Fixed Income Securities

Lecture Notes

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Hebrew University of Jerusalem
Chinese University of Hong Kong
Syllabus: Course Description

This course explores key concepts in understanding fixed income instruments.

It especially develops tools for valuing and modeling risk exposures of fixed income securities and their derivatives.

To make the material broadly accessible, concepts are, whenever possible, explained through hands-on applications and examples rather than through advanced mathematics.

Never-the-less, the course is quantitative and it requires good background in finance and statistical analysis as well as adequate analytical skills.
Syllabus: Topics

We will comprehensively cover topics related to fixed income instruments, including nominal yields, effective yields, yield to maturity, yield to call/put, spot rates, forward rates, present value, future value, mortgage payments, term structure of interest rates, bond price sensitivity to interest rate changes (duration and convexity), hedging, horizon analysis, credit risk, default probability, recovery rates, floaters, inverse floaters, swaps, forward rate agreements, Eurodollars, convertible bonds, callable bonds, interest rate models, risk neutral pricing, and fixed income arbitrage.
Syllabus: Motivation

Fixed income is important and relevant in the financial economic domain.

For one, much of the most successful and unsuccessful investment strategies implemented by highly qualified managed funds have involved fixed income instruments.

Domestically and internationally, pension and mutual funds, among others, have largely been specializing in bond investing.

Moreover, effective risk management is essential in the uncertain business environment in which interest rate and credit quality fluctuate often abruptly.

Such uncertainties invoke the use of fixed income derivatives.
Fixed income derivatives are standard instruments for managing financial risk.

More than 90% of the world's largest 500 companies use fixed income derivatives to manage interest rate and credit risk exposures.

Further, financial engineers keep inventing new fixed income derivatives to help firms transfer risk more effectively and selectively.

It is therefore critical for anyone involved in corporate finance or financial risk management to have deep-rooted understanding of interest rate risk and fixed income securities.
Syllabus: Resources

- **Instructor**
  I welcome students to see me whenever possible to discuss any aspect of the course.

- **Teaching Assistant**

- **Recommended text**: *Handbook of Fixed Income Securities* by Pietro Veronesi
Syllabus: Assessment

- **Home Assignments (20%)** – Problem sets will be assigned periodically. You have to submit all problem sets on time to be eligible for taking the final exam.

- **Final Exam (70%)** - The final exam will be based on the material and examples covered in class, problem sets, and assigned reading. The exam is closed books and closed notes. However, you will be allowed to bring in one piece of paper with handwritten or typed notes (double-sided, A4 size). You are not allowed to use any other notes. I will allow the use of non-programmable calculators during the exam.
Syllabus: Assessment

Class Participation (10%) - It is mandatory to attend all sessions. If you happen to miss a session, make sure to catch up, as you are fully responsible for class lectures, announcements, handouts, and discussions. If you miss more than three sessions without proper documentations - you will be unable to take this course for credit.
Overview, Risk Profile, and Trading Strategies
### Classifying Securities

<table>
<thead>
<tr>
<th>Basic Types</th>
<th>Major Subtypes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest-bearing</td>
<td>Money market instruments</td>
</tr>
<tr>
<td></td>
<td>Fixed-income securities</td>
</tr>
<tr>
<td>Equities</td>
<td>Ordinary and Preferred Stocks</td>
</tr>
<tr>
<td>Derivatives</td>
<td>Options, Futures, Swaptions</td>
</tr>
<tr>
<td></td>
<td>Interest rate derivatives</td>
</tr>
</tbody>
</table>
Recent Statistics

- In July 2017, the total US national debt was about $19.9 Trillion (source: http://www.usdebtclock.org/)

- As of 2017, there was $20.9 trillion of U.S. corporate (Financial and Non Financial) debt outstanding, and from 1996 to 2015 corporate bond issuance grew from $343 billion to $1.49 trillion (source: Securities Industry and Financial Markets Association).

- As of the end of 2016, there was $267 billions of Hong Kong corporate (Financial and Non Financial) debt outstanding.

- As of the end of 2016, there was $6.1 trillion of China corporate (Financial and Non Financial) debt outstanding.

- By recent estimates, the worldwide bond market’s total value amounts to $94.5 trillion, considerably exceeding the worldwide stock market value, which is about $66.5 trillion. http://www.bis.org/statistics/secstats.htm?m=6%7C33%7C615

- Why? Because bonds are issued by governments, agencies, states and municipalities (taxable and tax-exempt), mortgage-related and asset-backed, and corporations, while stocks are exclusively issued by corporations.
Bond vs. Stock Performance

- Stocks are perceived to be more risky than bonds and are thus expected to deliver higher average returns over an extended time period.

- Indeed, as you can observe from the next figures, stocks have overwhelmingly outperformed bonds over the 1925-2009 period, and much more so over the 1801 through 2001 period.

- But stocks are subject to massive short-term fluctuations.
A $1 Investment in Different Types of Portfolios, 1926—2009

Source: Global Financial Data (www.globalfinancialdata.com) and Professor Kenneth R. French, Dartmouth College. Professor Doron Avramov, Fixed Income Securities
The Very Long Run Perspective

**FIGURE 1.2** Financial Market History

*Total return indexes (1801–2001)*

- **Stocks**: $8.80 million
- **Bonds**: $13,975
- **Bills**: $4,455
- **Gold**: $14.67 (CPI)
- **CPI**: $14.38 (GOLD)

Asset Allocation VS. Stock and Bond Picking

- Various studies show that asset allocation accounts for more than 90% from the yield of the investment portfolio, while less than 10% originates from individual choice of a stock or a bond.

- Dalbar research company compared Senior Investment Managers versus benchmarks. The comparison was done between the years 1993 and 2012, and it was found that even sophisticated investors were far below benchmarks.
Asset Allocation VS. Stocks and Bond Picking

- In fact, the Investment Managers dedicated great efforts to detect stocks and bonds that, in their opinion, were cheap and dedicated little effort in managing the allocation of the assets.
- The following graph makes the case:
What is Risk Premium?

Risk premium (RP) is the average return of an investment over and above the risk free rate (proxied by T-bill rate).

