STOCHASTIC PROCESSES. SOLUTIONS TO HOME ASSIGNMENTS

7. Wiener Process and Stochastic Integral

Problem 7.1

Verify the axiomatic definition of Wiener process.

(1) $Z_t = \sqrt{\varepsilon}W_{t/\varepsilon}$. For any $\varepsilon > 0$, $Z_0 = 0$, the paths of Z_t are almost surely continuous (like W_t) and any vector

$$[Z_{t_1}, ..., Z_{t_k}] = \sqrt{\varepsilon}[W_{t_1/\varepsilon}, ..., W_{t_k/\varepsilon}]$$

is Gaussian. Moreover:

$$\mathbb{E}Z_t = 0, \quad \mathbb{E}Z_t Z_s = \varepsilon \mathbb{E}W_{t/\varepsilon}W_{s/\varepsilon} = \varepsilon \min(t/\varepsilon, s/\varepsilon) = \min(t, s)$$

(2) $Z'_t = W_{t+s} - W_s$ for any fixed $s > 0$. Clearly $Z'_0 = W_s - W_s = 0$. The continuity of Z'_t is directly implied by continuity of W_t. Any vector

$$[Z'_{t_1}, ..., Z'_{t_k}] = [W_{t_1+s} - W_s, ..., W_{t_k+s} - W_s]$$

is clearly Gaussian. Also $\mathbb{E}Z'_t = 0$ and

$$\begin{aligned} \mathbb{E}Z'_t Z'_u &= \mathbb{E}(W_{t+s} - W_s)(W_{u+s} - W_s) = \min(t + s, u + s) - \min(s, u + s) \\ &- \min(t + s, s) + \min(s, s) = \min(t, u) + s - s - s + s = \min(t, u) \end{aligned}$$

(3) $Z''_t = tW_{1/t}$. Let us verify that $\text{l.i.m.}_{t\to 0} Z''_t = 0$

$$\mathbb{E}(Z''_t)^2 = t^2/t = t \to 0, \quad t \to 0$$

So that if $Z''_0 = 0$ is defined, the process Z''_t has continuous trajectories almost surely. Further:

$$\mathbb{E}Z''_t Z''_s = \mathbb{E}tW_{1/t}sW_{1/s} = ts \min(1/t, 1/s) = ts/ \max(t, s) = \min(t, s)$$

Problem 7.2

Verify the *reflection* principle:

Proposition 7.1. *Let W_t be a Wiener process and $\tau_a = \inf\{t : W_t \geq a\}$. Then:*

$$\mathbb{P}\{W_t \leq x|\tau_a \leq t\} = \mathbb{P}\{W_t \geq 2a - x|\tau_a \leq t\} \tag{7.1}$$

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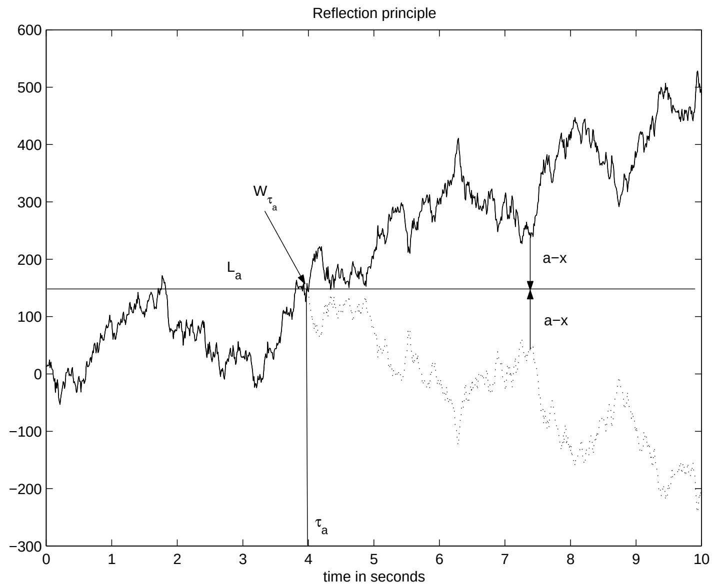

FIGURE 1. Geometrical interpretation of the reflection principle

Proof. Let $W_0^{\tau_a}$ be the events generated by $\{W_u, u \leq \tau_a\}$.

$$\begin{aligned} \mathbb{P}\{W_t \leq x | W_0^{\tau_a}\} &= \mathbb{P}\{W_t - W_{\tau_a} \leq x - a | W_0^{\tau_a}\} = \\ &\overset{\dagger}{=} \mathbb{P}\{W_t - W_{\tau_a} \geq a - x | W_0^{\tau_a}\} = \mathbb{P}\{W_t \geq 2a - x | W_0^{\tau_a}\} \end{aligned} \tag{7.2}$$

where the equality † is due to the fact that $W_t - a$ is distributed symmetrically around 0, conditioned on $W_0^{\tau_a}$ (e.g. $E(W_t - a | W_0^{\tau_a}) = 0$).

