STOCHASTIC PROCESSES. SOLUTIONS TO HOME ASSIGNMENTS

5. Gaussian Processes

Problem 5.1

First note that:

$$\begin{aligned}&-1/2(x-a-iK\lambda)^T K^{-1}(x-a-iK\lambda)=\\&=-1/2(x-a)^T K^{-1}(x-a)+i\lambda^T(x-a)+1/2\lambda^T K\lambda\end{aligned}$$

Using this fact:

$$\begin{aligned}&\Phi(\lambda)\triangleq\int_{\mathbb{R}^N} e^{i\lambda^T x}p(x)dx=\\&=\int_{\mathbb{R}^N}\exp\left\{i\lambda^T x\right\}\frac{1}{\sqrt{\det(2\pi K)}}\exp\left\{-1/2(x-a)^T K^{-1}(x-a)\right\}dx=\\&=\frac{1}{\sqrt{\det(2\pi K)}}\int_{\mathbb{R}^N}\exp\left\{-1/2(x-a-iK\lambda)^T K^{-1}(x-a-iK\lambda)+\right.\\&\left.+i\lambda^T a-1/2\lambda^T K\lambda\right\}dx=\exp\left\{i\lambda^T a-1/2\lambda^T K\lambda\right\}\times\\&\times\underbrace{\int_{\mathbb{R}^N}\frac{1}{\sqrt{\det(2\pi K)}}\exp\left\{-1/2(x-a-iK\lambda)^T K^{-1}(x-a-iK\lambda)dx\right\}}_{\triangleq 1}=\\&=\exp\left\{i\lambda^T a-1/2\lambda^T K\lambda\right\}\end{aligned}$$

Problem 5.2

Let $\varphi(\lambda)=\mathbb{E}e^{i\lambda\xi}=\mathbb{E}e^{i\lambda\eta}$. Set $X=\xi+\eta$ and $Y=\eta-\xi$, then by independence of X and Y:

$$\mathbb{E}e^{isX+itY}=\mathbb{E}e^{isX}\mathbb{E}e^{itY}=\varphi^2(s)\varphi(t)\varphi(-t)$$

On the other hand:

$$\mathbb{E}e^{isX+itY}=\mathbb{E}e^{i(s+t)\eta+i(s-t)\xi}=\varphi(s+t)\varphi(s-t)$$

so that $\varphi(\cdot)$ obeys an equation:

$$\varphi^2(s)\varphi(t)\varphi(-t)=\varphi(s+t)\varphi(s-t),\quad\forall s,t$$

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Assuming $\varphi(x)$ is smooth enough, differentiating twice w.r.t t and setting $t = 0$, the ODE for $\varphi(s)$ is obtained:

$$2\varphi^2(s)\big[\varphi''(0)\varphi(0) - (\varphi'(0))^2\big] = 2\big[\varphi''(s)\varphi(s) - (\varphi'(s))^2\big]$$

Without loss of generality assume $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi^2 = 1$, then $\varphi(0) = 1$, $\varphi'(0) = 0$, $\varphi''(0) = -1$ and the equation is obtained:

$$\begin{aligned}
&-\varphi^2(s) = \varphi''(s)\varphi(s) - \big[\varphi'(s)\big]^2 \\
&\varphi(0) = 1 \\
&\varphi'(0) = 0
\end{aligned}$$

which implies that:

$$(\log\varphi(s))'' = -1 \implies \varphi(s) = e^{-s^2/2}$$

i.e. ξ and η are Gaussian.

Problem 5.3

$$\begin{aligned}
\varphi(s) &:= \mathbb{E}e^{is\eta} = \mathbb{E}\exp\left\{is\frac{\xi_1 + \xi_2\xi_3}{\sqrt{1+\xi_3^2}}\right\} = \\
&= \mathbb{E}\mathbb{E}\left\{\exp\left(\frac{is\xi_1}{\sqrt{1+\xi_3^2}}\right)\Big|\xi_3\right\}\mathbb{E}\left\{\exp\left(\frac{is\xi_2\xi_3}{\sqrt{1+\xi_3^2}}\right)\Big|\xi_3\right\} = \\
&= \mathbb{E}\exp\left(\frac{-0.5s^2}{1+\xi_3^2}\right)\exp\left(\frac{-0.5s^2\xi_3^2}{1+\xi_3^2}\right) = e^{-s^2/2}
\end{aligned}$$

which implies that η is Gaussian with zero mean and unit variance.

