Comparing Local Risks by Acceptance and Rejection

Amnon Schreiber*

March 20, 2012

Abstract

It is said that risky asset $h$ acceptance dominates risky asset $k$ if any decision maker who rejects the investment in $h$ rejects also the investment in $k$. As Hart (2011) shows, acceptance dominance is an incomplete order on an ordinary set of gambles. We extend the definition of acceptance dominance order to risky assets whose values follow random processes. We call the risk that arises from investing in such assets, with a short investment time horizon, local risk. We show that for small investment time horizons, the acceptance dominance order is a complete order that can be represented by an index of local risk. Moreover, we show that the measures of riskiness proposed by Aumann & Serrano (2008), Foster & Hart (2009), and Schreiber (2011) all coincide with our index. We use the differential calculus as an analytical tool to present our results.

*The Department of Economics and the Center for the Study of Rationality, Hebrew University of Jerusalem, 91904 Jerusalem, Israel. Email: amnonschr@gmail.com
1 Introduction

The renewed interest of the literature in measures of riskiness, which started with the seminal work of Aumann & Serrano (2008), focuses on the problem of accepting or rejecting risky assets. In this literature, risky assets are characterized by random variables whose values are interpreted either as absolute returns (“gambles”) or relative returns (“securities”). Based on the simple decision problem of acceptance and rejection gambles, Hart (2011) defines an order on the set of gambles, called an acceptance dominance order, and Schreiber (2011) extends its definition to securities. Given two risky assets \( h \) and \( k \), either two gambles or two securities, it is said that \( s \) acceptance dominates \( k \) if any risk-averse decision maker who rejects the investment in \( h \) rejects also the investment in \( k \). On a regular set of risky assets—those which are characterized by a random variable—acceptance dominance is an incomplete order. Indeed, there are many cases of pairs of risky assets where none of them acceptance dominates the other.

In this paper we extend the acceptance dominance order to financial assets whose price follows continuous-time random processes. We say that a decision maker accepts an asset if she is better off investing in such an asset; otherwise she rejects it. In principle, accepting or rejecting such assets should depend on the investment time horizon. We limit our discussion only to short investment time horizons. We show that on this set of assets, the acceptance dominance order is a complete order that can be represented by a measure of riskiness. Moreover, we show that several measures of riskiness of regular assets that are compatible with the acceptance dominance order on regular assets induce the acceptance dominance order in the continuous-time setup. For instance, the Aumann & Serrano (2008) Economic Index of Riskiness, the Foster & Hart (2009) Operational Measure of Riskiness, and the Schreiber (2011) Economic Index of Relative Riskiness all coincide with our measure of riskiness in the continuous-time setup.

The paper is organized as follows. In Section 2 we define the acceptance dominance order on a set of risky assets whose value follows a continuous-time random process. We also propose a measure of riskiness of such assets that induces the acceptance dominance order. In Section 3 we present several measures of riskiness that somehow relate to the acceptance and rejection problem. The risky assets in Section 3 are regular assets, i.e., assets whose absolute or relative returns are characterized by a random variable. We show that several measures of riskiness, which originally were defined on regular
assets, coincide with our measure of riskiness on the set of assets in the continuous-time environment. Section 4 concludes. The main proofs are relegated to the Appendix.

2 Acceptance Dominance in a Continuous-Time Environment

In this paper a utility function is a von Neumann–Morgenstern utility function for money; i.e., is strictly monotonic, strictly concave, and twice continuously differentiable.

It is quite common in the financial literature to model securities by continuous-time random processes. A (continuous-time) security $s$ is an asset whose value (price) follows a continuous-time random process. The value of $s$ at time zero is given (denoted by $s_0$) and can be any real value. For any other time $t$, $0 < t < T$, the value of $s$ at $t$ ($s_t$) is the unique solution of a stochastic differential equation (SDE) of the form

$$ds_t = \mu_t dt + \sigma_t dW_t,$$

where $W$ is a standard (one-dimensional) Wiener process. A more rigorous description of the continuous-time framework is relegated to the Appendix. We denote the collection of all those securities by $S$.