Taking conditional expectation with respect to $\{\tau_a \leq t\}$ from both sides of (6.2), the desired result is obtained. □

By virtue of the reflection principle we have

$$\mathbb{P}\{W_t > a | \tau_a < t\} = \mathbb{P}\{W_t < a | \tau_a < t\} = 1/2$$

since $\mathbb{P}\{W_t > a | \tau_a < t\} + \mathbb{P}\{W_t < a | \tau_a < t\} \equiv 1$, $\mathbb{P}$-a.s.

Then:

$$1/2 \equiv \mathbb{P}\{W_t > a | \tau_a < t\} = \frac{\mathbb{P}\{\tau_a < t | W_t > a\} \mathbb{P}\{W_t > a\}}{\mathbb{P}\{\tau_a < t\}} = \frac{\mathbb{P}\{W_t > a\}}{\mathbb{P}\{\tau_a < t\}}$$

which implies

$$\mathbb{P}\{\tau_a < t\} = 2\mathbb{P}\{W_t > a\}.$$

So

$$\mathbb{P}\{\tau_a \leq t\} \;\; = \;\; 2\mathbb{P}\{W_t > a\} = \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-x^2/(2t)} dx = \sqrt{\frac{2}{\pi}} \int_{a/\sqrt{t}}^\infty e^{-z^2/2} dz$$

and finally:

$$p_\tau(t;a) = \frac{d}{dt}\mathbb{P}\{\tau_a \le t\} = ... = \frac{a}{\sqrt{2\pi t^3}}e^{-a^2/(2t)}$$

Since for large t, $p_\tau(t;a) \propto O(t^{-3/2})$, the first hitting time τ_a has infinite mean:

$$\mathbb{E}\tau_a = \int_0^\infty t p_\tau(t;a)dt = \infty$$

Problem 7.3

Assume $X_0 = 0$ for brevity, so that $m_t \equiv 0$. Note that for $t \ge s$

$$X_t = X_s + \int_s^t a_u X_u du + \int_s^t b_u dW_u$$

Multiply both sides by X_s and take expectation

$$K(t,s) = \mathbb{E}X_t X_s = V_s + \int_s^t a_u K(u,s)du$$

which leads to

$$K(t,s) = V_s \exp\left\{\int_s^t a_u du\right\},$$

where $V_s = \mathbb{E}X_s^2$.

To find V_t, apply the Ito formula to X_t^2:

$$d(X_t)^2 = +2X_t dX_t + b_t^2 dt =$$
$$2X_t^2 a_t dt + 2X_t b_t dW_t + b_t^2 dt$$

and take expectation:

$$\dot{V}_t = 2a_t V_t + b_t^2.$$

For X to be stationary one may require that $a_t \equiv a < 0$ and $b_t \equiv b$ and that $X_0 = 0$ and $\mathbb{E}X_0^2 = -b^2/(2a)$. Indeed in this case $V_t \equiv V = -b^2/(2a)$ and

$$K(t,s) = Ve^{a|t-s|}$$

Since X is Gaussian, stationarity in the wide sense implies stationarity. The spectral density is then

$$S(\lambda) = \int_{\mathbb{R}} K(v)e^{-i\lambda v}dv \propto \frac{1}{\lambda^2 + a^2}$$

Problem 7.4

Consider the following estimate

$$\widehat{\theta}_t(Y) = \frac{\int_0^t a_s dY_s}{\int_0^t a_s^2 ds}$$

and let $\Delta_t = \widehat{\theta}_t - \theta$. Then

$$\Delta_t = \frac{\theta \int_0^t a_s^2 ds + \int_0^t a_s dW_s}{\int_0^t a_s^2 ds} - \theta = \frac{\int_0^t a_s dW_s}{\int_0^t a_s^2 ds}$$

which suggests that

$$\mathbb{E}\Delta_t^2 = \frac{\mathbb{E}\left(\int_0^t a_s dW_s\right)^2}{\left(\int_0^t a_s^2 ds\right)^2} = \frac{1}{\int_0^t a_s^2 ds} \xrightarrow{t\to\infty} 0$$

under assumptions of the problem. So $\widehat{\theta}_t$ converges to θ at the rate independent of θ. By the way, this is nothing but the Maximum likelihood estimate of θ.