<table>
<thead>
<tr>
<th>Investment</th>
<th>Average Return (%)</th>
<th>Risk Premium (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large stocks</td>
<td>11.7</td>
<td>7.9</td>
</tr>
<tr>
<td>Small stocks</td>
<td>17.7</td>
<td>13.9</td>
</tr>
<tr>
<td>Long-term corporate bonds</td>
<td>6.5</td>
<td>2.7</td>
</tr>
<tr>
<td>Long-term government bonds</td>
<td>5.9</td>
<td>2.1</td>
</tr>
<tr>
<td>U.S treasury bills</td>
<td>3.8</td>
<td>0.0</td>
</tr>
</tbody>
</table>
What is the Sharpe Ratio?

The Sharpe Ratio – a common investment performance measure – divides the Risk Premium by the investment’s volatility.

<table>
<thead>
<tr>
<th>Series</th>
<th>RP (%)</th>
<th>Vol (%)</th>
<th>SR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large-company stocks</td>
<td>7.9</td>
<td>20.5</td>
<td>0.39</td>
</tr>
<tr>
<td>Small-company stocks</td>
<td>13.9</td>
<td>37.1</td>
<td>0.38</td>
</tr>
<tr>
<td>Long-term corporate bonds</td>
<td>2.7</td>
<td>7.0</td>
<td>0.39</td>
</tr>
<tr>
<td>Long-term government bonds</td>
<td>2.1</td>
<td>11.9</td>
<td>0.18</td>
</tr>
</tbody>
</table>
Over recent years bond yields have been historically low.

<table>
<thead>
<tr>
<th>Period</th>
<th>Japan Gov Bonds Yields</th>
<th>China Gov Bonds Yields</th>
<th>Hong Kong Gov Bonds Yields</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>July 14, 2016</td>
<td>July 14, 2016</td>
<td>July 14, 2016</td>
</tr>
<tr>
<td>1 year</td>
<td>-0.35%</td>
<td>2.32%</td>
<td>0.33%</td>
</tr>
<tr>
<td>2 years</td>
<td>-0.34%</td>
<td>2.51%</td>
<td>0.52%</td>
</tr>
<tr>
<td>5 years</td>
<td>-0.32%</td>
<td>2.70%</td>
<td>0.72%</td>
</tr>
<tr>
<td>10 years</td>
<td>-0.24%</td>
<td>2.87%</td>
<td>0.98%</td>
</tr>
<tr>
<td>30 years</td>
<td>0.18%</td>
<td>3.52%</td>
<td>--</td>
</tr>
</tbody>
</table>

Bond Yields: US and Israel

Over recent years bond yields have been historically low.

<table>
<thead>
<tr>
<th>Period</th>
<th>US Treasury Yields</th>
<th>Israeli Gov Bonds Yields</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>0.49%</td>
<td>1.20%</td>
</tr>
<tr>
<td>2 years</td>
<td>0.67%</td>
<td>1.36%</td>
</tr>
<tr>
<td>5 years</td>
<td>1.11%</td>
<td>1.87%</td>
</tr>
<tr>
<td>10 years</td>
<td>1.55%</td>
<td>2.33%</td>
</tr>
<tr>
<td>30 years</td>
<td>2.27%</td>
<td>2.92%</td>
</tr>
</tbody>
</table>

[http://www.boi.org.il/he/DataAndStatistics/Pages/MainPage.aspx?Level=3&Sid=42&SubjectType=2](http://www.boi.org.il/he/DataAndStatistics/Pages/MainPage.aspx?Level=3&Sid=42&SubjectType=2)
Typically, households do not directly trade debt instruments.

Rather, they invest through specialized funds, such as mutual funds and ETFs.

For instance, in the US only 10% of bonds outstanding are directly traded by households.

The fraction of equities directly held by households is considerably larger in the US and elsewhere.

Depending on its investment objectives, a bond fund may concentrate its investments in a particular type of debt security or some mix.
Bond Funds

The securities that funds hold vary in terms of:

- credit risk/rating
- duration
- type of issuer (government, corporation)
- indexation (foreign currency, CPI)
- embedded options (convertible, puttable, and callable)
- floating versus fixed rates
Get familiar with one of the largest worldwide bond mutual fund of The Vanguard Group ticker VBTLX

http://finance.yahoo.com/q?s=VBTLX

VBTLX manages around $178 billion, as of April 19, 2017, invested in hundreds of various positions.

The Vanguard Group overall manages approximately 3 trillion dollar.
“The Short of a lifetime?”

- Bill Gross known as "The king of bonds" is a founder of PIMCO and ran the PIMCO’s $270.0 billion Total Return Fund (PTTRX) until joining Janus on September 2014.

- On April 21, 2015 he noted: “It's just a question of when. It's certainly a trade that doesn't cost you anything in the short term, because it doesn't yield anything and it has the ultimate potential of a 10 or 15 percent (return) over a one or two year period of time ... German bunds are 'the short of a lifetime”

- Was he right?
Short Bund (German bond) ETF 23/1/15-19/7/17

17/4/2015
Risk in Bonds

- Indeed, a common misconception is that bonds exhibit little or no risk.

- However, debt securities are subject to various types of nontrivial risks.

- Historically, bond price fluctuations have often been immense.
Interest Rate Risk

- Interest rate risk reflects the fluctuations in bond price when interest rates move up and down.

- Bond prices are inversely related to interest rate levels.

- Nearly all bond funds are subject to this type of risk, but funds holding bonds with higher duration are more subject to this risk than funds holding lower-duration bonds.

- In the case of German Bunds – yields went down, bond prices went up, and bond short position went down, as shown earlier. This is a classical example of interest rate risk.
Credit Risk

- Credit risk reflects the possibility that the bond issuer gets into a state of default thereby failing to make coupon and/or principal payments.

- Credit risk establishes the major difference between government versus corporate bonds and between high versus low quality corporate bonds as reflected through bond ratings.

- Recently, the credit risk of government bonds of emerging markets has considerably advanced following the fall in commodity prices and the perceived worldwide recession.
Prepayment Risk

- Prepayment risk reflects the possibility that the bond issuer prepays the bond (when bonds are callable) at a time when interest rates fall.

- Funds holding callable bonds could be forced to reinvest the proceeds in lower yield bonds, which can severely hurt performance.
Other Risk Sources

- Liquidity risk. Both government and corporate bonds could be highly illiquid.

- Inflation risk (for non CPI indexed bonds)

- Currency risk

- Sovereign Risk - the risk that a foreign central bank will alter its foreign-exchange regulations.
Who Invests in Bonds?

