STOCHASTIC PROCESSES

7. Wiener Process and Stochastic Integral

Problem 7.1

Let $W = (W_t)_{t\geq 0}$ be the Wiener process. Show that

(1) $Z_t = \sqrt{\epsilon} W_{t/\epsilon}$, $\epsilon > 0$
(2) $Z'_t = W_{t+s} - W_s$, $s \geq 0$
(3) $Z''_t = t W_{1/t}$

are Wiener processes as well, i.e. verify the axiomatic definition of the Wiener process.

Problem 7.2 (*)

Let W be the Wiener process. Denote by τ_a the first time W crosses the line L_a, i.e.

$$\tau_a = \{\inf t : W_t \geq a\},$$

so that τ_a is a random variable.

(1) Verify the *reflection principle*:

$$\Pr\{W_t \leq x | \tau_a \leq t\} = \Pr\{W_t \geq 2a - x | \tau_a \leq t\}$$

Explain the origin of the name "reflection".

(2) Using the reflection principle, find probability density of τ_a:

$$p_\tau(t; a) = \frac{d \Pr\{\tau_a \leq t\}}{dt}$$

and verify that $\mathbb{E}\tau_a = \infty$.

Hint: Try to calculate $\mathbb{P}\{W_t > a; \tau_a < t\}$.

Problem 7.3

Let X_t be a random process, given by the Itô stochastic differential equation:

$$dX_t = a_t X_t dt + b_t dW_t$$

subject to the initial condition X_0 with $\mathbb{E}X_0 = m_0$ and $\mathbb{E}(X_0 - m_0)^2 = V_0$, independent of W. Assume that a_t and b_t are smooth deterministic functions.

Find the autocorrelation function $K(t,s) = E(X_t - \mathbb{E}X_t)(X_s - \mathbb{E}X_s)$. Find a sufficient condition for X_t to be a stationary process. Assume these conditions are satisfied and find spectral density function:

$$S_x(\lambda) = \int_{\infty}^{\infty} \mathbb{E}X_t X_0 e^{-i\lambda t} dt$$

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Problem 7.4

Let θ be some deterministic, but unknown parameter. Assume that it is observed in "white Gaussian noise" and the observations are given by

$$dY_t = a_t\theta dt + dW_t$$

subject to $Y_0 = 0$. Assume [1]:

$$\int_0^t a_s^2 ds > 0 \text{ for any } t \text{ and } \lim_{t\to\infty}\int_0^t a_s^2 ds = \infty$$

Propose a consistent estimate of θ from $Y_0^t = \{Y_s, 0 \le s \le t\}$, that is derive $\widehat{\theta}(Y_0^t)$ such that

$$\lim_{t\to\infty} \mathbb{E}(\theta - \widehat{\theta}(Y_0^t))^2 = 0.$$

The Itô Formula and SDE

Problem 7.5

With the help of the Itô formula, calculate the following

(1) $C(t) = \mathbb{E}\cos(W_t)$ and $S(t) = \mathbb{E}\sin(W_t)$
(2) $M_n(t) = \mathbb{E}W_t^n$, $n = 0, 1, ...$

Problem 7.6

It is customary in financial mathematics to model the asset prices by means of the diffusion model:

$$dX_t = rX_t dt + \sigma X_t dW_t, \quad s.t.\ X_0 > 0$$

where r is the safe asset interest rate (e.g. bank investments) and σ indicates the intensity of unsafe asset price variations (e.g. shares).

(1) Explain the origin of this model
(2) Solve for X_t explicitly and verify that $X_t \ge 0$, $t \ge 0$.

Problem 7.7 (*)

Suppose that $(W_t)_{t\ge0}$ and $(V_t)_{t\ge0}$ are two independent Wiener processes. Let $Z_t = \sqrt{W_t^2 + V_t^2}$, the distance from the origin of a Brownian particle on the plane. Show that P-a.s.

$$Z_3 = \sqrt{W_3^2 + V_3^2} \le \mathbb{E}(Z_4|V_3, W_3) \le \sqrt{2 + W_3^2 + V_3^2} = \sqrt{2 + Z_3^2}$$

Hint: the upper bound is a consequence of Jensen inequality and the lower bound can be obtained by Ito formula

Problem 7.8 (*)

Let $(X_t)_{t\ge0}$ be a continuous-time (i.e. $t \in \mathbb{R}^+$) Gaussian Markov process with zero mean and covariance function $R(t, s) = \mathbb{E}X_t X_s$.