Problem 7.5

(1) Apply Ito formula to $\xi_t = \cos(W_t)$ and to $\zeta_t = \sin(W_t)$:

$$d\xi_t = -\sin(W_t)dW_t - 1/2\cos(W_t)dt = -\zeta_t dW_t - 1/2\xi_t dt$$
$$d\zeta_t = \cos(W_t)dW_t - 1/2\sin(W_t)dt = \xi_t dW_t - 1/2\zeta_t dt$$

which implies

$$\dot{C}_t = -1/2C_t$$
$$\dot{S}_t = -1/2S_t$$

So

$$C_t = e^{-t/2}, \quad S_t \equiv 0$$

Let $P_n(t) = W_t^n$, then

$$dP_n(t) = nW_t^{n-1}dW_t + 1/2n(n-1)W_t^{n-2}dt$$

Taking expectation we find

$$\dot{M}_n(t) = 1/2n(n-1)M_{n-2}(t)$$

Now $M_1(t) = \mathbb{E}W_t \equiv 0$ - this implies that $M_k(t) \equiv 0$ for $k = 1, 3, 5, ...$ and $t \geq 0$. On the other hand, $M_2(t) = t$, so that $M_4(t) = 1/2 \cdot 4 \cdot 3 \int_0^t sds = 1/2 \cdot 4 \cdot 3t^2/2 = 3t^2$. Other moments are calculated similarly.

Note that in both cases application of Ito formula is easier than integration vs. Gaussian density.

Problem 7.6

(1) Heuristically, for $\delta > 0$ small enough,

$$X_{t+\delta} = X_t + rX_t\delta + \sigma X_t(W_{t+\delta} - W_t)$$

i.e. at time $t+\delta$ the change in asset price is built up by deterministic growth rate r (the positive term $rX_t\delta$) and stochastic risky part $\sigma X_t\xi$, where ξ is Gaussian random variable with variance δ. Of course, strictly speaking this is nonsense, since e.g. ξ_t can be negative enough to make X_t negative, which cannot be.

(2) Guess the answer

$$Z_t = X_0 \exp\left\{\sigma W_t + (r - 1/2\sigma^2)t\right\}$$

and verify it with Ito formula

$$dZ_t = Z_t\Big(\sigma dW_t + (r - 1/2\sigma^2)dt\Big) + 1/2 Z_t\sigma^2 dt = rZ_t dt + \sigma Z_t dW_t$$

and $Z_0 = X_0$. Clearly $Z_t > 0$ with probability one.

Note: This model stands behind the famous Black-Scholes formulae for option pricing.

Problem 7.7

(1) Note that

$$Z_4 = \sqrt{W_4^2 + V_4^2} = \sqrt{(W_4 - W_3 + W_3)^2 + (V_4 - V_3 + V_3)^2}$$

where $(W_4 - W_3, W_3, V_4 - V_3, V_3)$ is a Gaussian vector with independent entries. So

$$\mathbb{E}(Z_4|W_3, V_3) = \widetilde{\mathbb{E}}\sqrt{(\widetilde{\xi} + W_3)^2 + (\widetilde{\theta} + V_3)^2} \tag{7.3}$$

where expectation $\widetilde{\mathbb{E}}$ is with respect to the vector[1]$(\widetilde{\xi}, \widetilde{\theta})$, a pair of auxiliary Gaussian random variables, independent and with zero means and unit variances. Now use Jensen inequality to obtain the upper bound

$$\mathbb{E}(Z_4|W_3, V_3) \le \sqrt{\widetilde{\mathbb{E}}(\widetilde{\xi} + W_3)^2 + \widetilde{\mathbb{E}}(\widetilde{\theta} + V_3)^2} =$$
$$\sqrt{\widetilde{\mathbb{E}}\widetilde{\xi}^2 + +\widetilde{\mathbb{E}}\widetilde{\theta}^2 + W_3^2 + V_3^2} = \sqrt{2 + W_3^2 + V_3^2}$$