- There is also a common misconception that only solid institutions e.g., retirement savers, insurance companies, commercial banks, central banks, educational institutions, endowments, and pension funds, undertake bond positions.

- However, hedge funds, hedged-mutual funds, as well as day traders are intensely active in fixed income trading.
Hedge Funds

- HFs are private investment partnerships of high wealth individuals or qualified investors (pension funds).
- In the US, HFs are usually up to 500-investor partnership.
- Hedge funds have existed for a long time.
- The first HF was started by Alfred W. Jones in 1949.
- Hedge funds often make headlines because of spectacular losses or spectacular gains.
- The typical compensation contract of a hedge fund manager makes extremely high compensation possible – but only if the investors experience large returns.
Hedge Funds

- A comparison between hedge funds and mutual funds is useful for further understanding what hedge funds are.

- While it is a common knowledge that mutual funds do not beat the market on average – there is some evidence that hedge funds do. But information on hedge funds is incomplete and subject to too-many biases. Moreover, recent years have not been glamorous for the hedge fund industry.

- General partners in hedge funds typically invest significant fraction of their own wealth in the fund.
Mutual vs. Hedge Funds

- Minimum investment: MF small; HF large (around $1 M)
- Investors: MFs unlimited; HF's limited partnership.
- Availability: MFs are publicly available; HF's to accredited investors.
- Liquidity: MFs daily liquidity and redemptions; HF's liquidity varies from monthly to annually. Indeed, HF restrict the ability of investors to withdraw funds; thus they are able to maintain positions in illiquid assets.
Mutual vs. Hedge Funds

- Fees: MFs limits imposed by the SEC; HFs – no limits. Plus, HFs could charge asymmetric (convex) management fees based on performance. MFs cannot.

- One major difference is the nature of trading strategies. Hedge funds are more speculative and sophisticated. While mutual funds typically undertake solid long positions, hedge funds are involved in pair trading, buying and writing derivatives, day trading, etc.

- Correspondingly, mutual fund returns are typically highly correlated with the market returns, while hedge funds could record low correlation (the ambition is to be market neutral).
Mutual vs. Hedge Funds

- Regulation: MFs are regulated by the SEC. Their reports are detailed and strict. On the other hand, U.S. hedge funds are exempt from many of the standard registration and reporting requirements because they only accept accredited investors.

- In 2010, regulations were enacted in the US and European Union, which introduced additional hedge fund reporting requirements. These included the U.S.'s Dodd-Frank Wall Street Reform Act and European Alternative Investment Fund Managers Directive.
Mutual vs. Hedge Funds

- Hedge funds are more prone to fraud. The most notable one is Bernard Madoff who committed a Ponzi scheme. Total loss about $17 billion – the largest financial fraud in US history. A Ponzi scheme is a fraudulent investment operation where the operator pays returns to its investors from new capital financed by new investor inflows, rather than from profits.
Arbitrage in the Eyes of Hedge Funds

- There are several “fixed-income arbitrage” strategies undertaken by hedge funds and other highly qualified traders.

- In the hedge fund world, arbitrage refers to the simultaneous purchase and sale of two similar securities whose prices are not in sync with what the trader believes to be their “true value.”

- Acting on the assumption that prices will ultimately converge to true values, the trader will sell short the perceived overpriced security and buy the perceived underpriced security.

- If prices indeed converge to the perceived true values, then the trade can be liquidated at a profit.
Arbitrage in the Eyes of Hedge Funds

- This is not a risk-free arbitrage – in fact, it is quite risky which has frequently taken financial institutions out of business.

- In other words, it is an *ex ante* (*perceived*) arbitrage – but it may be an *ex post* (*realized*) disaster.

- The arbitrage strategy is related to the notion of pairs trading.

- Ideally, undertaking simultaneous long and short positions integrates out the market risk. But market risk is still there.
To understand fixed-income arbitrage, it is important to understand standard as well as more complex fixed-income securities (e.g., convertible bonds, floaters, CDS, etc.).

It is also essential to master important concepts such as the yield curve, option pricing, and implied versus realized volatility.

This course attempts to deliver the tools for understanding these topical concepts.

In what follow, classical fixed income arbitrage strategies are explained briefly.
Hedge Funds Investing in Bonds: Convertible Arbitrage

- The strategy involves the simultaneous purchase of convertible securities and the same-amount short sale of the same issuer's common stock.

- The premise of the strategy is that the convertible bond is often under priced relative to the underlying stock, for reasons that range from illiquidity to market psychology.
Taking two Wrong Positions on General Motors

Many convertible arbitrageurs suffered losses in early 2005 when the GM’s bond was downgraded at the same time Kirk Kerkorian (an American businessman) was making a bid offer for GM's stock. Since most arbitrageurs were long GM debt and short the equity, they suffered losses on both the long and short legs.
The 1987 Crash

- Many convertible "arbs" sustained big losses in the “crash of '87”.

- In theory, when a stock price falls, the associated convertible bond will fall less, because it is protected by its value as a fixed-income instrument as it pays interest periodically.

- In the 1987 stock market crash, however, many convertible bonds declined more than the underlying stocks, apparently for liquidity reasons (the market for the stocks being much more liquid than the relatively small market for the bonds).
The 1987 Crash – S&P500 and DJ
The 1987 Crash – DJ on Black Monday

The 1987 Stock Market Crash Trading
Dow Jones Industrial Averages - UK Time

2.40pm - Dow Futures Trading at 2045 - More than 150 below spot Index that is failing to give a proper price

UK 9am - Dow Pre-open Trading at Previous Closing Price of 2247 - Market NOT discounting a Crash - Time to SELL SHORT!

7.40pm - Broker No longer taking orders to SELL

Mid-day - Pre-Open Dow Quote is 1570 - Time to take profits and move stops!

(c) 2007 MarketOracle.co.uk

Black Monday - 19th October 1987

Tuesday - 20th October 1987
Capital Structure Arbitrage

- Capital structure arbitrage strategies exploit the perceived lack of co-ordination between various claims on a company, such as its bond and stock.

- A trader who believes that the bond of a company is overpriced relative to its stock, would short the bond and buy its stock.

- Capital structure arbitrage may also involve trading junior vs. senior debt – the notion is to buy the (perceived) cheap and sell the (perceived) expensive instruments.
Yield Curve Arbitrage

- The yield curve is a graphical representation of the relation between bond yields and time to maturity.

