STOCHASTIC PROCESSES. SOLUTIONS

4. Conditional Expectation

Problem 4.1

Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a bounded measurable function. Then

$$\begin{aligned} E\big(g(\xi_n)|\xi_1, ..., \xi_{n-2}\big) =& E\Big(E\big(g(\xi_n)|\xi_1, ..., \xi_{n-1}\big)\big|\xi_1, ..., \xi_{n-2}\Big) = \\ & E\Big(E\big(g(\xi_n)|\xi_{n-1}\big)\big|\xi_1, ..., \xi_{n-2}\Big) = \\ & E\Big(E\big(g(\xi_n)|\xi_{n-1}\big)\big|\xi_1, ..., \xi_{n-2}\Big) = \\ & E\Big(E\big(g(\xi_n)|\xi_{n-1}\big)\big|\xi_{n-2}\Big) = \\ & E\Big(E\big(g(\xi_n)|\xi_1, ..., \xi_{n-1}\big)\big|\xi_{n-2}\Big) = \\ & E\big(g(\xi_n)|\xi_{n-2}\big) \end{aligned}$$

and by induction for any $m < n$,

$$E\big(g(\xi_n)|\xi_1, ..., \xi_m\big) = E\big(g(\xi_n)|\xi_m\big), \quad P - a.s. \tag{4.1}$$

Let $n > m > \ell$

$$\begin{aligned} E\big(g(\xi_n)|\xi_\ell\big) = E\Big(E\big(g(\xi_n)|\xi_1, ..., \xi_m\big)\big|\xi_1, ..., \xi_\ell\Big) = \\ E\Big(E\big(g(\xi_n)|\xi_m\big)\big|\xi_1, ..., \xi_\ell\Big) = E\Big(E\big(g(\xi_n)|\xi_m\big)\big|\xi_\ell\Big) \end{aligned}$$

In terms of densities the latter reads

$$\int_{\mathbb{R}} g(u) f_{\xi_n|\xi_\ell}(u, \xi_\ell) du = \int_{\mathbb{R}} \int_{\mathbb{R}} g(u) f_{\xi_n|\xi_m}(u, r) f_{\xi_m|\xi_\ell}(r, \xi_\ell) dr,$$

and the required equality follows from arbitrariness of g.

Problem 4.2

a) Given X_2 has unifrom distribution on $[0, X_1]$, conditioned on X_1

$$f_{X_2|X_1}(s, t) = \begin{cases} \frac{1}{t} & s \in [0, t] \\ 0 & \text{otherwise.} \end{cases}$$

Date: Summer, 2004.

Similarly

$$f_{X_3|X_2}(s,t) = \begin{cases} \frac{1}{t} & s \in [0,t] \\ 0 & \text{otherwise.} \end{cases}$$

By the Chapman-Kolmogorov equation

$$\begin{aligned} f_{X_3|X_1}(s,t) = \int_{\mathbb{R}} f_{X_3|X_2}(s,u) f_{X_2|X_1}(u,t) du = \\ \int_{\mathbb{R}} \frac{1}{u} I(0 \le s \le u) \frac{1}{t} I(0 \le u \le t) du = \\ I(0 \le s \le t) \int_s^t \frac{1}{ut} du = \begin{cases} \frac{1}{t} \log(t/s), & s \in [0,t] \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

b) Following (a) we can write:

$$f_{X_{n+2}|X_n}(s,t) = \begin{cases} \frac{1}{t} \log\left(\frac{t}{s}\right) & s \in [0,t] \\ 0 & \text{otherwise} \end{cases}$$