Investors can decide on the number of units of a security that they buy. Buying $x$ units of security $s$ with investment time horizon $t$, where the initial wealth is $w$, causes the wealth to be distributed as $w - xs_0 + xs_t$. For instance, if one invests all one’s initial wealth in the security, one’s wealth will be distributed as $ws_t/s_0$ (assuming that $s_0 \neq 0$). Alternatively, buying only one unit of $s$ causes the wealth to be distributed as $w - s_0 + s_t$.

The question whether an agent benefits from the investment in a certain security depends, among other parameters, on the investment time horizon which is the length of time that an investment is held before it is liquidated. We focus on only short investment time horizons. We say that an agent with utility $u_i$ and initial wealth $w$ accepts $x$ units of security $s$, if there exists

---

1The drift $\mu_t$ and the diffusion $\sigma_t$ are both functions of $s_t$ and $t$, i.e., $\mu_t = \mu(s_t, t)$ and $\sigma_t = \sigma(s_t, t)$. In addition, we assume that the drift and the diffusion are both continuous functions and that $\sigma_t \neq 0$.

2The security value can be interpreted as if the discount factor were taken into account.
$T^* > 0$ such that

$$E[u_i(w - x s_0 + x s_t)] > u_i(w),$$

for all $t, 0 < t < T^*$.

Hart (2011) defines an (incomplete) order of risky assets (gambles) which he calls “acceptance dominance”. Originally, acceptance dominance is defined in relation to gambles that are characterized by random variables. Here, we extend its definition to securities in the continuous-time framework. Let $s$ and $k$ in $S$ be two securities.

**Definition 1.** $s$ acceptance dominates $k$, denoted as $s \succeq_A k$, if any decision maker who accepts $x$ units of $k$ at $w$ accepts also $x$ units of $s$ at $w$, for any value of $x > 0$ and $w$.

The result presented here is that the acceptance dominance order in the continuous-time setup is a complete order and can be represented by a measure of riskiness. In general, a measure of riskiness is a real-valued function defined on risky assets. The riskiness of a regular asset—an asset that is characterized by a random variable—is simply a real number. By contrast, if assets are characterized by random processes (as in our continuous-time framework), the riskiness is defined locally, as it may change over time. If $s$ is a security, we define the local riskiness of $s$ at time zero to be as follows:

$$R_l(s) = \frac{\sigma^2(s)}{2\mu(s)},$$

where $R_l(s)$ is the local riskiness of $s$ at time zero.\(^3\) In addition, $\sigma(s) \equiv \sigma_0(s)$ and $\mu(s) \equiv \mu_0(s)$.

Obviously, $R_l$ induces a complete order on the set of securities $S$. Our claim is that the acceptance dominance order and the order that is induced by $R_l$ are equivalent. Formally, let $s$ and $k$ be two securities in $S$. Then,

**Theorem 2.** $s >_A k \iff R_l(k) > R_l(s)$.

Theorem 2.1 implies, roughly speaking, that any decision maker who benefits from investing in security $k$ with a short investment time horizon will benefit also from investing in $s$ with a short investment time horizon.

\(^3\)For simplicity we present the local riskiness of the security at $t = 0$. More generally, the local riskiness of security $s$ at $t$ is defined by $R_l(s, t) = \sigma_t(s)/(2\mu_t(s))$, where $R_l(s, t)$ is the local riskiness of security $s$ at time $t$. 
3 Local Riskiness

3.1 Riskiness of Gambles and Securities

In the previous section we defined an order of riskiness on a set of continuous-time securities. Several recent papers deal with other types of risky assets, namely, gambles and securities. A gamble $g$ is a real-valued random variable with positive expectation and some negative values. A security $r$ is a real-valued random variable with a geometric mean greater than one and some values lower than one.\footnote{The definitions of a gamble and a security are taken from Aumann \& Serrano (2008) and Schreiber (2011), respectively.}

We say that a decision maker accepts gamble $g$ if she is better off investing in $g$, i.e., if $E[u(w+g)] > E(u(w))$; otherwise she rejects it. Similarly, we say that a decision maker accepts the investment in $r$ (or simply accepts $r$), if she is better off investing all her initial wealth\footnote{In principle, the investment amount can be any real number. Following Schreiber (2011), we restricted the investment amount to be exactly the initial wealth.} $w$ in $r$, i.e., if $E[u(wr)] > u(w)$; otherwise she rejects it. Based on the problem of accepting or rejecting gambles, Hart (2011) defines an order on the set of gambles, called acceptance dominance order, and Schreiber (2011) extends its definition to the set of securities: if $h$ and $k$ are two risky assets (two gambles or two securities), we say that $h$ acceptance dominates $k$ ($h \geq_A k$), if any decision maker who rejects $h$ at $w$ rejects also $k$ at $w$.

