**STOCHASTIC PROCESSES**
**SOLUTIONS TO HOME ASSIGNMENTS**

## 1. Basics of mathematical probability

**Problem 1.1**

(a) By definition
$$P(A) = \int_{\omega\in A} dP(\omega) = \int_{\omega\in\Omega} I_A(\omega)dP(\omega) = \mathbb{E}I_A$$

(b) Since by definition any $\omega$ is in $\Omega$ and not in $\emptyset$, we have $I_\Omega(\omega) \equiv 1$ and $I_\emptyset(\omega) \equiv 0$

(c)
$$\omega' \in A \implies \begin{cases} I_A(\omega') = 1 \\ I_{\bar{A}}(\omega') = 0 \end{cases} \implies I_A(\omega') + I_{\bar{A}}(\omega') = 1$$
Similarly
$$\omega'' \in \bar{A} \implies \begin{cases} I_A(\omega'') = 0 \\ I_{\bar{A}}(\omega'') = 1 \end{cases} \implies I_A(\omega'') + I_{\bar{A}}(\omega'') = 1$$
so for any $\omega \in \Omega$
$$I_A(\omega) + I_{\bar{A}}(\omega) \equiv 1$$

(d) $I_A(\omega)I_B(\omega) = 1$ if and only if $I_A(\omega) = 1$ and $I_B(\omega) = 1$, that is $\omega \in A$ *and* $\omega \in B$, in other words $\omega \in A \cap B$

(e) $\omega \in A \cup B$ if and only if $\omega \in A$ *or* $\omega \in B$:
$$\begin{cases} \omega \in A \\ \omega \notin B \end{cases} \implies I_{A\cup B} = I_A + I_B - I_{A\cap B} = 1 + 0 - 0 = 1$$
$$\begin{cases} \omega \notin A \\ \omega \in B \end{cases} \implies I_{A\cup B} = I_A + I_B - I_{A\cap B} = 0 + 1 - 0 = 1$$
$$\begin{cases} \omega \in A \\ \omega \in B \end{cases} \implies I_{A\cup B} = I_A + I_B - I_{A\cap B} = 1 + 1 - 1 = 1$$
$$\begin{cases} \omega \notin A \\ \omega \notin B \end{cases} \implies I_{A\cup B} = I_A + I_B - I_{A\cap B} = 0 + 0 - 0 = 0$$
that is $I_{A\cup B}(\omega) \equiv I_A(\omega) + I_B(\omega) - I_{A\cap B}(\omega)$

(f) By (c) $I_{\bar{A}_i} = 1 - I_{A_i}$. Further by (d)
$$\prod_{i=1}^n (1 - I_{A_i}) = I_{\cap_{i=1}^n \bar{A}_i}$$
and again by (c):
$$1 - \prod_{i=1}^n (1 - I_{A_i}) = I_{\overline{\cap_{i=1}^n \bar{A}_i}}$$

By Morgan rules from the basic set theory

$$\cup_{i=1}^n A_i = \overline{\cap_{i=1}^n \overline{A}_i}$$

and the desired result holds.

(g) Directly implied by (f) (note that $I_{A_i} I_{A_j} = 0$ for $i \neq j$)

(h)

$$(I_A - I_B)^2 = I_A + I_B - 2I_{A\cap B} = I_{A\cup B} - I_{A\cap B}$$

Clearly the above equals 1 if and only if $\omega \in A \cup B$ and $\omega \notin A \cap B$, which is the definition of $A \triangle B$.

**Problem 1.2**

1) $X$ is a simple r.v. so by definition $\mathbb{E}X = P(\omega \le 1/2) = 1/2$. In the case of $Y$, the Lebesgue integral

$$EY = \int_\Omega Y(\omega) d\lambda(\omega)$$

coincides with the usual integral

$$EY = \int_0^1 s^2 ds = 1/3.$$

2) The distribution of $X$ is

$$F(x) = P(X \le x) = \begin{cases} 0 & x < 0 \\ 1/2 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

so

$$EX = \int_{\mathbb{R}} x dF(x) = 0 \cdot \Delta F(0) + 1 \cdot \Delta F(1) = 1/2.$$

First find the distribution function of $Y$: for $x < 0$, $G(x) = P(Y < x) \equiv 0$; for $x \ge 0$

$$G(x) = P(Y \le x) = P(\omega^2 \le x) = P(\omega \le \sqrt{x}) = \begin{cases} \sqrt{x}, & x < 1 \\ 1, & x \ge 1 \end{cases}.$$

So

$$EY = \int_{\mathbb{R}} s dG(s) = \int_0^1 s d\sqrt{s} = \int_0^1 s \frac{1}{2\sqrt{s}} ds = \frac{1}{2}\int_0^1 \sqrt{s} ds = 1/3.$$