Proceed by induction: assume (guess by iterating for $k = 3, 4$, etc.) that the formula

$$f_{X_{n+k}|X_n}(s,t) = \begin{cases} \frac{1}{t(k-1)!} \log^{k-1}\left(\frac{t}{s}\right) & s \in [0,t] \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

holds for some $k \ge 1$. By the Chapman-Kolmogorov equation:

$$\begin{aligned} f_{X_{n+k+1}|X_n}(s,t) = \int_{\mathbb{R}} f_{X_{n+k+1}|X_{n+k}}(s,u) f_{X_{n+k}|X_n}(u,t) du = \\ \int_{\mathbb{R}} \frac{1}{u} I(0 \le s \le u) \frac{1}{t(k-1)!} \log^{k-1}\left(\frac{t}{u}\right) I(0 \le u \le t) du = \\ I(0 \le s \le t) \int_s^t \frac{1}{u} \frac{1}{t(k-1)!} \log^{k-1}\left(\frac{t}{u}\right) du = \\ \frac{1}{tk!} \log^k\left(\frac{t}{s}\right) I(0 \le s \le t) \end{aligned}$$

which verifies (4.2).

c) Rewrite equation (4.2) for $n \ge 2$ as follows:

$$f_{X_n|X_1}(s,t) = \begin{cases} \frac{1}{t(n-2)!} \log^{n-2}\left(\frac{t}{s}\right) & s \in [0,t] \\ 0 & \text{otherwise} \end{cases}$$

Now

$$
\begin{aligned}
f_{X_n}(s) = &\int_{\mathbb{R}} f_{X_n|X_1}(s,u) f_{X_1}(u) du = \\
&\int_{\mathbb{R}} \frac{1}{t(n-2)!} \log^{n-2}\left(\frac{t}{s}\right) I(0 \le s \le t) I(0 \le u \le 1) du = \\
&I(0 \le s \le 1) \int_s^1 \frac{1}{u(n-2)!} \log^{n-2}\left(\frac{u}{s}\right) du = \\
&\begin{cases} \frac{1}{(n-1)!} \log^{n-1}\left(\frac{1}{s}\right), & s \in [0,1] \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

d) Intuitively X_n converges to zero. Let's look at the distribution of X_n

$$
F_n(t) = P(X_n \le t) = \int_0^t f_{X_n}(u) du = \int_0^t \frac{(-1)^{n-1}}{(n-1)!} \log^{n-1}(u) du
$$

$$
= \frac{(-1)^{n-1}}{(n-1)!} \left\{ t \log^{n-1}(t) - (n-1) \int_0^t \log^{n-2}(x) dx \right\} =
$$

$$
t \sum_{k=0}^{n-1} \frac{1}{k!} \log^k \left(\frac{1}{t}\right) \xrightarrow{n\to\infty} t \exp\{-\log(t)\} = 1, \quad \forall t > 0.
$$

Thus

$$
\lim_{n\to\infty} P(X_n > \varepsilon) = \lim_{n\to\infty} \big(1 - P(X_n \le \varepsilon)\big) = 0, \quad \forall \varepsilon > 0.
$$

i.e. X_n converges in probability. Since $X_n \le 1$, the sequence converges in $\mathbb{L}^p$ for any $p \ge 1$.

Problem 4.3

Clearly

$$
\{S_n, S_{n+1}, ...\} = \{S_n, \xi_{n+1}, \xi_{n+2}, ...\}
$$

So

$$
E(\xi_1 | S_n, S_{n+1}, ...) = E(\xi_1 | S_n, \xi_{n+1}, ...) = E(\xi_1 | S_n)
$$

Since ξ_i are i.i.d. r.v. we have (why?)

$$
E(\xi_k | S_n) = E(\xi_m | S_n) \ \ \forall \ \ k, m \le n, \quad P - a.s.
$$

so that

$$
nE(\xi_1 | S_n) = \sum_{i=1}^n E(\xi_i | S_n) = E(\sum_{i=1}^n \xi_i | S_n) = E\left(S_n | S_n\right) = S_n
$$

that is

$$
E(\xi_1 | S_n) = \frac{1}{n} S_n
$$

Problem 4.4

Let ξ be the distance from the center of the needle to the left boundary and θ be the angle, formed by the needle and the horizontal axis.