A measure of riskiness is compatible with the acceptance dominance order if $h \geq_A k$ implies that $k$ is riskier than $h$. Since the acceptance dominance order is an incomplete order on the set of gambles and on the set of securities, there exist different measures of riskiness that are compatible with the acceptance dominance order. Here we present three such measures, two of them defined on gambles and one defined on securities.

Given a gamble $g$, the Aumann–Serrano index of riskiness, $R^{AS}$, is defined implicitly by

$$E\left[\exp\left(-\frac{g}{R^{AS}(g)}\right)\right] = 1,$$

and the Foster–Hart measure of riskiness, $R^{FH}$, is defined implicitly by

$$E\left[\log\left(1 + \frac{g}{R^{FH}(g)}\right)\right] = 0.$$
Although based on quite different considerations, the two measures turn out to be similar in many ways, and to share several useful properties such as monotonicity with respect to stochastic dominance; see Hart (2011). The practical use of these two measures is analyzed by Kadan & Liu (2011) who show how these two measures can be applied to address the problem of tail events and rare disasters.

The Schreiber (2011) index of riskiness is defined on securities rather than on gambles. If \( r \) is a security, the Schreiber measure of riskiness, denoted by \( R^S \), is defined implicitly by

\[
E \left[ r^{-1/R^S(r)} \right] = 1.
\] (5)

Schreiber (2011) defines the adjusted riskiness as

\[
\tilde{R}^S(r) = R^S(r)/(1 + R^S(r)).
\]

It is easy to see that \( \tilde{R}^S \) can be defined implicitly by the equation

\[
E \left[ r^{1-1/\tilde{R}^S(r)} \right] = 1.
\] (6)

Note that \( R^S \) and \( \tilde{R}^S \) are ordinally equivalent; i.e., given any two securities, the two measures will always agree on which one of the securities is riskier.

### 3.2 Riskiness and Local Riskiness

Let \( s \) be a continuous time security. Recall that \( s_t \)—the value of \( s \) at time \( t \)—is a random variable. Given a measure of riskiness of regular assets, \( R^* \), whose domain is a set of random variables, \( R^* \) can be used to measure the riskiness of \( s_t \) for all \( t, 0 < t < T \). We define the local riskiness of \( s \) at time zero, based on \( R^* \), as the limit of the riskiness of \( s \) as \( t \) goes to zero. The local riskiness of \( s \) is not well defined for all measures of riskiness and all securities, as some measures of riskiness are not well defined on all random variables.\(^6\) We denote the local riskiness by \( R^*_l(s) \), which can be formally defined as follows:

\[
R^*_l(s) \equiv \lim_{t \to 0} R^*(s_t),
\] (7)

\(^6\)For instance, Aumann & Serrano (2008) and Foster & Hart (2009) relate only to gambles that have a positive expectation and take negative values with a positive probability. If we dispense with these constraints, equations (3) and (4) may have no solution. See
Note that even if $R^*$ is well defined on $s_t$ for all $t$, $0 < t < T^*$, it does not guarantee that the expression in (7) is well defined.

The following theorem shows the connection between the local riskiness based on the above-mentioned measures of riskiness and the local riskiness as we defined it in Section 2.

