STOCHASTIC PROCESSES

4. Conditional Expectation

Problem 4.1

Suppose $\xi = (\xi_n)_{n\ge 1}$ is a Markov process, i.e. for any bounded function $g : \mathbb{R} \mapsto \mathbb{R}$ and any $n \ge 2$

$$E\big(g(\xi_n)|\xi_1, ..., \xi_{n-1}\big) = E\big(g(\xi_n)|\xi_{n-1}\big), \quad P-a.s.$$

Introduce the generic notation for conditional densities [1]

$$f_{\xi_n|\xi_m}(s,t) = \frac{\partial}{\partial s} P\big(\xi_n \le s|\xi_m = t\big).$$

Derive the Chapman-Kolmogorov equation

$$f_{\xi_n|\xi_\ell}(s,t) = \int_{\mathbb{R}} f_{\xi_n|\xi_m}(s,u) f_{\xi_m|\xi_\ell}(u,t)du, \quad \ell < m < n$$

Problem 4.2

Consider a random sequence $\{X_n\}_{n\ge 1}$, where X_1 is uniformly distributed on $[0,1]$ and for $n > 1$ X_n has conditionally uniform distribution on $[0, X_{n-1}]$, given $\sigma\{X_1, ..., X_{n-1}\}$.

(a) Calculate the following conditional probability densities:

$$f_{X_2|X_1}(s,t), f_{X_3|X_2}(s,t), f_{X_3|X_1}(s,t).$$

(b) Derive a general expression for $f_{X_{n+k}|X_n}(s,t)$ for $k > 0$.

(c) From the expression derived in b) calculate the probability density $f_{X_n}(x)$.

(d) Does the sequence X converge? If it does, in which sense and what is the limit?

Problem 4.3

Let $\xi_1, \xi_2, ...$ be an i.i.d. sequence. Show that:

$$\mathbb{E}(\xi_1|S_n, S_{n+1}, ...) = \frac{S_n}{n}$$

where $S_n = \xi_1 + ... + \xi_n$.

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[1] recall that for a pair of random variables X, Y, $E(X|Y = y)$ denotes a function $\psi(y)$, such that $E(X|Y) = \psi(Y)$.

Problem 4.4

(Buffon's needle)

A needle of unit length is thrown at random on a (vertical) "corridor" of unit width and infinite length, so that at least half the needle falls *inside* the corridor. Find the probability of the event:

$A = \{\omega$: the center of the needle falls inside the corridor and the needle crosses corridor's boundary $\}$.

Problem 4.5

Let $X \sim U[0,1]$ and η be a r.v. given by:

$$\eta = \begin{cases} X & X \le 0.5 \\ 0.5 & X > 0.5 \end{cases}$$

Find $\mathbb{E}(X|\eta)$.

Problem 4.6

a) Consider an event A that does not depend on itself, i.e. A and A are independent. Show that:

$$\mathbb{P}\{A\} = 1 \quad \text{or} \quad \mathbb{P}\{A\} = 0$$

b) Let A be an event so that $\mathbb{P}\{A\} = 1$ or $\mathbb{P}\{A\} = 0$. Show that A and any other event B are independent.
c) Show that a r.v. $\xi(\omega)$ doesn't depend on itself if and only if $\xi(\omega) \equiv \text{const}$.

Problem 4.7

Consider the probability space $([0,1], \mathcal{B}, \lambda)$, with λ being the Lebesgue measure on $\mathcal{B}$ and particularly

$$\lambda\big([a,b]\big) = b - a, \quad b \ge a.$$

It is well known that each point $\omega \in \Omega$ can be represented by an infinite binary sequence, so that

$$\omega = \frac{x_1}{2} + \frac{x_2}{2^2} + \dots$$

where $x_i \in \{0,1\}$, $i = 1, 2, \dots$. Define a sequence of r.v. $\xi_1(\omega), \xi_2(\omega), \dots$ by:

$$\xi_n(\omega) = x_n$$

Show that $\{\xi_i(\omega)\}_{i=1}^n$ are *independent* binary r.v. for any n.

Problem 4.8

Let $Y(\omega)$ be a positive random variable with probability density:

$$f(y) = \frac{1}{\sqrt{2\pi}} \frac{e^{-y/2}}{\sqrt{y}}, \quad y > 0$$

Define the conditional density of $X(\omega)$ given fixed $Y(\omega) = y$:

$$f(x|y) = \frac{\sqrt{y}}{\sqrt{2\pi}} e^{-yx^2/2}$$

Does the formula $\mathbb{E}\big(\mathbb{E}(X|Y)\big) = \mathbb{E}X$ hold ? If not, explain why.

Problem 4.9

Let X and Z be a pair of independent r.v. and $\mathbb{E}|X| < \infty$. Then $\mathbb{E}(X|Z) = EX$ with probability one. Does the formula

$$\mathbb{E}(X|Z, Y) = \mathbb{E}(X|Y)$$

holds for any r.v. Y ? To answer this question consider the following example. On the probability space $([0,1], \mathcal{B}, \lambda)$ consider the r.v.

$$X(\omega) = I_{[0,1/2]}(\omega), \quad Y(\omega) = I_{[0,3/4]}(\omega), \quad Z(\omega) = I_{[1/4,3/4]}(\omega)$$

Problem 4.10

Random variables ξ_1 and ξ_2 are *conditionally* independent with respect to ξ_3 if for any pair of bounded functions $f(x)$ and $g(x)$

$$\mathbb{E}[f(\xi_1)g(\xi_2)|\xi_3] = \mathbb{E}[f(\xi_1)|\xi_3]\mathbb{E}[g(\xi_2)|\xi_3] \tag{4.1}$$

Show that (4.1) holds *if and only if*

$$\mathbb{E}[f(\xi_1)|\xi_2, \xi_3] = \mathbb{E}[f(\xi_1)|\xi_3]$$

for any bounded $f(x)$.

Problem 4.11

Let X_1 and X_2 be two random variables such that, $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_2 = 0$. Suppose we can find a linear combination $Y = X_1 + \alpha X_2$, which is independent of X_2. Show that $\mathbb{E}(X_1|X_2) = -\alpha X_2$.

Problem 4.12

Let $(X_n)_{n\geq 0}$ be a sequence of random variables. Assume that the random variables Y and Z are measurable with respect to $\sigma\{X_0, ..., X_n\}$ and $\sigma\{X_n, X_{n+1}, ...\}$ respectively. Show that the following are equivalent:

(i) $\mathbb{E}\big(Z|X_0, ..., X_k\big) = \mathbb{E}\big(Z|X_k\big)$, if $k \leq n$.
(ii) $\mathbb{E}\big(ZY|X_n\big) = \mathbb{E}\big(Z|X_n\big)\mathbb{E}\big(Y|X_n\big)$

Note that (i) is the Markov property and due to the equivalence it can be defined by means of (ii), which means that for Markov process, ”the future and the past are conditionally independent, given the present”