Problem 5.4

By virtue of Cauchy-Schwarz inequality

$$\left[\mathbb{E}\left(\frac{p'(\xi)}{p(\xi)}\xi\right)\right]^2 \le \mathbb{E}\left(\frac{p'(\xi)}{p(\xi)}\right)^2\mathbb{E}\xi^2 = I_\xi\mathbb{E}\xi^2$$

but

$$\begin{aligned}
\mathbb{E}\left(\frac{p'(\xi)}{p(\xi)}\xi\right) &= \int_{-\infty}^{\infty}\frac{p'(x)}{p(x)}xp(x)dx = \int_{-\infty}^{\infty}p'(x)xdx = \\
&= p(x)x\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}p(x)dx = -1
\end{aligned}$$

So $\mathbb{E}\xi^2 \ge 1/I_\xi$. The equality is attained only if $p'(x)/p(x) = Cx$ for some constant C. This implies that

$$p(x) \propto e^{Cx^2/2}$$

i.e. ξ is Gaussian.

Problem 5.5

Use the property of the characteristic function[1]:

$$\frac{\partial^4}{\partial\lambda_1\partial\lambda_2\partial\lambda_3\partial\lambda_4}\varphi(\lambda)\Big|_{\lambda=0} = \mathbb{E}\xi_1\xi_2\xi_3\xi_4$$

For Gaussian vector ξ (with zero mean):

$$\varphi(\lambda) = \mathbb{E}\exp\{i\lambda^*\xi\} = \exp\{-1/2\lambda^* R\lambda\} = \exp\left\{-1/2\sum_{i,j}\lambda_i R_{ij}\lambda_j\right\}$$

where $R_{ij} = \mathbb{E}\xi_i\xi_j$, $1 \le i,j \le 4$.

$$\begin{aligned}\frac{\partial}{\partial\lambda_1}\varphi(\lambda) &= -\varphi(\lambda)\sum_i \lambda_i R_{1i} \\ \frac{\partial^2}{\partial\lambda_1\partial\lambda_2}\varphi(\lambda) &= \varphi(\lambda)\sum_i \lambda_i R_{1i}\sum_\ell \lambda_\ell R_{2\ell} - \varphi(\lambda)R_{12} := A + B\end{aligned}$$

Clearly

$$\frac{\partial^2}{\partial\lambda_3\partial\lambda_4}B = -R_{12}\varphi(\lambda)\sum_k \lambda_k R_{3k}\sum_m \lambda_m R_{4m} + \varphi(\lambda)R_{12}R_{34}$$

so that

$$\frac{\partial^2}{\partial\lambda_3\partial\lambda_4}B(\lambda)\Big|_{\lambda=0} = R_{12}R_{34}\varphi(0) \tag{5.1}$$

Further,

$$\begin{aligned}\frac{\partial}{\partial\lambda_3}A(\lambda) &= -\varphi(\lambda)\sum_i \lambda_i R_{1i}\sum_j \lambda_j R_{2j}\sum_\ell \lambda_\ell R_{3\ell} + \\ &\quad +\varphi(\lambda)\left[R_{13}\sum_j \lambda_j R_{2j} + R_{23}\sum_i \lambda_i R_{1i}\right]\end{aligned}$$

$$\frac{\partial^2}{\partial\lambda_3\lambda_4}A(\lambda)\Big|_{\lambda=0} = \varphi(0)[R_{13}R_{24} + R_{23}R_{14}] \tag{5.2}$$

Eq. (5.1) and (5.2) imply the desired formula.

[1]why is it correct?