- When the yield curve is flat, shorter- and longer-term yields are close.

- When the yield curve is heavily sloped, there is a greater gap between short- and long-term yields.

- Yield curve arbitrage strategies are designed to profit from shifts in the steepness of or kinks in the yield curve by taking long and short positions on various maturities.
Yield Curve Arbitrage

- This could take the form of a butterfly trade, where the investor goes long five-year bonds and goes short two and ten-year bonds. Or, it may take the form of a spread trade, where, the investor goes short the front end of the curve and goes long the back end of the curve.

- The strategy requires the investor to identify some points along the yield curve that are “expensive” or “cheap.”
CDS (Credit Default Swap)

- Many of the bond arbitrage strategies involve trading CDS.
- CDS are complex financial instruments similar to insurance contracts in that they provide the buyer with protection against specific risks.
- They fall into the derivatives category and are traded over the counter (OTC) – not a formal exchange.
- For example, you buy a corporate bond of Ford.
- You are hopeful Ford will pay you back interest and principle.
- However, default is not a zero probability event, and you invested a lot of money, so you attempt to add protection.
Hence, you ask your insurance company (IC) to sell you insurance against the possible default of Ford bonds. The IC charges you a fee for that insurance, just as it would if you were buying car insurance or homeowner’s insurance. But there is a catch. With car or homeowner types of insurance you have to actually own the asset in order to insure it. Yet, with CDS, the IC is selling insurance on Ford bonds to anyone interested, and, even worse, it does not hold the required reserves to accommodate the event of default.
AIG and CDS Trading

- AIG was a giant CDS seller. GS was a major buyer.
- In 2008, AIG presented the second highest loss in US history – 99.3 Billion Dollar, in large due to its CDS trading.
- Nowadays, after its impressive recovery (thank to the $180 Billion Federal Government bailout) AIG is traded for $63 - it was traded for above $2000 per share prior to the financial crises.
- A recommended movie about the global financial meltdown is “Inside Job.”
AIG and CDS Trading

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- In 2008, AIG presented the second highest loss in US history – 99.3 Billion Dollar, in large due to its CDS trading.
- Nowadays, after its impressive recovery (thank to the $180 Billion Federal Government bailout) AIG is traded for $53 - it was traded for above $2000 per share prior to the financial crises.
- A recommended movie about the global financial meltdown is “Inside Job.”
AIG at the Crises

Professor Doron Avramov, Fixed Income Securities
In what Follows

- The arbitrage strategies explained over the next slides are more complex to grasp given that we are just starting the course – they are presented here as a way of motivation.

- We aim to understand the relevant concepts towards the end of this course.

- The uninterested reader could skip the descriptions until the topic: Fixed Income Arbitrage is Risky.
Fixed Income Arbitrage: Swap Spread (SS)

→ One of the most popular fixed-income arbitrage strategies is called “swap-spread arbitrage.”

→ While swap-spread arbitrage is too complex a topic to explain in full here, it involves taking a bet on the direction of credit default swap rates and other interest rates, such as the interest rate of U.S. Treasuries or the LIBOR.
Fixed Income Arbitrage: SS

- A typical swap-spread arb trade would consist of a fixed receiver swap and a short position in a par Treasury bond of the same maturity.

- The proceeds of the sale of the Treasury bond would be invested in a margin account earning the repo rate.

- This trade is a simple bet that the difference between the swap rate and the coupon rate will be more than the difference between the Libor and the repo rate. The trade could be engineered in the opposite direction, depending upon the spreads.
Fixed Income Arbitrage: SS

- The arbitrager:
  1. Receives a fixed coupon rate CMS (constant maturity swap rate).
  2. Pays the floating Libor rate.
  3. Pays the fixed coupon rate of the Treasury bond CMT (constant maturity Treasury rate).
  4. Receives the repo rate from the margin account.

- Combining the cash flows shows that the arbitrager receives a fixed annuity of $SS = CMS - CMT$ and pays the floating spread $St = Libor - Repo$

- SS arb strategy is profitable if the fixed annuity of SS received will be larger than the floating spread St paid, or $SS - St > 0$. 
Fixed Income Arbitrage: Volatility

- In their simplest form volatility arbitrage strategies profit from the tendency of implied volatilities to exceed expected realized volatilities.

- This is done by selling options of fixed income instruments and then delta-hedging the exposure to the underlying asset.

- Hedge funds typically implement vastly more complex volatility arbitrage strategies.

- Relevant terminologies in that context are variance Swaps, log contracts, and futures contracts on the VIX.
Mortgage Arbitrage

- The mortgage-backed security (MBS) strategy consists of buying MBS passthroughs, financing them with dollar rolls (cheap financing) and hedging their interest rate exposure with swaps.

- A passthrough is a MBS that passes all the interest and principal cash flows of a pool of mortgages (after servicing and guarantee fees) to the passthrough investors.

- The main risk of a MBS passthrough is prepayment risk. Homeowners refinance their mortgages when interest rates drop.

- To compensate for possible losses, due to prepayment, investors require higher yields to hold callable securities.
Fixed Income Arbitrage is Risky

- Ideally, since FIA strategies consist of long and short positions, they would cancel out systematic market risks such as changes to the yield curve.

- Managers are free to hedge away specific risk exposures, including risks due to changes in interest rates, creditworthiness, foreign exchange risks, though the extent of hedging employed varies greatly among hedge fund managers.
Fixed Income Arbitrage is Risky

- Of course, managers could take directional positions instead of relying solely on pure neutral hedges.

- Practically, fixed-income arbitrageurs must be willing to accept significant risk.

- That’s because fixed-income arbitrage typically provides relatively small returns, but can potentially lead to huge losses.

- In fact, some refer to fixed-income arbitrage as "picking up nickels in front of a steamroller." Some disagree!
It is instrumental to mention Long Term Capital Management (LTCM) in the FIA context.

LTCM was a hedge fund that in the 1990s realized average annual returns greater than 40%.

Some of its successful trades involved undertaking long and short positions in bonds.

However, in 1998, when some of its bets moved against it, the fund had to be rescued by prominent Wall Street banks with a $3.5 billion package orchestrated by the Federal Reserve Board.
In August-September 1998 LTCM lost about $4.5 billion.

The largest source of losses for LTCM was the swap spread positions - about $1.6 billion.