**Problem 1.3**

The sequence $Y_n$ starts with a string of $b$'s till the first occurrence of $X_n = 0$. From this point on the sequence stays at the value $a$. Verify $\mathbb{L}^2$ convergence

$$E\{(Y_n - a)^2\} = (b-a)^2 P\Big( \cap_{k=1}^n \{X_k = 1\}\Big) = (b-a)^2 2^{-n} \xrightarrow{n\to\infty} 0.$$

This implies convergence in probability. The latter can be verified directly: for $|b - a| > \varepsilon > 0$

$$\mathbb{P}\{|Y_n - a| \ge \varepsilon\} = \mathbb{P}\{Y_n \neq a\} = \mathbb{P}\big( \cap_{k=1}^n \{X_k = 1\}\big) = 2^{-n} \xrightarrow{n\to\infty} 0.$$

$P$-a.s. convergence is implied by the Borel-Cantelli Lemma, since

$$\sum_{n=1}^{\infty} P(|Y_n - a| \geq \varepsilon) < \infty.$$

Alternatively

$$P(\lim_{n\to\infty} Y_n \neq a) = P(\cap_{n\geq 1}\{X_n = 1\}) =$$
$$P(\lim_{m\to\infty} \cap_{n\leq m}\{X_n = 1\}) = \lim_{m\to\infty} P(\cap_{n\leq m}\{X_n = 1\}) = \lim_{m\to\infty} 2^{-m} = 0.$$

**Problem 1.4**

1. Use the elementary inequality $(x+y)^2 \leq 2x^2 + 2y^2$ for any real $x, y$

$$E\big(aX_n + bY_n - aX - bY\big)^2 = E\big(a(X_n - X) + b(Y_n - Y)\big)^2 \leq$$
$$2a^2 E(X_n - X)^2 + 2b^2 E(Y_n - Y)^2 \xrightarrow{n\to\infty} 0$$

2. By the Jensen inequality

$$\big|EX_n - EX\big| \leq E\big|X_n - X\big| \leq \sqrt{E(X_n - X)^2} \xrightarrow{n\to\infty} 0$$

3. Use the Cauchy-Shwartz inequality

$$\big|EXY - EX_nY_n\big| \leq E|XY - X_nY_n| =$$
$$E|Y(X - X_n) + X(Y - Y_n) - (X - X_n)(Y - Y_n)| \leq$$
$$E|Y(X - X_n)| + E|X(Y - Y_n)| + E|(X - X_n)(Y - Y_n)| \leq$$
$$\sqrt{EY^2 E(X - X_n)^2} + \sqrt{EX^2 E(Y - Y_n)^2} + \sqrt{E(X - X_n)^2 E(Y - Y_n)^2} \xrightarrow{n\to\infty} 0$$

The second statement is the particular case of the first.

4. Use the previous results

$$EV \overset{2}{=} \lim_{n\to\infty} EV_n = \lim_{n\to\infty} EX_nY_n \overset{3}{=} EXY.$$

**Problem 1.5**

Note that

$$S_n = \sum_{k=1}^{n} Z_k = \sum_{k=1}^{n} U^k = nI(U = 1) + I(U \neq 1)\frac{U(1 - U^n)}{1 - U}$$

so $\{S_n \not\to U/(1-U)\} = \{U = 1\}$ and since $P(U = 1) = 0$, $S_n$ converges $P$-a.s. to $U/(1-U)$. Convergence in probability follows from $P$-a.s. convergence.

**Problem 1.6**

$$E\{|X_n - a|^2\} = E\{|X_n - a_n + a_n - a|^2\} \leq 2E\{|X_n - a_n|^2\} + 2E\{|a_n - a|^2\} \xrightarrow{n\to\infty} 0.$$

**Problem 1.7**

1.

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n}\xi_i \implies \mathbb{E}\mu_n = 0$$

$$\mathbb{E}\mu_n^2 = 1/n^2\mathbb{E}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\xi_j = 1/n^2\sum_{j,i}\delta(i-j)\sigma^2 = \sigma^2/n \overset{n\to\infty}{\longrightarrow} 0$$

which implies that $\lim_{n\to\infty}\mu_n = 0$ in $\mathbb{L}^2$.