**Theorem 3.1.**

1. Let $s \in S$ be a security in the continuous-time environment such that $s_0 = 0$ and $R^{AS}(s_t)$ is well defined for any $t$, $0 < t < T^*$. Then,

$$R^{AS}_l(s) = R_l(s).$$

(8)

2. Let $s \in S$ be a security in the continuous-time environment such that $s_0 = 0$ and $R^{FH}(s_t)$ is well defined for any $t$, $0 < t < T^*$. Then,

$$R^{FH}_l(s) = R_l(s).$$

(9)

3. Let $s \in S$ be a security in the continuous-time environment such that $s_0 = 1$ and $R^S(s_t)$ is well defined for any $t$, $0 < t < T^*$. Then,

$$R^S_l(s) = R_l(s).$$

(10)

The decision problem, which underlies the measures of riskiness proposed by Aumann–Serrano and Foster–Hart, concerns whether to accept or reject a gamble whose price is zero. Theorem 3.1 asserts that for securities whose price (at $t = 0$) is zero, $R^{AS}_l$ and $R^{FH}_l$ coincide with $R_l$. It is possible to normalize any security in $S$ by subtracting their value at $t = 0$, and to get a more general result that relates to any security in $S$. Similarly, Schulze (2010) who studies on which distributions the Aumann–Serrano index is well defined. Similarly, Schreiber (2011) defines his measure of riskiness on the set of securities whose geometric mean is greater than one and that takes values lower than one with positive probability. On other random variables, equations (5) and (1) may not have a real solution. The idea that a measure of riskiness is not well defined on the whole set of risky assets should come as no surprise, as almost any other familiar measure of riskiness is not well defined on the entire set of random variables. For instance, it is quite common in the financial literature (and in the industry) to measure the riskiness of a security by its variance. But there are random variables that have no variance, such as random variables that have the Cauchy distribution.
the decision problem, which underlies the measure of riskiness of Schreiber, concerns whether to accept or reject a security whose price is 1. Hence, the third part of Theorem 3.1 relates only to securities whose price (at \( t = 0 \)) is one. For a more general result that relates to any security in \( S \), we could normalize the prices of securities by dividing their values at each point in time by their value at \( t = 0 \) (assuming that it is not zero).

4 Examples

In this section we explore several types of random processes and show what is the local risk of securities that follow these processes.

4.1 The Arithmetic Brownian Motion

The SDE that describes the Arithmetic Brownian Motion is

\[
ds = \mu dt + \sigma dW, \tag{11}
\]

where \( \mu \) and \( \sigma \) are constant over time.

The local riskiness of \( s \), which is fixed in time, equals \( \sigma^2/2\mu \) (for \( \mu \neq 0 \)). If \( t > 0 \), then \( s_t \) is a future value of the security \( s \) relative to time 0. The distribution of \( s_t \) given \( s_0 \) is normal with mean \( s_0 + \mu t \) and standard deviation \( \sigma \sqrt{t} \). As we showed before, if \( s_0 = 0 \), the Aumann–Serrano index coincides locally with our measure of local riskiness. It is interesting to note that, as Aumann & Serrano (2008) show, if \( g \) is a gamble distributed normally, then \( R^{AS} = \text{Var} \ g/2Eg \).

4.2 Geometric Brownian Motion

The SDE that describes the Geometric Brownian Motion is

\[
ds = \mu s dt + \sigma s dW, \tag{12}
\]

where \( \mu \) and \( \sigma \) are constant over time.

At time zero the local riskiness of \( s \) equals \( s_0 \sigma^2/2\mu \) (for \( \mu \neq 0 \)). The conditional distribution of \( s_t \) given \( s_0 \) is log normal with parameters \( \ln s_0 + \mu t - 0.5\sigma^2 t \) and \( \sigma \sqrt{t} \) (\( \ln s_t \) is normally distributed).
4.3 Mean Reverting Process

The SDE that describes the Mean Reverting Process is

\[ ds = \kappa (\mu - s) dt + \sigma \sqrt{s} dW, \]

where \( \mu, \sigma \), and \( \kappa \) are constant over time. We assume that \( \kappa \) and \( \mu \) are positive.

At time zero, the local riskiness of \( s \) equals \( \sqrt{s_0} \sigma^2/2\kappa(\mu - s_0) \). The conditional distribution of \( s_t \) given \( s_0 \) is non-central \( \chi^2 \). The mean of \( s \) is \( (s_t - \mu) \exp(-\kappa t) + \mu \), and the variance of the distribution is \( s_t(\sigma^2/\kappa)(\exp[-\kappa t] - \exp[-2\kappa t]) + \mu(\sigma^2/(2\kappa))(1 - \exp[-\kappa t])^2 \).