Since the needle is dropped at random it is natural to assume that ξ and θ are distributed uniformly on $[0,1]$ and $-[\pi/2, \pi/2]$.

Introduce a set:

$$B = \left\{(\theta, \xi) : |\theta| \le \frac{\pi}{2}, \xi \in \left[0, \frac{1}{2}\cos\theta\right] \cup \left[1 - \frac{1}{2}\cos\theta, 1\right]\right\}$$

Obviously the needle crosses one of the boundaries if and only if B happens. Then the desired probability is:

$$p = EI_B(\omega) = EE\left(I_B(\omega)|\theta\right) == \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} E\left(I_B(\omega)|\theta = a\right) da$$

The inner conditional probability is

$$E\left(I_B(\omega)|\theta = a\right) = P\left\{\omega : \xi \in \left[0, \frac{1}{2}\cos a\right] \cup \left[1 - \frac{1}{2}\cos a, 1\right]\right\} = \cos a$$

and so

$$p = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos a da = \frac{2}{\pi}.$$

Problem 4.5

By definition

$$E(X|\eta) = g(\eta)$$

such that

$$Ef(\eta)(X - g(\eta)) = 0$$

for any bounded function $f(x)$. Then

$$Ef(\eta)[X - g(\eta)] = \int_0^{1/2} f(s)[s - g(s)]ds + \int_{1/2}^{1} f(1/2)[s - g(1/2)]ds = 0$$

By uniqueness of cond. expectation we conclude:

$$g(s) = \begin{cases} s & 0 \le s < 1/2 \\ 3/4 & s \ge 1/2 \end{cases}$$

Note that there exist many versions of conditional expectation, e.g.

$$\widetilde{g}(s) = \begin{cases} s & 0 \le s < 1/2 \\ \frac{3}{2}s & s \ge 1/2 \end{cases}$$

Clearly $g(\eta) = \widetilde{g}(\eta)$ P-a.s.

Problem 4.6

a) If A does not depend on A:

$$P(A \cap A) = P^2(A)$$

But on the other hand $A \cap A = A$ so $P(A \cap A) = P(A)$ and thus

$$P^2(A) = P(A) \Longrightarrow \begin{cases} P(A) = 0 \\ P(A) = 1 \end{cases}$$

b) Consider the case $P(A) = 0$. Clearly

$$\left.\begin{array}{l} P(A \cap B) \leq P(A) = 0 \\ P(A \cap B) \geq 0 \end{array}\right\} \Longrightarrow \ P(A \cap B) = 0$$

But also $P(A)P(B) = 0$, so $P(A \cap B) = P(A)P(B)$ i.e. the result holds.

Now consider the other case $P(A) = 1$ Since

$$\left.\begin{array}{l} P(A \cup B) \geq P(A) = 1 \\ P(A \cup B) \leq 1 \end{array}\right\} \Longrightarrow \ P(A \cup B) = 1$$

we have

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = P(B)$$

But $P(A) = 1$, and hence $P(A \cap B) = P(B)P(A)$

c) Assume $\xi(\omega) \equiv C$. Define a set (event)

$$A(x) = \{\omega : \xi(\omega) \leq x\}$$

Obviously

$$P(A(x)) = \begin{cases} 1 & if \ \ x \geq C \\ 0 & if \ \ x < C \end{cases}$$

Then by virtue of (b) $A(x)$ is independent of any other event and in particular of itself. This implies that $\xi(\omega)$ doesn't depend on itself. Now assume that $\xi(\omega)$ does not depend on itself. i.e.

$$P\{\xi \leq x_1 \cap \xi \leq x_2\} = P(\xi \leq x_1)P(\xi \leq x_2) \qquad \forall \ x_1, x_2$$

in particular for $x_1 = x_2 = x$ the event $\{\xi \leq x\}$ is independent of itself. By (a) $P\{\xi \leq x\} = 1$ or $P\{\xi \leq x\} = 0$ this implies that $\xi(\omega) \equiv$ const P-a.s.