Problem 5.6

(1) Verify that $g_n(x)$ is a probability density function, i.e. that $g_n(x) \geq 0$ for all $x \in \mathbb{R}^n$ and that $\int_{\mathbb{R}^n} g_n(x)dx = 1$. The function $g_n(x)$ is non-negative if

$$\left|(x_k - a)\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-(x_k - a)^2/(2\sigma^2)\right\}\right| \leq 1, \quad x_k \in \mathbb{R}, \quad k = 1, ..., n$$

or equivalently if

$$|\varphi(y)| = \left|\frac{y}{\sqrt{2\pi\sigma^2}} e^{-y^2/2\sigma^2}\right| \leq 1, \quad y \in \mathbb{R}$$

The latter holds if $\max_t |\varphi(t)| \leq 1$. The maximum and the points where it is attained can be found explicitly:

$$\varphi'(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \left\{e^{-y^2/(2\sigma^2)} - \frac{y^2}{\sigma^2} e^{-y^2/(2\sigma^2)}\right\}$$

$$\varphi'(y) = 0 \implies y_{1,2} = \pm\sigma$$

and

$$|\varphi(\sigma)| = \frac{1}{\sqrt{2\pi}} e^{-1/2} < 1$$

This implies that $g_n(x) > 0$ for all x.

Note that $f(x) = 1/\sqrt{2\pi\sigma^2} \exp\{-1/2(x - a)^2/\sigma^2\}$ is symmetric around a so that

$$\int_{\mathbb{R}} (x - a) f^2(x) dx = 0$$

which implies $\int_{\mathbb{R}^n} g_n(x)dx = 1$.

(2) Let $\widetilde{X}$ be a subvector of X with k components. Denote its pdf by $\widetilde{g}_k(x)$, $k < n$, i.e.

$$\widetilde{g}_k(x) = \int_{\mathbb{R}^{n-k}} g_n(x_1, ..., x_n) dx_{i_1} ... dx_{i_{n-k}}$$

where $\mathcal{J} := \{i_1, ..., i_{n-k}\}$ are the indices of the components, which do not appear at $\widetilde{X}$. It is not difficult to see that

$$\widetilde{g}_k(x) = \prod_{j \notin \mathcal{J}} f(x_j)$$

i.e. the distribution of any subvector of X is Gaussian !

(3) On the other hand, X is clearly not Gaussian. This leads to a conclusion: a vector is Gaussian only if *any* combination of its components is Gaussian.

Problem 5.7

(1) Clearly $\int_{\mathbb{R}^2} g(x,y)dxdy = c_1 + c_2 = 1$ and $c_1 > 0, c_2 > 0 \implies g(x,y) > 0$, so that $g(x,y)$ is a prob. density.

(2)
$$f(x) = \int_{\mathbb{R}} g(x,y)dy = c_1\varphi(x) + c_2\varphi(x) = \varphi(x)$$
where $\varphi(x)$ denotes standard Gaussian density. Similarly $f(y) = \varphi(y)$.

(3) $\mathbb{E}XY = c_1\rho_1 + c_2\rho_2$. Choose e.g. $c_1 = |\rho_2|$ and $c_2 = |\rho_1|$. If $\rho_1\rho_2 < 0$, then $\mathbb{E}XY = 0$. But X and Y are dependent, since $g(x,y) \neq f(x)f(y)$.

(4) A pair of r.v. can be dependent, even if each one is Gaussian (separately) and they are uncorrelated. However, if in addition they are jointly Gaussian, their independence follows.

Problem 5.8

(1) First prove the auxiliary result.

Lemma 5.1. *if α and β are independent Gaussian random variables with zero mean and variances σ_α^2 and σ_β^2, then $\gamma = \alpha\beta/\sqrt{\alpha^2+\beta^2}$ is a Gaussian r.v. with zero mean and variance $\sigma_\alpha^2\sigma_\beta^2/(\sigma_\alpha+\sigma_\beta)^2$.*

Proof. (there are other elegant proofs!) Note that $\gamma^{-2} = \alpha^{-2} + \beta^{-2}$. Let $\psi_\alpha(s) = \mathbb{E}(e^{is/\alpha^2})$:

$$\begin{aligned}\psi_\alpha(s) &= \frac{1}{\sqrt{2\pi\sigma_\alpha^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{is}{x^2} - \frac{x^2}{2\sigma_\alpha^2}\right\}dx = \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp\left\{-\frac{is}{z^2 2\sigma_\alpha^2} - z^2\right\}dz = h\left(\sqrt{\frac{s}{2\sigma_\alpha^2}}\right)\end{aligned}$$

where
$$h(t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp\left\{-\frac{it^2}{z^2} - z^2\right\}dz$$
It is easily seen that $h'(t) = -2\sqrt{i}h(t)$, so $h(t) = C\exp\{-2\sqrt{i}t\}$. Since $h(0) = 1$ we finally conclude that $h(t) = \exp\{-2\sqrt{i}t\}$.