In the financial literature, the mean-reverting process is used to model interest rates or inflation rates that may have stable long-run values.

5 Conclusion

We extended the acceptance dominance order, which originally was defined by Hart (2011) on gambles, to include risky assets whose values follow continuous-time random processes. We presented a measure of local riskiness of such assets that induces the acceptance dominance order. This shows that on this set of assets, acceptance dominance is a complete order. In addition, we showed that several measures of riskiness defined on random variables coincide locally with our measure of local riskiness.

The focus on risky assets whose random returns evolve continuously over time enables us to analyze cases where the risks are in some sense small, by focusing only on short investment time horizons. Studying situations of decision making under risk, where the risks are small, has been done already by different methods. For example, Pratt (1964) showed that if the distribution of the returns is sufficiently concentrated, which means that the third absolute central moment is sufficiently small compared with the variance, then for any decision maker, the magnitude of the so-called risk premium is correlated with the level of the decision maker’s risk aversion. Another similar interpretation of the risk-aversion measures has been developed independently by Arrow (1965). In addition, Samuelson (1970) showed that the classic mean-variance analysis, initiated by Markowitz (1959), applies approximately to all utility functions, in situations that involve what he calls “compact” distribution. An additional interesting analysis of small risks was
proposed by Shorrer (2011), who showed that defining the risk aversion of decision makers by the problem of accepting and rejecting small gambles is just equivalent to the well-known Arrow–Pratt measure of absolute risk aversion. According to him, gambles have small risk if the support of the gambles is small.

A The Securities Model

The uncertainty in this model is generated by $K$ standard Wiener processes $W^1, \ldots, W^K$ defined on a filtered probability space $(\Omega, F_T, F, P)$ that satisfies the so-called usual conditions. The filtration $F = (F_t)_{t \in [0,T]}$ is the augmentation of the natural filtration $F^W$, generated by the vector $W = \{W(t) = W^1(t)\ldots W^K(t), t \in [t_0, T]\}$ of standard Wiener processes; see Karatzas & Shreve (1998).

Let $S$ be the set of securities whose prices follow continuous-time random processes. Let $s \in S$ be a security. The value of $s$ at time zero is given, denoted as $s_0$. For any other value of time $t$, $T > t > 0$, $s_t$ is the unique (strong) solution of a stochastic differential equation is described by

$$ds_t = \mu_t dt + \sigma_t dW_t,$$  \hspace{1cm} (14)

where $\mu_t \equiv \mu(s_t, t)$ and $\sigma_t \equiv \sigma(s_t, t)$ are two continuous functions and $\forall t, \mu_t > 0$. We assume also that $\sigma_t \neq 0$ a.s..

B Proofs

Throughout the proofs we shall use Ito’s lemma several times. It is worthwhile to recall a simple version of this lemma. If $s$ is a random process that can be described by the SDE

$$ds = \mu dt + \sigma dW, \hspace{1cm} (15)$$

and $f(s, t)$ is a twice differentiable function of two variables, then

$$df_{s,t} = [\mu_t f_s + 0.5\sigma^2 f_{ss} + f_t]dt + \sigma f_s dW,$$ \hspace{1cm} (16)

where $f_s$ and $f_{ss}$ are the first and second derivatives of $f$ in relation to $s$, and $f_t$ is the first derivative of $f$ in relation to $t$. 

10
Proof of Theorem 2.1.

Given a security $s$, an agent benefits from buying $x$ units of $s$ with investment time horizon $t$ if $EU(w - xs_0 + xs_t) - U(w) > 0$. According to Ito’s lemma,

$$EU(w - xs_0 + xs_t) - U(w) = E\left[\int_0^t xU'(w - xs_0 + xs_t)\mu_k + \frac{1}{2}x^2U''(w - xs_0 + xs_t)\sigma_k^2dk\right].$$  \hspace{1cm} (17)