2.

$$\begin{aligned} S_n &= \frac{1}{n-1}\sum_{i=1}^{n}(\xi_i-\mu_n)^2 = \frac{1}{n-1}\left\{\sum_{i=1}^{n}\xi_i^2 - 2\mu_n\sum_{i=1}^{n}\xi_i + n\mu_n^2\right\} = \\ &= \frac{1}{n-1}\left\{\sum_{i=1}^{n}\xi_i^2 - n\mu_n^2\right\} \end{aligned}$$

So

$$\mathbb{E}S_n = \frac{1}{n-1}\left\{\sum_{i=1}^{n}\mathbb{E}\xi_i^2 - n\mathbb{E}\mu_n^2\right\} = \frac{1}{n-1}\left\{n\sigma^2-\sigma^2\right\} = \sigma^2$$

Further $\mathbb{E}(S_n-\mathbb{E}S_n)^2 = \mathbb{E}S_n^2 - (\mathbb{E}S_n)^2$ and:

$$\mathbb{E}S_n^2 = \frac{1}{(n-1)^2}\mathbb{E}\sum_{i=1}^{n}\sum_{j=1}^{n}(\xi_i-\mu_n)^2(\xi_j-\mu_n)^2 \tag{1.1}$$

Note that $(\xi_i-\mu_n)$ is a Gaussian r.v. By virtue of a well known formula [1] we obtain:

$$\begin{aligned} \mathbb{E}(\xi_i-\mu_n)^2(\xi_j-\mu_n)^2 &= \mathbb{E}(\xi_i-\mu_n)^2\mathbb{E}(\xi_j-\mu_n)^2 \\ &\quad +2\left[\mathbb{E}(\xi_i-\mu_n)(\xi_j-\mu_n)\right]^2 \end{aligned} \tag{1.2}$$

The last term in (1.2) is simplified:

$$\begin{aligned} \mathbb{E}(\xi_i-\mu_n)(\xi_j-\mu_n) &= \sigma^2\delta(i-j) + \mathbb{E}\mu_n^2 - \mathbb{E}\xi_i\mu_n - \mathbb{E}\xi_j\mu_n = \\ &= \sigma^2\delta(i-j) + \sigma^2/n - \sigma^2/n - \sigma^2/n \\ &= \sigma^2\delta(i-j) - \sigma^2/n \end{aligned} \tag{1.3}$$

[1] for a Gaussian vector $X = [X_1X_2X_3X_4]$ with zero mean

$$\mathbb{E}X_1X_2X_3X_4 = \mathbb{E}X_1X_2\mathbb{E}X_3X_4 + \mathbb{E}X_1X_3\mathbb{E}X_2X_4 + \mathbb{E}X_1X_4\mathbb{E}X_2X_3$$

Combining (1.1), (1.2) and (1.3) we arrive at:

$$\begin{aligned}
\mathbb{E}S_n^2 &= \frac{1}{(n-1)^2}\left\{\sum_i\sum_j \mathbb{E}(\xi_i-\mu_n)^2\mathbb{E}(\xi_j-\mu_n)^2+\right.\\
&\left. +2\sum_i\sum_j\left(\sigma^2\delta(i-j)-\sigma^2/n\right)^2\right\}=\\
&= \frac{1}{(n-1)^2}\Big\{(n-1)^2(\mathbb{E}S_n)^2+\\
&2\sigma^4\sum_i\sum_j\left(\delta(i-j)-\frac{2}{n}\delta(i-j)+\frac{1}{n^2}\right)\Big\}=\\
&= (\mathbb{E}S_n)^2+\frac{2\sigma^4}{(n-1)^2}\left\{n-\frac{2}{n}n+\frac{1}{n^2}n^2\right\}=\\
&= (\mathbb{E}S_n)^2+\frac{2\sigma^4}{n-1} \qquad (1.4)
\end{aligned}$$

Hence

$$\mathrm{Var}(S_n)=\frac{2\sigma^4}{n-1}\overset{n\to\infty}{\longrightarrow}0$$

which implies $\lim_{n\to\infty} S_n=\sigma^2$ in $\mathbb{L}^2$.

3. Define $Z_j=\xi_j-\mu_n$. Note that $S_n$ is an explicit functional of $Z_j$, $j=1,...,n$. Obviously, independence of $\{Z_j\}_{j=1}^n$ and $\mu_n$ implies independence of $S_n$ and $\mu_n$. The sequence $Z_j$ is Gaussian, so it suffices to show that:

$$\mathbb{E}Z_j\mu_n=0,\quad 1\le j\le n$$

which is easily verified

$$\mathbb{E}Z_j\mu_n=\mathbb{E}(\xi_j-\mu_n)\mu_n=\mathbb{E}\xi_j\mu_n-\mathbb{E}\mu_n^2=\frac{1}{n}\sum_i\mathbb{E}\xi_i\xi_j-\frac{\sigma^2}{n}=0$$