LTCM lost about $1.3 billion on volatility arbitrage positions.
Other Trading Strategies

- Momentum and momentum spillover are two simple trading strategies.
- Bond momentum means that out of a large cross section of bonds you buy the winners and sell the losers.
- Momentum spillover – buy long bonds of winning stocks and sell short bonds of loosing stocks.
To Summarize this Section

- The fixed income domain is quantitatively complex.
- There are various distinct types of debt instruments.
- Debt instruments are subject to multiple types of risk.
- Both solid and speculative investors trade debt instruments.
- Many of the most successful and unsuccessful trades of hedge funds have been involving debt instruments.
- Fixed income arbitrage is not really a risk-free arbitrage.
- Some more simple trading strategies such as momentum and momentum spillover have been successful in beating bond benchmarks.
Bond Pricing
Selected Topics

- What is the time value of money?
- What is the compounding effect?
- How to calculate the value of a mortgage as well as the amortization schedule?
- What are yield to maturity (YTM) and spot rates?
- How to calculate the price of zero coupon bonds?
- How to calculate the price of coupon bonds?
- What is the relation between YTM and bond price?
- What is horizon analysis?
Time Value of Money

You will get $1,100 in one year. Annual interest rate (effective) is 10%.

What is the present value (PV) of the future payment?

\[ PV = \frac{1,100}{1.1} = 1,000 \]
PV of an Ordinary Annuity

- When the same dollar amount of money is received each period or paid each period the series is referred to as an annuity.
- When the first payment is received one period away from now the annuity is called an ordinary annuity.
- When the first payment is immediate the annuity is called annuity due.
- The present value of the ordinary annuity is:

\[ PV = A \left[ \frac{1 - \frac{1}{(1 + r)^n}}{r} \right] \]
PV of an Ordinary Annuity

- In the PV formula, A is the annuity amount, r is the interest rate, and n is the number of periods.
- The term in brackets is the present value of ordinary annuity of $1 for n periods.
- Example of present value of an ordinary annuity using annual interest

If A=100, r=0.09 and n=8 then

\[ PV = A \left[ \frac{1}{r} - \frac{1}{(1+r)^n} \right] = $100 \left[ \frac{1 - \frac{1}{(1+0.09)^8}}{0.09} \right] = $100 \left[ \frac{1 - \frac{1}{1.99256}}{0.09} \right] = $553.48 \]
PV and Compounding Frequency

Consider a 5% annual interest rate (APR=annual % rate)

<table>
<thead>
<tr>
<th>Frequency</th>
<th>m</th>
<th>APR</th>
<th>AER</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual</td>
<td>1</td>
<td>5%</td>
<td>5.0000%</td>
</tr>
<tr>
<td>Semi-annual</td>
<td>2</td>
<td>5%</td>
<td>5.0625%</td>
</tr>
<tr>
<td>Quarterly</td>
<td>4</td>
<td>5%</td>
<td>5.0945%</td>
</tr>
<tr>
<td>Monthly</td>
<td>12</td>
<td>5%</td>
<td>5.1162%</td>
</tr>
<tr>
<td>Weekly</td>
<td>52</td>
<td>5%</td>
<td>5.1246%</td>
</tr>
<tr>
<td>Daily</td>
<td>365</td>
<td>5%</td>
<td>5.1267%</td>
</tr>
<tr>
<td>Continuous</td>
<td>infinity</td>
<td>5%</td>
<td>5.1271%</td>
</tr>
</tbody>
</table>
PV and Fixed-Rate Mortgage

- Suppose a 30-year $100,000 mortgage loan is financed at a fixed interest rate (APR) of 8%. What is the monthly payment?

\[
\text{Monthly Payment} = \frac{100,000 \times 0.08/12}{1 - 1/(1 + 0.08/12)^{360}} = 733.76
\]

- Notice that you overall pay \(733.76 \times 30 \times 12 = 264,154\) for this loan, the majority of which -- \$164,154 -- constitutes interest payments.

- What is the monthly fixed amount assuming APR=6%?
Amortization of Mortgage

What are the interest and principal payments in the first and second months?

- In the first month:
  - Interest payment = $100,000 \times 0.08/12 = $666.67
  - Principal payment = $733.76 – $666.67 = $67.09
  - New principal = $100,000 – $67.09 = $99,932.91

- In the second month:
  - Interest payment = $99,932.91 \times 0.08/12 = $666.22
  - Principal payment = $733.76 – $666.22 = $67.54
  - New principal = $99,932.91 – $67.54 = $99,865.37
Amortization for the Long Run

- Then, in the next month the interest will be based on the new principal, $99,865.37.

- The principal payment is the difference between the fixed amount $733.76 and the interest computed on the new principal... and so on and so forth till the end.

- Notice that interest payments diminish while principal payments increase through time.
Amortization of Mortgage

- The graph below illustrates how monthly installments are divided between interest and principal over time.
- As computed earlier, total monthly installment is $733.76
Future Value

Suppose that a pension fund manager invests $10,000,000 in a financial instrument that promises to pay 9.2% per year for six years where the compounding is semi-annually.

What is the future value of the investment?

\[ FV = 10,000,000 \times 1.046^{12} = 17,154,600 \]

What if the compounding is annual?

\[ FV = 10,000,000 \times 1.092^6 = 16,956,500 \]
Future Value (FV) of an Ordinary Annuity

- When the same amount of money is invested periodically, it is referred to as an annuity.
- When the first investment occurs one period from now, it is referred to as an ordinary annuity.
- Clearly, it should be the case that $FV = PV(1 + r)^n$.
  where $r$ is the interest rate and $n$ is the number of payments.
- Indeed, the future value of an ordinary annuity is

$$FV = A \left[ \frac{(1 + r)^n - 1}{r} \right]$$

where $A$ is the amount of the annuity (in dollars).
Example of Future Value of an Ordinary Annuity using Annual Interest

If $A = \$2,000,000$, $r = 0.08$, and $n = 15$, then

$$FV = A \left[ \frac{(1 + r)^n - 1}{r} \right] \rightarrow FV = \$2,000,000 \left[ \frac{3.17217 - 1}{0.08} \right]$$

$$= \$2,000,000[27.152125] = \$54,304,250$$

Notice that $15 \times (\$2,000,000) = \$30,000,000$, which means the balance of $\$54,304,250 - \$30,000,000 = \$24,304,250$ is the interest earned by reinvesting these annual interest payments.
Consider the same example. But now we assume semiannual interest payments. APR=0.08. The FV is, of course, higher.