The lower bound can be obtained by means of Ito formula. Let $R(x, y) = \sqrt{x^2 + y^2}$ Clearly

$$\frac{\partial}{\partial x}R(x, y) = R_x(x, y) = \frac{x}{R}$$

and

$$R_y = \frac{y}{R}, \quad R_{xx} = \frac{1}{R} - \frac{x^2}{R^3}, \qquad R_{yy} = \frac{1}{R} - \frac{y^2}{R^3}$$

[1]here V_3 and W_3 are hold fixed and the equality in (6.3) is of course P-a.s. Make sure you understand this point

and Ito formula gives

$$dZ_t = \frac{W_t}{Z_t}dW_t + \frac{V_t}{Z_t}dV_t + \frac{1}{2}\left(\frac{1}{Z_t} - \frac{W_t^2}{Z_t^3}\right)dt + \frac{1}{2}\left(\frac{1}{Z_t} - \frac{V_t^2}{Z_t^3}\right)dt =$$
$$= \frac{1}{Z_t}dt - \frac{1}{2}\frac{V_t^2 + W_t^2}{Z_t^3}dt + \frac{W_t}{Z_t}dW_t + \frac{V_t}{Z_t}dV_t =$$
$$= \frac{1}{2}\frac{1}{Z_t}dt + \frac{W_t}{Z_t}dW_t + \frac{V_t}{Z_t}dV_t$$

and hence

$$Z_4 = Z_3 + \int_3^4 \frac{1}{2Z_s}ds + \int_3^4 \left(\frac{W_s}{Z_s}dW_s + \frac{V_s}{Z_s}dV_s\right)$$

Taking conditional expectation from both sides gives the lower bound

$$\mathbb{E}(Z_4|V_3, W_3) = Z_3 + \mathbb{E}\left(\int_3^4 \frac{1}{2Z_s}ds\big|V_3, W_3\right) \geq Z_3$$

Problem 7.8

a. Let $P(x, t; y, s)$ denote the transition distribution of $(X_t)_{t\geq 0}$, i.e.

$$P(x, t; y, s) = \mathbb{P}(X_t \leq x|X_s)\big|_{X_s := y}$$

Any Markov process obeys Chapman-Kolmogorov equation:

$$P(x, t; y, \tau) = \int_{z\in\mathbb{R}} P(x, t; z, s)dP(z, s; y, \tau) \tag{7.4}$$

Since $(X_t)_{t\geq 0}$ is a Gaussian process (assuming $R(t, t) > 0$):

$$\mathbb{E}(X_t|X_\tau)\big|_{X_\tau := y} = \frac{R(t, \tau)}{R(\tau, \tau)}y \tag{7.5}$$

On the other hand

$$\mathbb{E}(X_t|X_\tau)\big|_{X_\tau := y} = \int_{x\in\mathbb{R}} xdP(x, t; y, \tau) = \tag{7.6}$$
$$= \int_{x\in\mathbb{R}} x \int_{z\in\mathbb{R}} dP(x, t; z, s)dP(z, s; y, \tau) = \int_{z\in\mathbb{R}} \frac{R(t, s)}{R(s, s)}zdP(z, s; y, \tau) =$$
$$= \frac{R(t, s)}{R(s, s)}\frac{R(s, \tau)}{R(\tau, \tau)}y$$

Comparing (6.5) and (6.6), we conclude that for any $t \geq s \geq \tau$

$$R(t, \tau) = \frac{R(t, s)R(s, \tau)}{R(s, s)} \tag{7.7}$$

b. Let $R(t, \tau)$ be a solution of eq. (6.7). Since $R(t, \tau)$ satisfies (6.7) for *any* $s \in [\tau, t]$, fix some $s' \in [\tau, t]$ and define e.g. $f(t) := R(t, s')$ and $g(\tau) := R(s', \tau)/R(s', s')$. Now set $R^\circ(u, v) := f\big(\max(u, v)\big)g\big(\min(u, v)\big)$.