**Problem 1.8**

First let us establish:

$$\eta_n(\xi-\xi_n)\overset{P}{\longrightarrow}0 \qquad (1.5)$$

Fix a constant $C>0$ then

$$\begin{aligned}
&\mathbb{P}\big(|\eta_n(\xi_n-\xi)|>\varepsilon\big)=\mathbb{P}\big(\{|\eta_n(\xi_n-\xi)|>\varepsilon\}\cap\{|\eta_n|\ge C\}\}\big)+\\
&+\mathbb{P}\big(\{|\eta_n(\xi_n-\xi)|>\varepsilon\}\cap\{|\eta_n|<C\}\big)\le\\
&\le\mathbb{P}\big(|\eta_n|\ge C\big)+\mathbb{P}\big(C|\xi_n-\xi|>\varepsilon\big)\\
&\le\mathbb{P}\big(|\eta_n-\eta|\ge C/2\big)+\mathbb{P}\big(|\eta|\ge C/2\big)+\mathbb{P}\big(C|\xi_n-\xi|>\varepsilon\big)
\end{aligned}$$

which implies that:

$$\mathbb{P}\big(|\eta_n(\xi_n-\xi)|>\varepsilon\big)\to\mathbb{P}\big(|\eta|\ge C/2\big),\quad n\to\infty$$

Since $C$ can be chosen arbitrary large, (1.5) holds.

The desired result follows

$$\mathbb{P}\{|\xi_n\eta_n-\xi\eta|>\varepsilon\}\le\mathbb{P}\{|\xi_n(\eta_n-\eta)|>\varepsilon/2\}+\mathbb{P}\{|\eta(\xi_n-\xi)|>\varepsilon/2\}$$

**Problem 1.9**

**1.** Using Chebyshev inequality, for any $a \in \mathbb{R}$ and $\lambda \in \Lambda \subset \mathbb{R}^+$

$$\mathbb{P}\big(\xi \ge a\big) = \mathbb{P}\big(e^{\lambda\xi} \ge e^{\lambda a}\big) \le \frac{\mathbb{E}e^{\lambda\xi}}{e^{\lambda a}} = e^{\psi(\lambda)-a\lambda}$$

Minimizing the upper bound with respect to $\lambda$ gives

$$\mathbb{P}\big(\xi \ge a\big) \le e^{\inf_{\lambda\in\Lambda}\big\{\psi(\lambda)-a\lambda\big\}} = e^{-\sup_{\lambda\in\Lambda}\big\{a\lambda-\psi(\lambda)\big\}} = e^{-I(a)}$$

**2.** Let $S_n = \sum_{k=1}^n \xi_k$, then

$$\psi_n(\lambda) := \log \mathbb{E}e^{\lambda\sum_{k=1}^n \xi_n} = \log \prod_{k=1}^n \mathbb{E}e^{\lambda\xi_k} = n\psi(\lambda)$$

so that

$$\mathbb{P}\left(\frac{1}{n}\sum_{k=1}^n \xi_k \ge a\right) = \mathbb{P}\left(\sum_{k=1}^n \xi_k \ge an\right) \le e^{-\sup_{\lambda\in\Lambda}\big(\lambda(na)-n\psi(\lambda)\big)} = e^{-nI(a)}$$

**3.**

(i) Let $\xi_1$ be a Gaussian r.v. Since $\xi_1$ has a symmetric distribution we have

$$\mathbb{P}\left(\Big|\frac{1}{n}\sum_{k=1}^n \xi_k\Big| \ge a\right) = \mathbb{P}\left(\frac{1}{n}\sum_{k=1}^n \xi_k \ge a\right) + \mathbb{P}\left(\frac{1}{n}\sum_{k=1}^n \xi_k \le -a\right) =$$

$$= 2\mathbb{P}\left(\frac{1}{n}\sum_{k=1}^n \xi_k \ge a\right) \le 2e^{-nI(a)} \tag{1.6}$$

In this case $\Lambda = \mathbb{R}^+$ and $\psi(\lambda) = \log \mathbb{E}e^{\lambda\xi_1} = 1/2\lambda^2$, so that for $a > 0$,

$$I(a) = \sup_{\lambda\in\mathbb{R}^+}(\lambda a - 1/2\lambda^2) = a^2/2$$

Note that $S_n$ is Gaussian with zero mean and variance $1/n$. So in the special case of Gaussian r.v. this result can be obtained directly, making use of well-known bounds for integrals of Gaussian densities.

(ii) Let $\xi_1$ be symmetric Bernoulli r.v. with values in $\{-1, 1\}$. Due to symmetry (1.6) holds.

$$\psi(\lambda) = \log\cosh(\lambda)$$

so that

$$I(a) = \sup_{\lambda\in\mathbb{R}^+}\big\{\lambda a - \log\cosh(\lambda)\big\} := \sup_{\lambda\in\mathbb{R}^+} H(\lambda, a)$$

Note that $H(0, a) = 0$ and if $0 < a < 1$, for $\lambda >> 1$, $H(\lambda, a) \sim \lambda a - \lambda$. Since $\{\lambda \in \mathbb{R}^+ : H(\lambda, a) > 0\} \ne \emptyset$ and $H(\lambda, a)$ is differentiable

$$I(a) = H(\tanh^{-1}(a), a) = \tanh^{-1}(a)a - \log\cosh\big(\tanh^{-1}(a)\big)$$