If

\[
A = \frac{\$2,000,000}{2} = \$1,000,000, \quad r = \frac{0.08}{2} = 0.04, \quad n = 2 \times 15 = 30
\]

then

\[
FV = A \left[ \frac{(1 + r)^n - 1}{r} \right] \rightarrow FV = \$1,000,000 \left[ \frac{(1 + 0.04)^{30} - 1}{0.04} \right]
\]

\[
= \$1,000,000 \left[ \frac{3.2434 - 1}{0.04} \right] = \$1,000,000[56.085] = \$56,085,000
\]
Bond Pricing

- Using the PV formula bond pricing follows straightforwardly.
- We first price zero coupon bonds then coupon bonds.
- Spot rates are the engine of pricing.
Pricing Zero Coupon Bonds

- A zero-coupon bond (also called a discount bond) is a debt instrument bought at a price lower than its face value, with the face value repaid at the time of maturity.

- It does not make periodic coupon payments which explains the term zero-coupon bond.

- Examples of zero-coupon bonds include U.S. T-Bills, U.S. saving bonds, long-term zero-coupon bonds, and any type of coupon bond that has been stripped of its components.
STRIPS

- **STRIPS: Separate Trading of Registered Interest and Principal of Securities**

- STRIPS are effectively zero coupon bonds (zeroes).

- STRIPS were originally derived from 10-year T-notes and 30-year T-bonds
  - A 30-year T-bond can be separated into 61 strips - 60 semiannual coupons + a single face value payment
### STRIPS’ Quotes

#### Treasury Bond, Stripped Principal:

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Bid</th>
<th>Asked</th>
<th>Change</th>
<th>Asked yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>2016 May 15</td>
<td>99.965</td>
<td>99.967</td>
<td>0.006</td>
<td>0.26</td>
</tr>
<tr>
<td>2020 Feb 15</td>
<td>95.771</td>
<td>95.808</td>
<td>0.320</td>
<td>1.11</td>
</tr>
<tr>
<td>2045 Feb 15</td>
<td>45.468</td>
<td>45.598</td>
<td>0.408</td>
<td>2.74</td>
</tr>
</tbody>
</table>

#### Treasury Note, Stripped Principal:

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Bid</th>
<th>Asked</th>
<th>Change</th>
<th>Asked yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>2016 Jun 15</td>
<td>99.947</td>
<td>99.949</td>
<td>0.180</td>
<td>0.24</td>
</tr>
<tr>
<td>2020 Feb 29</td>
<td>95.861</td>
<td>95.898</td>
<td>0.287</td>
<td>1.07</td>
</tr>
<tr>
<td>2025 Feb 15</td>
<td>85.242</td>
<td>85.317</td>
<td>0.481</td>
<td>1.80</td>
</tr>
</tbody>
</table>

#### Stripped Coupon Interest:

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Bid</th>
<th>Asked</th>
<th>Change</th>
<th>Asked yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>2016 May 15</td>
<td>99.965</td>
<td>99.966</td>
<td>0.006</td>
<td>0.27</td>
</tr>
<tr>
<td>2020 Feb 15</td>
<td>95.568</td>
<td>95.605</td>
<td>0.319</td>
<td>1.16</td>
</tr>
<tr>
<td>2045 Feb 15</td>
<td>45.536</td>
<td>45.565</td>
<td>0.408</td>
<td>2.74</td>
</tr>
</tbody>
</table>
Types of Zero Coupon Bonds

- Some zero coupon bonds are inflation indexed, but the vast majority are nominal.

- Zero coupon bonds may be long or short term investments.

- Long: typically ten to fifteen years.

- Short: maturities of less than one year and are called bills.

- Long and short zero coupon bonds can be held until maturity or sold on secondary bond markets.
Spot Rates: The Engine of Bond Pricing

- Consider a 1-year zero-coupon bond with a face value of $100 and a spot rate of 5%.

- The price of the bond is the PV of the face value which is $95.24.

- If the spot rate is instead 6%, the bond price is $94.34.
Zero Coupon Pricing

The price of a 10-year zero-coupon bond with a $100 face value is

\[ 61.4 = \frac{100}{(1.05)^{10}} \]

If the 10-year spot rate is 6% the price drops to 55.8 following the same formulation.
Yields and Bond Prices

Note that:

a. The bond price is inversely related to the spot rate.

b. The 10-year bond is about as ten times as sensitive (in percentage terms) to a change from a 5% to 6% rate as the 1-year bond.
A 2-year Treasury Note with a 6% coupon rate and a face value of $100 is essentially a package of

- a $3 zero-coupon bond with a 0.5 year maturity
- a $3 zero-coupon bond with a 1 year maturity
- a $3 zero-coupon bond with a 1.5 year maturity
- a $103 zero-coupon bond with a 2 year maturity

If we know the spot rates for each of these four zero-coupon bonds, then we can value the four components separately, and add up their values.
Pricing a Coupon Bond

- If the price of the 2-year bond was higher (beyond transaction costs) than the value of this “portfolio” of zero-coupon bonds, then one could exploit this arbitrage opportunity.

- In particular, sell short the coupon bond and buy long the zero-coupon bonds – this is arbitrage.

- You earn a risk-free profit!
Spot Rates are Expressed as Annualized Rates on a bond-equivalent yield basis

- 0.5 year: 4.4% which is 2.2% six-month effective
- 1.0 year: 4.8% which is 4.8576% one-year effective
- 1.5 year: 5.2% which is 5.2676% one year effective
- 2.0 year: 5.6% which is 5.6784% one-year effective

Value of a 6% coupon 2-year T-Note:

\[
P = \frac{3}{(1.022)} + \frac{3}{(1.048576)} + \frac{3}{(1.052676)^{1.5}} + \frac{103}{(1.056784)^2}
\]
\[
= 3 (0.978 + 0.954 + 0.926 + 0.895) + 100 (0.895)
\]
\[
= 100.76
\]
No Arbitrage Bond Pricing: An Example

Given information about bonds A, B, and C having the same default risk

<table>
<thead>
<tr>
<th>Bond</th>
<th>Price</th>
<th>CF year 1</th>
<th>CF year 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>96</td>
<td>100</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>91</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>C</td>
<td>?</td>
<td>10</td>
<td>110</td>
</tr>
</tbody>
</table>

what is the no-arbitrage price of bond C? (No arbitrage does not imply that pricing is rational).
The objective is to find the discount factors $d_1$ and $d_2$ for one and two years.