Problem 4.7

First note that for any i

$$\lambda\{\xi_i(\omega) = 1\} = \lambda\{\omega \in [0, 1) : \ i - th \ bit \ of \ \omega \ is \ 1\} = \frac{1}{2}$$

This holds since there is a one-to-one correspondence between any number x with i-th bit equal to 1, to exactly one other number y with the same bit equal to 0, i.e.

$$x - y = \left(\frac{1}{2}\right)^i.$$

Now let us consider a binary vector $[a_1, ..., a_n]$ with $a_i \in \{0, 1\}$, then:

$$\begin{aligned}\lambda\{\xi_1 = a_1, ..., \xi_n = a_n\} &= \lambda\left\{\omega : \sum_{i=1}^{n} \frac{a_i}{2^i} \le \omega < \sum_{i=1}^{n} \frac{a_i}{2^i} + \frac{1}{2^n}\right\} = \\ &= \frac{1}{2^n} = \prod_{i=1}^{n} P\{\xi_i = a_i\}\end{aligned}$$

which together with the fact that $\lambda\{\xi_i = a_i\} = \frac{1}{2}$ proves the independency of $\{\xi_i\}$.

Problem 4.8

Since $f(x|y)$ is an even function of x we find that

$$E(X|Y) = \int_R x f(x|y) dx = 0$$

and thus

$$EE(X|Y) = 0$$

Let us find EX. The density of X is given by

$$f(x) = \int_{R_+} f(y) f(x/y) dy = \frac{1}{2\pi} \int_0^{\infty} e^{-y/2(x^2+1)} dy = \frac{1/\pi}{x^2+1}$$

that is X has Cauchy distribution and thus EX is not well defined and $EE(X|Y) \neq EX$. In fact it is consistent with the definition of $E(X|Y)$, which requires [1] $E|X| < \infty$.

Problem 4.9

First let us check that X and Z are indeed independent: this is verified by straight forward calculations:

$$\lambda\{X = i \cap Z = j\} = \lambda\{X = i\}\lambda\{Z = j\}, \qquad \begin{matrix} i = 0, 1 \\ j = 0, 1 \end{matrix}$$

$$\left(\begin{aligned} e.g. \quad \lambda\{X = 1, Z = 0\} = \lambda\{[0, 1/4]\} &= \tfrac{1}{4} \\ \lambda\{X = 1\} = \lambda\{[0, 1/2]\} &= \tfrac{1}{2} \\ \lambda\{Z = 0\} = \lambda\{[0, 1/4] \cup [3/4, 1]\} &= \tfrac{1}{2} \end{aligned}\right)$$

Find the conditional expectation:

$$E(X|Y) \equiv g(Y)$$

so that

[1] or at least $\min(EX^-, EX^+) < \infty$, where $X^+ = \max(0, X)$ and $X^- = -\min(0, X)$

$$E\left(X-g(Y)\right)\varphi(Y)=0 \qquad \forall\, \varphi \text{ bounded} \tag{4.3}$$

The left hand side is found explicitly

$$\int_0^{1/2}\left[1-g(1)\right]\varphi(1)ds+\int_{1/2}^{3/4}\left[0-g(1)\right]\varphi(1)ds+\int_{3/4}^{1}\left[0-g(0)\right]\varphi(0)ds$$
$$=\frac{1}{2}\varphi(1)\left(1-g(1)\right)+\frac{1}{4}\varphi(1)\left(-g(1)\right)-\frac{1}{4}\varphi(0)g(0)=$$
$$=\varphi(1)\left(\frac{1}{2}-\frac{3}{4}g(1)\right)-\frac{1}{4}g(0)\varphi(0)$$

