STOCHASTIC PROCESSES

6. Non-Linear Filtering Of Markov Processes

Problem 6.1 *(Counting observations)*

A random switch $\theta_n \in \{0,1\}$, $n \geq 0$ is a discrete-time two-state Markov chain with transition matrix:

$$\Lambda = \begin{bmatrix} \lambda_1 & 1-\lambda_1 \\ 1-\lambda_2 & \lambda_2 \end{bmatrix}$$

i.e. $\mathbb{P}\{\theta_n = j|\theta_{n-1} = i\} = \Lambda_{i,j}$, $i, j \in \{0,1\}$. Assume that $\theta_0 = 1$.

A counter ξ_n, counts arrivals (of e.g. particles) from two independent sources with different intensities α and β. The counter is connected according to the state of the switch θ_n to one source or another, so that:

$$\xi_n = \xi_{n-1} + I(\theta_n = 1)\varepsilon_n^\alpha + I(\theta_n = 0)\varepsilon_n^\beta, \quad n = 1, 2, ...$$

subject $\xi_0 = 0$. Here β and α are constants from interval $(0,1)$ and $\varepsilon_n^\gamma \in \{0,1\}$ denotes an i.i.d. sequence with $\mathbb{P}\{\varepsilon_n^\gamma = 1\} = \gamma$ $(0 < \gamma < 1)$.

a) Find the optimal estimate of the switch state, given the counter data up to current moment, i.e. derive the recursion for $\pi_n(\xi) = \mathbb{E}(\theta_n|\xi_0^n)$.

b) Explain the meaning and examine the behaviour of the filter in the limit cases:

(1) $\alpha = 1$ and $\beta = 0$ (simultaneously).
(2) $\lambda_1 = 1$ and $\lambda_2 = 0$ (and vice versa).
(3) $\lambda_1 = \lambda_2 = 1$

Problem 6.2 *(*)(Conditionally Gaussian Filter)*

Let signal/observation model $(X_n, Y_n)_{n\geq 0}$:

$$\begin{aligned} X_n &= a_0(Y_0^{n-1}) + a_1(Y_0^{n-1})X_{n-1} + b\varepsilon_n, \quad n = 1, 2, ... \\ Y_n &= A_0(Y_0^{n-1}) + A_1(Y_0^{n-1})X_{n-1} + B\xi_n \end{aligned}$$

where b and B are constants and $A_i(Y_0^{n-1})$ and $a_i(Y_0^{n-1})$, $i = 0, 1$ are some functionals of the vector $[Y_0, Y_1, ..., Y_{n-1}]$. $(\varepsilon_n)_{n\geq 1}$ and $(\xi_n)_{n\geq 1}$ are independent i.i.d. standard Gaussian random sequences. The initial condition (X_0, Y_0) is also a standard Gaussian vector with unit covariance matrix.

(1) Is the pair of processes $(X_n, Y_n)_{n\geq 0}$ necessarily Gaussian ? Prove or verify your answer by example.

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(2) Find the recursion for $\widehat{X}_n = \mathbb{E}(X_n|Y_0^n)$ and $P_n = \mathbb{E}\big[(X_n - \widehat{X}_n)^2|Y_0^n\big]$. Is the obtained filter linear ? time invariant ? asymptotically time invariant (i.e. time invariant as $n \to \infty$)?

Hint: prove first that X_n is Gaussian, conditioned on Y_0^n.

(3) Verify that in case of $a_i(Y_0^{n-1}) \equiv a_i$ and $A_i(Y_0^{n-1}) \equiv A_i$, $i = 0, 1$ (a_i and A_i constants) your solution coincides with the Kalman filter.

Problem 6.3 *(*)Extended Kalman Filter*

Let $(\theta_n, \xi_n)_{n\ge 0}$ be a pair of processes, generated by the system:

$$\begin{aligned}\theta_n &= h(\theta_{n-1}) + u_n \\ \xi_n &= g(\theta_{n-1}) + v_n, \quad n \ge 1\end{aligned}$$

The noises v_n and u_n are zero mean independent Gaussian sequences with $\mathbb{E}v_n^2 = B^2 > 0$ and $\mathbb{E}u_n^2 = b^2$. Functions $g(x)$ and $h(x)$ are twice differentiable.

Assume that some reasonable unbiased estimate $\widetilde{\theta}_n$ is available, so that $\mathbb{E}(\theta_n - \widetilde{\theta}_n)^2 \ll 1$.

(a) Expand $g(\cdot)$ and $h(\cdot)$ into series of powers of the filtering error around the estimate $\widetilde{\theta}_n$. By neglecting the higher powers, formally derive Conditionally Gaussian Kalman filter equations for the obtained linear model. (This *heuristic* device is called Extended Kalman Filter)

(b) Explain why the EKF may fail with e.g. $h(x) = \tanh(x^3)$

Problem 6.4

Consider a signal/observation pair $(\theta, \xi_n)_{n\ge 1}$, where θ is a random variable distributed uniformly on $[0, 1]$ and (ξ_n) is a sequence generated by:

$$\xi_n = \theta U_n$$

where $(U_n)_{n\ge 1}$ is a sequence of i.i.d. random variables with uniform distribution on $[0, 1]$. θ and U are independent.

(a) Derive a Kalman filter for $\widehat{\theta}_n = \widehat{\mathbb{E}}(\theta|\xi_1^n)$.

(b) Find the corresponding mean square error $P_n = \mathbb{E}(\theta - \widehat{\theta}_n)^2$. Show that it converges to zero as $n \to \infty$ and determine the rate of convergence [1]

(c) Consider the recursive filtering estimate $(\widetilde{\theta}_n)_{n\ge 0}$, generated by

$$\widetilde{\theta}_n = \max(\widetilde{\theta}_{n-1}, \xi_n), \quad \widetilde{\theta}_0 = 0$$

Find the corresponding mean square error, $Q_n = \mathbb{E}(\theta - \widetilde{\theta}_n)^2$.

[1] i.e. find a sequence of r_n, such that $\lim_{n\to\infty} r_n P_n$ exists and positive

(d) Show that Q_n converges to zero and find the rate of convergence. Does this filter give better accuracy, compared to Kalman filter, uniformly in n ? Asymptotically in n?

(e) Verify whether $\widetilde{\theta}_n$ is the optimal in the mean square sense filtering estimate. If not, find the optimal estimate $\bar{\theta}_n = \mathbb{E}(\theta|\xi_1^n)$.