Since the expression $EU(w - xs_0 + xs_t) - U(w) > 0$ is continuous over time, the agent accepts $s$ if the limit of $EU(w - xs_0 + xs_t) - U(w)$ is positive, as $t$ goes to zero. Following (17), this condition can be written as

$$\lim_{t \to 0} E\left[\int_0^t xU'(w - xs_0 + xs_t)\mu_k + \frac{1}{2}x^2U''(w - xs_0 + xs_t)\sigma_k^2ds/t \right] > 0.$$  \hspace{1cm} (18)

It follows from (18) that the question whether an agent accepts or rejects $x$ units of a security depends on three parameters only: $x$, the riskiness of the security $R_t$, and the Absolute Risk Aversion of Pratt and Arrow. Hence, if a decision maker accepts $x$ units of a security $s$, she also accepts $x$ units of a security $k$ if $R_t(k) < R_t(s)$ (at the same initial wealth level).

The following lemma will be useful in the proof of Theorem 3.1.

Lemma B.1. Let $F_t(R)$ be a set of continuous real-valued functions, defined on the set of real numbers, where $t > 0$. Let $F_0(R)$ be a function and let $R_0 > 0$ be a number, such that:

1. $\lim_{t \to 0} F_t(R) = F_0(R)$ for all $R$.

2. $F(R_0) = 0$

3. There exists an environment of $R_0$, in which $F_0$ is monotonic (increasing or decreasing).

Then,

1. $\exists \epsilon \ s.t. \forall t < \epsilon \exists R_t \ s.t. \ F_t(R_t) = 0$.

2. $\lim_{t \to 0} R_t = R_0$.
Proof. Since F equals zero only on $R_0$, without loss of generality we can assume that $\forall \delta > 0 \ F(R_0 + \delta) > 0$ and $F(R_0 - \delta) < 0$. Let $\delta > 0$ be a positive number and let $\epsilon_1 = \min\{|F(R_0) - F(R_0 + \delta)|, |F(R_0) - F(R_0 - \delta)|\}$. There exists $\epsilon$ s.t. $\forall \epsilon < \epsilon_1$:

$$|F(R_0 + \delta) - F_t(R_0 + \delta)| < \epsilon_1$$
$$|F(R_0 - \delta) - F_t(R_0 - \delta)| < \epsilon_1.$$

Hence, $\forall \epsilon < \epsilon_1 F_t(R_0 + \delta) > 0$ and $F_t(R_0 - \delta) < 0$. Since $F_t$ are continuous, there exists $R_t \in (R_0 - \delta, R_0 + \delta)$ s.t. $F_t(R_t) = 0$.

Let $f$ be a real-valued function defined on the real numbers. In addition assume that its first derivative is positive and that its second derivative is negative. Defining the measure of riskiness $R^f$ on gambles implicitly by the equation

$$Ef(g/R^f(g)) = f(0), \quad (19)$$

the following lemma will be useful for the proof of Theorem 3.1.

**Lemma B.2.** Let $s$ be a security in the continuous-time environment. If there exists $0 < T^*$ such that $R^f$ is well defined on $s_t$ for all $t$, $0 < t < T^*$, then

$$R^f_t(s) = \frac{\sigma_0(s)}{2\mu_0(s)},$$

where $R^f_t$ is the local riskiness defined by $R^f$; see (7).

**Proof.** Given a security $s$, we look at the stochastic process $f((s_k)/X)$, where $0 < k < T$. Using Ito’s lemma, $f$ can be described by the SDE

$$df_k = \left(\frac{1}{X} \mu_k f'_k + \frac{1}{2} \frac{1}{X^2} \sigma_k^2 f''_k\right) dk + \frac{1}{X} \sigma_k f'_k dW, \quad (20)$$

where $f'_k$ denotes the first derivative of $f$ at the point $s_k/X$ and $f''_k$ denotes the second derivative of $f$ at the point $s_k/X$. Taking the expectation in (20), we get:

$$E_0 \left[f(s_t/X)\right] = f(0) + E_0 \left[\int_0^t \left(\frac{1}{X} \mu_k f'_k + \frac{1}{2} \frac{1}{X^2} \sigma_k^2 f''_k\right) dk\right]. \quad (21)$$
Substituting $X = R(s_t)$, by definition $E\left[ f(s_t/R(s_t)) \right] = f(0)$ for all $t$, and we get

$$E_0 \left[ \int_0^t \left( \frac{1}{R(s_t)} \mu_k f_k' + \frac{1}{2} \frac{1}{R(s_t)^2} \sigma_k^2 f_k'' \right) dk \right] = 0. \quad (22)$$

It is left to show that the limit of $R(s_t)$ is $\frac{2\mu_0}{\sigma_0}$ as $t$ goes to zero.