[1] Note that a_t is not necessarily positive for any t.

(a) Assuming that $R(s,s) > 0$, show that for any $t > s > \tau$:

$$R(t,\tau) = \frac{R(t,s)R(s,\tau)}{R(s,s)} \tag{7.1}$$

Hint: any Markov process satisfies Chapman-Kolmogorov equation

(b) Show that any solution of (7.1) has the form

$$R(t,s) = f\big(\max(t,s)\big)g\big(\min(t,s)\big) \tag{7.2}$$

(c) Construct a Markov Gaussian process with covariance function of the form (7.2).

Hint: try the construction $f(t)W_{g(t)/f(t)}$, where $(W_t)_{t\geq 0}$ is the Wiener process.

(d) Let $(X_t)_{t\geq 0}$ be a Gaussian process with zero mean and $\mathbb{E}X_tX_s = e^{-|t-s|}$. Express X_t in the form:

$$X_t := f(t)W_{g(t)/f(t)}$$

where $(W_t)_{t\geq 0}$ is the Wiener process.

(e) Let $(X_t)_{t\geq 0}$ be a mean square continuous [2] and stationary Gaussian-Markov process. Find its covariance function.

Problem 7.9

Let X and Y be the solution of

$$\begin{aligned} dX_t &= -0.5X_tdt - Y_tdB_t \\ dY_t &= -0.5Y_tdt + X_tdB_t. \end{aligned}$$

subject to $X_0 = x$ and $Y_0 = y$ with B_t being a Wiener process (Brownian motion).

(a) Show that $X_t^2 + Y_t^2 \equiv x^2 + y^2$ for all $t \geq 0$, i.e. the vector (X_t, Y_t) revolves on a circle.

(b) Find the SDE, satisfied by $\theta_t = \arctan(X_t/Y_t)$.

Problem 7.10

Verify the solution of the following SDE's

(a) $X_t = B_t/(t+1)$ solves

$$dX_t = -\frac{1}{1+t}X_tdt + \frac{1}{1+t}dB_t, \quad X_0 = 0$$

(b) $X_1(t) = X_1(0) + t + B_1$ and $X_2(t) = X_2(0) + X_1(0)B_2(t) + \int_0^t sdB_2(s) + \int_0^t B_1(s)dB_2(s)$ solve

$$\begin{aligned} dX_1 &= dt + dB_1 \\ dX_2 &= X_1dB_2 \end{aligned}$$

[2] The process $(\xi_t)_{t\geq 0}$ is *mean square continuous* if $\text{l.i.m.}_{h\to\pm 0} X_{t\pm h} = X_t$

$$\lim_{h\to 0} \mathbb{E}(X_{t+h} - X_t)^2 = 0$$

It is not difficult to show that X_t is m.s. continuous if and only if $R(t,t)$ is continuous, where $R(s,t) = \mathbb{E}X_tX_s$. Note that this does not imply continuity of the trajectories of X (as in the case of e.g. the Poisson process)!

(c) $X_t = e^{-t}X_0 + e^{-t}B_t$ solves

$$dX_t = -X_t dt + e^{-t}dB_t.$$

(d) $Y_t = \exp(aB_t - 0.5a^2t)\big[Y_0 + r\int_0^t \exp(-aB_s + 0.5a^2s)ds\big]$ solve

$$dY = rdt + aY dB_t.$$

(e) The processes $X_1(t) = X_1(0)\cosh(t) + X_2(0)\sinh(t) + \int_0^t a\cosh(t-s)dB_1 + \int_0^t b\sinh(t-s)dB_2$ and $X_2(t) = X_1(0)\sinh(t) + X_2(0)\cosh(t) + \int_0^t a\sinh(t-s)dB_1 + \int_0^t b\cosh(t-s)dB_2$ solve

$$dX_1 = X_2 dt + adB_1$$
$$dX_2 = X_1 dt + bdB_2,$$

which can be seen as stochastically excited vibrating string equations.