Consequently $\psi_\alpha(s) = \exp\{-2\sqrt{is/(2\sigma_\alpha^2)}\}$ and analogously $\psi_\beta(s) = \exp\{-2\sqrt{is/(2\sigma_\beta^2)}\}$. Then since α and β are independent, we have:

$$\begin{aligned}\psi_\gamma(s) &\overset{\triangle}{=} \mathbb{E}\left(e^{is/\gamma^2}\right) = \psi_\beta(s)\psi_\alpha(s) = \exp\{-\sqrt{2is}(1/\sigma_\beta + 1/\sigma_\alpha)\} = \\ &= \exp\left\{-\sqrt{2is}\left(\frac{\sigma_\beta\sigma_\alpha}{\sigma_\beta+\sigma_\alpha}\right)^{-1}\right\}\end{aligned} \tag{5.3}$$

Note that γ has a symmetric density (Why ?), so the distribution of γ is determined by the distribution of $1/\gamma^2$. The latter and (5.3) allow to conclude that γ is Gaussian. □

Assume that X_{n-1} is Gaussian, then clearly X_n is Gaussian, since ξ_n and X_{n-1} are independent. Since the initial condition is Gaussian, we conclude that X_n is a Gaussian r.v. for each n.

(2) The process $(X_n)_{n\geq 0}$ is *not* Gaussian. By contradiction, assume that $[X_0, X_1]$ is a Gaussian vector. Then since $\mathbb{E}X_1X_0 = 0$ they are independent and hence we expect that $\mathbb{E}(X_1^2|X_0) = \mathbb{E}X_1^2$ is not a function of X_0. Let's prove that the latter does not hold:

$$\mathbb{E}(X_1^2|X_0) = \mathbb{E}\left(\frac{X_0^2\xi_1^2}{X_0^2+\xi_1^2}\middle|X_0\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{X_0^2z^2}{X_0^2+z^2}e^{-z^2/2}dz \triangleq H(X_0)$$

Obviously $H(X_0) \neq \mathit{const}$: $H(0) = 0$ and $H(1) \neq 0$.

(3) $m_n = \mathbb{E}X_n \equiv 0$ and

$$V_n = \frac{V_{n-1}\sigma_\xi^2}{(\sqrt{V_{n-1}}+\sigma_\xi)^2}, \quad V_0 = 1$$

(4) Show that $\lim_{n\to\infty} V_n = 0$ and then $X_n \to 0$ as $n \to \infty$ in mean square sense and hence also in the mean and in probability. Let $Q_n = 1/V_n$ then

$$Q_n = (\sigma_\xi^{-1} + \sqrt{Q_{n-1}})^2$$

Define an auxiliary sequence:

$$\widetilde{Q}_n = \widetilde{Q}_{n-1} + \sigma_\xi^{-2}, \quad \widetilde{Q}_0 = Q_0$$

By induction we show that $Q_n \geq \widetilde{Q}_n$ for $n \geq 0$: assume that $Q_{n-1} \geq \widetilde{Q}_{n-1}$ then

$$Q_n = \sigma_\xi^{-2} + Q_{n-1} + 2\sigma_\xi^{-1}\sqrt{Q_{n-1}} \geq \sigma_\xi^{-2} + Q_{n-1} \geq \sigma_\xi^{-2} + \widetilde{Q}_{n-1} = \widetilde{Q}_n$$

Clearly $\widetilde{Q}_n \to \infty$, which implies $Q_n \to \infty$ as $n \to \infty$, and thus $V_n \to 0$.