Let $F_t$ be a set of functions defined by

$$F_t(R) = E\left[ \int_0^t \left( \frac{1}{R} \mu_k f_k' + \frac{1}{2} \frac{1}{R^2} \sigma_k^2 f_k'' \right) dk \right]/t, \quad (23)$$

and let $F(R)$ be defined as the limit of $F_t(R)$ as $t$ goes to zero:

$$F(R) = \frac{1}{R} \mu_0 f'(0) + \frac{1}{2} \frac{1}{R^2} \sigma_0^2 f''(0). \quad (24)$$

Now, let $R_0$ be s.t.

$$F(R_0) = 0 \Rightarrow R_0 = -\frac{f''(0)}{2f'(0)} \frac{\sigma_i^2}{\mu_i}.$$

It follows from Lemma 1 that there exists $\epsilon > 0$ s.t. for all $t$, $0 < t < \epsilon$ there exists $R_t$ s.t. $F_t(R_t) = 0$ and $\lim_{t \to 0} R_t = R_0$. Defining $R(s_t) = R_t$ completes the proof.

**Proof of theorem 3.1.** The proof of the first two parts of the theorem follow from lemma (B.2). The proof of the third part of the theorem is as follows. Let $s \in S$ be a security and assume that $s_0 = 1$. We look at the stochastic process $f(s_k) = s_k^{1-\alpha}$, where $0 < k < T$, and $\alpha = 1/R$. Using Ito’s lemma, $f(s_k)$ can be described by the SDE as

$$df_k = \left[ (1 - \alpha) \mu_k s_k^{-\alpha} - \frac{1}{2} \alpha (1 - \alpha) \sigma_k^2 s_k^{-(\alpha+1)} \right] dk + \sigma_k \alpha s_k^{-\alpha} dW, \quad (25)$$

where $f(s_0) = s_0^{1-\alpha}$.

Taking the expectation in (25), we get:

$$E_0 \left[ s_t^{1-\alpha} \right] = s_0 + E_0 \left[ \int_0^t \left( (1 - \alpha) \mu_k s_k^{-\alpha} - \frac{1}{2} \alpha (1 - \alpha) \sigma_k^2 s_k^{-(\alpha+1)} \right) dk \right]. \quad (26)$$
For a given value of $t$, we substitute $\alpha = 1/R(s_t)$ and write (26) as follows:

$$E_0\left[\int_0^t \left((1 - \alpha_t)\mu_k s_k^{-\alpha_t} - \frac{1}{2}\alpha_t(1 - \alpha_t)\sigma_k^2 s_k^{-(\alpha_t+1)}\right) dk\right] = 0. \quad (27)$$

Let $F_t$ be a set of real-valued functions, defined by

$$F_t(R) = E_0\left[\int_0^t \left((1 - 1/R)\mu_k s_k^{-1/R} - \frac{1}{2}(1/R)(1 - 1/R)\sigma_k^2 s_k^{-(1/R+1)}\right) dk\right]/t, \quad (28)$$

and let $F(R)$ be defined as the limit of $F_t(R)$ as $t$ goes to zero:

$$F(R) = (1 - 1/R)\mu_0 - \frac{1}{2}(1/R)(1 - 1/R)\sigma_0^2. \quad (29)$$

Now, let $R_0$ be s.t.

$$F(R_0) = 0 \Rightarrow R_0 = \frac{\sigma_0^2}{2\mu_0}. \quad (29)$$

It follows from Lemma 1 that there exists $\epsilon > 0$ s.t. for all $t, 0 < t < \epsilon$, there exists $R_t$ s.t. $F_t(R_t) = 0$ and $\lim_{t \to 0} R_t = R_0$. Defining $R(s_t) = R_t$ completes the proof.

**References**