Solve the two-equation two-unknown system:

$96 = 100 \ d_1$
$91 = 100 \ d_2,$

which yields $d_1 = 0.96$ and $d_2 = 0.91$.

The price of bond C is simply given by

$10 \ d_1 + 110 \ d_2 = 109.7.$
Yield-To-Maturity (YTM)

- Whereas spot rates are macro-wide variables common to similar bonds, YTM is bond specific and is applied to all future cash flows.

- Given a bond’s specified cash flows, $C_1, C_2, \ldots C_n$, and its market price, $P$, the promised yield-to-maturity (YTM) can be calculated as:

$$P = \frac{C_1}{(1 + y)^1} + \frac{C_2}{(1 + y)^2} + \frac{C_3}{(1 + y)^3} + \ldots + \frac{C_n}{(1 + y)^n}$$

or

$$P = \sum_{t=1}^{n} \frac{C_t}{(1 + y)^t}$$
Yield-To-Maturity (YTM)

- A bond’s promised YTM is thus the internal rate of return that makes the present discounted value of the bond’s cash flows equal to their price.

- The YTM calculations assume that
  a. all cash flows received during the life of the bond are reinvested at the YTM.
  b. the bond is held until maturity.
  c. all promised cash flows are paid.
Calculating YTM for Premium Bond

- An 8% 30-year coupon bond with face value of $1,000 is currently priced at $1,276.76. What must be the YTM?

\[
$1,276.76 = \sum_{t=1}^{60} \frac{40}{(1 + y)^t} + \frac{1,000}{(1 + y)^{60}}
\]

- Using Excel, you get y=3%; thus the YTM=6% on an annualized bond-equivalent yield basis.
Computing YTM when Payments are Contingent

In the previous example, YTM is computed assuming that all payments are known in advance.

- What if some payments are contingent?
  - For instance, what if there are embedded options, flowing coupons, etc?
Example

- Cost: 101
- Promised cash-flow:
  - After 1 year 6
  - After 2 years 7, callable at 100
  - After 3 years 8
  - After 4 years 9
  - After 5 years 110
Yield to Call Calculation

\[ 101 = \frac{6}{1 + y} + \frac{107}{(1 + y)^2} \]

\[ y = 5.94\% \]
Yield to Worst

- It is common practice to take the lowest among YTM, yield to call, and yield to put.

- If you hold the bond till maturity then YTM=7.6% (check!!).

- In our case, yield to worst = $\min(7.6\%, 5.94\%) = 5.94\%$. 
YTM vs. Holding Period Average Return

- YTM is often considered to be the holding period average return.

- This is correct only if:
  a. the investor holds the bond till maturity
  b. coupons and principal are paid as promised
  c. the reinvestment rate of coupons is equal to the YTM, something highly unlikely.
Computing Holding Period Average Return

Consider a four-year 100,000 face value bond paying 4% coupon in the end of the year.

Market price is 100,000, the YTM is thus 4%.

Assume that rates are diminishing. In particular, you invest the first coupon for 2% for the next three years, the second coupon for 1% for the next two years, and the last coupon for 0% for the next year.
YTM is not the Holding Period Average Return

- The future value of the investment is 116,325.
- The average annual return is thus 3.85%<4%.

Assume now that you sell the bond after two years.
- The bond price is the present value (YTM=1%) of two coupon payments and the face value 105,911.
- The account has accumulated the amount
  \[4,000 \times 1.02 + 4,000 + 105,911 = 113,991\]
- Thus, the annual two-year return is 6.77%.
Ex ante vs. Ex post Measures

- The actual return can be computed only after selling the bond or after receiving the entire cash flow streams, coupon and principle payments, if the bond is held till maturity.

- Notice that the holding period return is an ex-post, or realized, measure while YTM is an ex ante, or expected, concept.
On the Relationship Between Bond Prices and Yields
Bond Price Characteristics

a. Price and YTM move in opposite directions.

b. Long-term bonds are more price sensitive than short-term bonds to changing interest rates.

c. Prices are “convex functions” of yield. Translation: a 1% decrease in yield raises a bond’s price by an amount that is greater than the fall in price when the bond’s yield increases by 1%. Moreover, the larger the convexity measure the more expensive the bond is. We will revisit the notion of convexity later.
Why would the Bond Price Change?

- Changing risk-free spot rates: macro level.
- Changing credit risk: firm, industry, or market levels.
- Changing liquidity.
- The bond approaches maturity – then the price of premium or discount bonds converges to the face value.
There are several problems with the notion of YTM, which is a prominent criterion in fixed income investing.

The YTM computation exhibits two shortcomings:
- it assumes reinvestment of coupons at the same rate as the YTM.
- it does not take account of the investment horizon.

For instance, an investor with a five-year horizon can pick one of four bonds whose characteristics are displayed on the next page.

What is the best choice?
Is YTM a Robust Criterion?

<table>
<thead>
<tr>
<th>Bond</th>
<th>Coupon</th>
<th>T</th>
<th>YTM</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5</td>
<td>3</td>
<td>9.0</td>
</tr>
<tr>
<td>B</td>
<td>6</td>
<td>20</td>
<td>8.6</td>
</tr>
<tr>
<td>C</td>
<td>11</td>
<td>15</td>
<td>9.2</td>
</tr>
<tr>
<td>D</td>
<td>8</td>
<td>5</td>
<td>8.0</td>
</tr>
</tbody>
</table>
So - YTM is not the Ultimate Criterion

- Perhaps you should invest in bond C – it delivers the highest YTM. However, you have to sell in five year. Plus the coupon is high - hence the reinvestment risk is considerable.

- Perhaps bond A – second highest YTM? However, you have to find a new investment after 3 year.

- Perhaps bond D? It matches the holding period but has the lowest yield to maturity and second largest coupon.

- The YTM does not really help.
What can you Do?

- Form expectations about future rates.

- To illustrate, suppose that your investment horizon is 3 years.

- A 20 year 8% coupon bond is selling for 828.4 for 1000 face value, meaning YTM=10%.

- What is the expected (not actual) annual return over the three-year investment horizon?
Expected Bond Return

a. The coupon will be re-invested in 6%
b. YTM in the end of three years is 7%.