If we choose $g(x)$ so that $g(1)=\frac{2}{3}$ and $g(0)=0$, the eq. (4.3) will hold for any bounded φ. So one of the versions of the required cond. expectation is

$$E(X|Y)=\begin{cases} 2/3 & \omega\in[0,3/4] \\ 0 & \text{otherwise}\end{cases} \equiv 2/3I(Y(\omega)=1)$$

Similarly

$$E(X|Y,Z)=\begin{cases} 0 & \omega\in[3/4,1] \\ 1/2 & \omega\in[1/4,3/4) \\ 1 & \omega\in[0,1/4)\end{cases}$$

Clearly $E(X|Z,Y)\neq E(X|Y)$ in spite of X and Z are independent.

Problem 4.10

Assume $E[f(\xi_1)|\xi_2,\xi_3]=E[f(\xi_1)|\xi_3]$ with probability one. This means that for any bounded $\psi(x,y)$

$$E\big[f(\xi_1)-E[f(\xi_1)|\xi_3]\big]\psi(\xi_2,\xi_3)=0 \tag{4.4}$$

Take special $\psi(x,y)=\phi(x)\rho(y)$, then:

$$Ef(\xi_1)\phi(\xi_2)\rho(\xi_3)=EE[f(\xi_1)|\xi_3]\phi(\xi_2)\rho(\xi_3)$$

or

$$Ef(\xi_1)\phi(\xi_2)\rho(\xi_3)=EE[f(\xi_1)|\xi_3]E[\phi(\xi_2)|\xi_3]\rho(\xi_3)$$

that is:

$$E\big[f(\xi_1)\phi(\xi_2)-E[f(\xi_1)|\xi_3]E[\phi(\xi_2)|\xi_3]\big]\rho(\xi_3)=0 \tag{4.5}$$

which by definition gives:

$$E[f(\xi_1)\phi(\xi_2)|\xi_3]=E[f(\xi_1)|\xi_3]E[\phi(\xi_2)|\xi_3]$$

Now assume that ξ_1 and ξ_2 are independent, conditioned on ξ_3, i.e. assume that (4.5) holds for any bounded $f(x)$, $\phi(x)$ and $\rho(x)$. In fact, it is sufficient to verify [2] (4.4) for any $\psi(x,y)=\phi(x)\rho(y)$:

$$E\big[f(\xi_1)-E[f(\xi_1)|\xi_3]\big]\phi(\xi_2)\rho(\xi_3) = E\big[E[f(\xi_1)|\xi_3]E[\phi(\xi_2)|\xi_3]\rho(\xi_3)\big] - \\ -E\big[E[f(\xi_1)|\xi_3]E[\phi(\xi_2)|\xi_3]\rho(\xi_3)\big]=0$$

Problem 4.11

Since Y is independent of X_2, $E(Y|X_2)=EY=EX_1+\alpha EX_2=0$. But $X_1=Y-\alpha X_2$, so $E(X_1|X_2)=E(Y|X_2)-\alpha X_2=-\alpha X_2$.

Problem 4.12

Show that (i) implies (ii).

$$E(ZY|X_n)=E\big(E(ZY|X_0^n)\big|X_n\big)=E\big(YE(Z|X_0^n)\big|X_n\big)= \\ =E\big(YE(Z|X_n)\big|X_n\big)=E(Z|X_n)E(Y|X_n)$$

To show that (ii) implies (i), let η be an arbitrary random variable, generated by $\{X_0,...,X_k\}$, $k\le n$ then

$$E\eta E\big(Z|X_k\big)=EE(\eta|X_k)E(Z|X_k)\stackrel{\dagger}{=}EE(\eta Z|X_k)=E\eta Z$$

where the equality † is due to (ii). By the choice of η, the latter equation is nothing but definition of $E(Z|X_0^k)$.

[2]since any bounded two dimensional function can be approximated uniformly by series of one dimensional functions