It is straightforward to check that for any $t \geq \tau$, $R^\circ(u,v)$ satisfies (6.7) and also $R(t,\tau) \equiv R^\circ(t,\tau)$.

c. The objective is to construct a Gaussian Markov process $(Z_t)_{t\geq 0}$, with covariance function $R(t,\tau) = f\big(\max(t,\tau)\big)g\big(\min(t,\tau)\big)$, where $f(t)$ and $g(t)$ are some specified functions. Define $\nu(t) = g(t)/f(t)$. We claim that $\nu(t)$ is a positive ($R(t,t) = f(t)g(t) > 0$, so $g(t)/f(t) > 0$ as well) nondecreasing function. Indeed by virtue of Cauchy-Schwarz inequality

$$R(t,\tau) \leq \sqrt{R(t,t)R(\tau,\tau)}$$

i.e. e.g. $t \geq \tau$

$$f(t)g(\tau) \leq \sqrt{f(t)g(t)f(\tau)g(\tau)} \implies 1 \leq \sqrt{\nu(t)}\sqrt{1/\nu(\tau)} \implies \nu(\tau) \leq \nu(t)$$

Let W_t be the Wiener process. Define $Z_t = f(t)W_{\nu(t)}$. Since $f(t)$ and $g(t)$ are some deterministic functions, Z_t is Gaussian and for $t \geq \tau$

$$R_z(t,\tau) = \mathbb{E}Z_tZ_\tau = f(t)f(\tau)\min(\nu(t),\nu(\tau)) = f(t)f(\tau)\nu(\tau) = f(t)g(\tau)$$

by the same arguments, flipping t and τ, one arrives at the desired form of the correlation function:

$$R_z(t,\tau) = f\big(\max(t,\tau)\big)g\big(\min(t,\tau)\big)$$

Since $\nu(t)$ is non decreasing, Z_t is Markov, for any bounded function $\varphi(x) : \mathbb{R} \to \mathbb{R}$ and for any $\tau \leq t$

$$\begin{aligned}\mathbb{E}\big(\varphi(Z_t)|Z_s, s \leq \tau\big) &= \mathbb{E}\big(\varphi(f(t)W_{\nu(t)})|f(s)W_{\nu(s)}, s \leq \tau\big) = \\ &= \mathbb{E}\big(\varphi(f(t)W_{\nu(t)})|f(s)W_{\nu(\tau)}\big) = \mathbb{E}\big(\varphi(Z_t)|Z_\tau\big)\end{aligned}$$

d. Note that

$$e^{-|t-s|} = e^{-\max(t,s)}e^{\min(t,s)}$$

Following the results of the previous questions,

$$X_t = e^{-t}W_{e^{2t}}$$

where $(W_t)_{t\geq 0}$ is the Wiener process.

e. Any Gaussian Markov process satisfies (6.7). Since X_t is stationary, $R(t,s) = R(t-s)$. Set $\rho(t-s) = R(t-s)/R(0)$, then

$$R(t-\tau) = \frac{R(t-s)R(s-\tau)}{R(0)} \implies \rho(t-\tau) = \rho(t-s)\rho(s-\tau)$$

or by appropriate change of variables

$$\rho(u+v) = \rho(u)\rho(v)$$

The solution of this equation in the class of continuous functions is well know to be

$$\rho(t) = e^{-\lambda|t|}$$

where $\lambda > 0$ is some constant, which is proved as follows. Fix integers m and n, then

$$\rho(m/n) = \rho(1/n + ... + 1/n) = \rho^m(1/n) \tag{7.8}$$

In particular

$$\rho(1) = \rho^n(1/n) \tag{7.9}$$

Combine (6.8) and (6.9) to obtain:

$$\rho(m/n) = \rho(1)^{m/n} \tag{7.10}$$

Since m and n have been chosen arbitrary and since $\rho(t)$ is continuous (6.10) holds for any $t > 0$, i.e.

$$\rho(t) = \rho(1)^t \implies \rho(t) = e^{\lambda t}$$

where $\lambda = \log(\rho(1))$. Note that $\rho(1) = R(1)/R(0) < 1$, so that $\lambda < 0$. By symmetry we obtain the desired result.

Problem 7.9

a) Apply the Ito formula to $r_t^2 = X_t^2 + Y_t^2$

$$\begin{aligned} dr_t^2 =& 2X_t dX_t + 2Y_t dY_t + X_t^2 dt + Y_t^2 dt = \\ & - X_t^2 dt - 2X_t Y_t dB_t - Y_t^2 dt + 2X_t Y_t dt + X_t^2 dt + Y_t^2 dt \equiv 0, \end{aligned}$$

that is $r_t^2 = r_0^2 = x^2 + y^2$.

b) Analogously applying the Ito formula to $\theta_t = \arctan(X_t/Y_t)$ one gets

$$d\theta_t = dB_t$$

subject to $\theta_0 = \arctan(x/y)$. That is the process (X_t, Y_t) may be regarded as a Brownian motion on a circle, i.e. e^{iB_t}.

Problem 7.10

Immediate implication of the Ito formula.