observed in the white noise $Y_n = X_n + \varepsilon_n$.

- (a) Construct a recursive model for the $(X_n)_{n \geq 0}$, suitable for the Kalman filter setup, i.e. find deterministic sequences A_n and B_n , so that:

$$X_n = A_n X_{n-1} + B_n \eta_n,$$

where $E\eta_n = 0$, $E\eta_n\eta_m = \delta_{n,m}$ and $EX_m\eta_n = 0$ for $m < n$. Note that $(\eta_n)_{n \geq 1}$ may generally depend, even explicitly, on $(X_n)_{n \geq 1}$, being uncorrelated with it !

- (b) Derive Kalman filter for the obtained model.
 (c) Does the obtained recursion realizes the orthogonal projection on the subspace spanned by $\{Y_1, \dots, Y_n\}$?

Problem 3.4

Let $(X_n, Y_n)_{n \geq 1}$ satisfy the recursion:

$$\begin{aligned} X_n &= aX_{n-1} + \varepsilon_n \\ Y_n &= \eta_n X_{n-1} + \xi_n \end{aligned}$$

subject to standard¹ r.v. X_0 and where $(\xi_n)_{n \geq 1}$, $(\varepsilon_n)_{n \geq 1}$ and $(\eta_n)_{n \geq 1}$ are independent white noises with

$$E\xi_1^2 = \sigma_\xi^2, \quad E\varepsilon_1^2 = \sigma_\varepsilon^2, \quad E\eta_1 = \mu \neq 0, \quad E(\eta_1 - \mu)^2 = \sigma_\eta^2$$

Derive the recursion for orthogonal projection of X_n on $\{Y_k, 0 \leq k \leq n\}$, i.e. $\widehat{E}(X_n | Y_0^n)$.

Problem 3.5

Let $(\theta_n, \xi_n)_{n \geq 0}$ be a pair of signals with the observable component ξ_n . Assume that θ_n has ARMA(p, q) structure, namely:

$$\theta_n = - \sum_{k=1}^p a_k \theta_{n-k} + \sum_{m=0}^q b_m \varepsilon_{n-m}, \quad n \geq p, \quad b_0 = 1$$

subject to random initial conditions $\theta_0, \dots, \theta_{p-1}$, where $(\varepsilon_n)_{n \geq 0}$ is the white noise with variance σ^2 . The observations are given by:

$$\xi_n = \theta_n + v_n$$

where $v = (v_n)_{n \geq 1}$ is a white noise sequence with $E v_n^2 = \sigma_v^2$, independent of θ . Derive the Kalman filter for θ_n with observations ξ_0^n .

Problem 3.6

"Information Filter"

- (a) (*Matrix Inversion Lemma*) Let $A > 0$ and $B > 0$ be two M -by- M matrices related by:

$$A = B^{-1} + CD^{-1}C^*$$

where D is another positive definite N -by- N matrix and C is an M -by- N matrix. Show that:

$$A^{-1} = B - BC(D + C^*BC)^{-1}C^*B$$

¹standard means zero mean and unit covariance

(b) Recall the Riccati equation from the Kalman filter:

$$P_{n+1} = aP_n a^* + bb^* - aP_n A^* (AP_n A^* + BB^*)^{-1} AP_n a^*$$

Assuming that a and BB^* are non-singular, derive recursion for the information matrix $J_n = P_n^{-1}$ (in a number of problems J_n is easier to calculate numerically than P_n)

Problem 3.7

Show that x_k generated by ($t = 0, \dots, k - 1$)

$$x_{t+1} = x_t + \frac{\gamma_t a_{t+1}^*}{\alpha + a_{t+1} \gamma_t a_{t+1}^*} (y_{t+1} - a_{t+1} x_t), \quad x_0 = x$$

$$\gamma_{t+1} = \gamma_t - \frac{\gamma_t a_{t+1}^* a_{t+1} \gamma_t}{\alpha + a_{t+1} \gamma_t a_{t+1}^*}, \quad \gamma_0 = I$$

is the unique solution of linear equations $(\alpha I + A^* A)x = A^* y$, where $a_t, t = 1, \dots, k$ are rows of A , α is a positive constant and y_t are components of y . Verify that

$$\gamma_k = (\alpha I + A^* A)^{-1} \alpha.$$