- FV of three-year coupons: \( FV(40\,3\%,\,6) = 258.74 \)
- Sell the bond after three years for
  \[ PV(40\,3.5\%,\,34) + pv(1000\,3.5\%,\,34) = 1098.5 \]
- Uppercase letters denote present or future values for a stream of payments (annuity).
- Lower case letters correspond to a single payment.
- You get 1357.25 in the end of three years. You bought the bond for 828.4 – the semi-annual expected return is thus 8.58%.
End of Chapter Questions

- Here are several questions that exercise the content covered in this chapter.
- I will give such questions in the end of each chapter that follows.
- Some of these questions are taken from past year final exams.
- Home assignment: solve the unanswered questions.
- Problem sets should be solved on an individual basis.
Question 1

You buy 20 Million Dollar bond for 15 years with a coupon of 10% paid once a year. The coupon payments are deposited in a risk-free cash account with 8% per year.

What is the amount accumulated after 15 years?

Answer:

- FV of coupon payments is 54,304,228
- Principle 20,000,000
- Total 74,304,228
Question 2

What if the coupon is paid twice per year?

Answer:

\[ FV = 1,000,000 \times \frac{(1.04)^{30} - 1}{0.04} = 56,084,938 \]

Total 76,084,938
Question 3

A zero coupon bond with a par value of $100,000 due in two years from now is selling today for $97,000.

What is the effective annual yield?

Answer:

The two-year yield is

\[
\frac{100,000 - 97,000}{97,000} = 3.09\%.
\]

The annual yield is

\[
(1 + 0.0309)^{\frac{1}{2}} - 1 = 1.53\%
\]
Questions 4, 5, 6

Consider the fixed mortgage loan example from the page of Amortization of Mortgage.

- **Q4**: What is the amount of principle after 10 years (120 monthly payments)?

- **Q5**: What is the market value of the mortgage after 10 years given that the market interest rate has risen to 9% (APR)?

- **Q6**: Plot the fixed payment as a function of APR (ranging from 3% to 10% in intervals of 0.5%).
Given information about bonds A, B, and C having the same default risk

<table>
<thead>
<tr>
<th>Bond</th>
<th>Price</th>
<th>CF - 1</th>
<th>CF - 2</th>
<th>CF - III</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>90</td>
<td>4</td>
<td>4</td>
<td>104</td>
</tr>
<tr>
<td>B</td>
<td>110</td>
<td>6</td>
<td>6</td>
<td>106</td>
</tr>
<tr>
<td>C</td>
<td>?</td>
<td>12</td>
<td>12</td>
<td>112</td>
</tr>
</tbody>
</table>

What is the no arbitrage price of bond C?
Questions 8, 9

You observe the following $100 face value, default free, no callable Treasury securities. Coupons are paid semi-annually.

<table>
<thead>
<tr>
<th>Price</th>
<th>Coupon</th>
<th>Maturity</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>97.2006</td>
<td>0.000%</td>
<td>0.5</td>
<td>T-bill</td>
</tr>
<tr>
<td>94.4337</td>
<td>0.000%</td>
<td>1.0</td>
<td>T-bill</td>
</tr>
<tr>
<td>100.1265</td>
<td>6.250%</td>
<td>2.0</td>
<td>T note</td>
</tr>
<tr>
<td>100.3400</td>
<td>6.375%</td>
<td>2.0</td>
<td>T note</td>
</tr>
</tbody>
</table>

**Q8:** What is the annual three-period (1.5-year) spot rate?

**Q9:** What is the annual four-period (two-year) spot rate?
More Questions to Practice

Q: The portfolio manager of a tax-exempt fund is considering investing $500,000 in a debt instrument that pays an annual interest rate of 5.7% for four years. At the end of four years, the portfolio manager plans to reinvest the proceeds for three more years and expects that for the three-year period, an annual interest rate of 7.2% can be earned. What is the future value of this investment?

A: At the end of year four, the portfolio manager’s amount is given by 
\[ P_4 = 500,000(1.057)^4 = 500,000(1.248245382) = 624,122.66 \]. The requested future value is 
\[ P_7 = 624,122.66(1.072)^3 = 624,122.66(1.231925248) = 768,872.47 \].
More Practice Questions

Q: Suppose that a portfolio manager purchases $10 million of par value of an eight-year bond that has a coupon rate of 7% and pays interest once per year. The first annual coupon payment will be made one year from now. How much will the portfolio manager have if she (1) holds the bond until it matures eight years from now, and (2) can reinvest all the annual interest payments at an annual interest rate of 6.2%?
A: In the end of the eight year the amount accumulated is equal to the future value of coupon paid plus the face value.

Annual interest payment: $10,000,000 \times 0.07 = \$700,000$

\[ FV = 700,000 \times \frac{(1+0.062)^8 - 1}{0.062} = \$6,978,160.38 \]

Then

\[ \$10,000,000 + \$6,978,160.38 = \$16,978,160.38 \]
Q: Following are four government bonds maturing at or before the end of Year 4. Their prices today and cash flows at the end of each year are as follows:

<table>
<thead>
<tr>
<th>Bond</th>
<th>Price</th>
<th>CF Year 1</th>
<th>CF Year 2</th>
<th>CF Year 3</th>
<th>CF Year 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>127.2</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>110</td>
</tr>
<tr>
<td>B</td>
<td>18.8</td>
<td>0</td>
<td>20</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>55.8</td>
<td>0</td>
<td>30</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td>D</td>
<td>27.8</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

Given that you can buy or sell any amount of these bonds, what is the no-arbitrage price of a new bond E that matures at the end of year 4 and provides cash flows of 50, 20, 20, and 150 on years 1 through 4, respectively?
More Practice Questions

A:

\[10D_1 + 10D_2 + 10D_3 + 110D_4 = 127.2\]
\[20D_2 = 18.8\]
\[30D_2 + 30D_3 + 10D_4 = 55.8\]
\[10D_1 + 10D_3 + 10D_4 = 27.8\]

Get: \(D_1 = 1.26, D_2 = 0.96, D_3 = 0.62, D_4 = 0.9.\)
Bond Price Volatility
Contents for this Chapter

- We already know that:
  - the higher the yield the lower the bond price
  - the higher the time to maturity the more sensitive the bond price is to interest rate changes.
- Broadly, we would like to formalize the fluctuations of bond prices in response to changing interest rates.
- Duration and convexity are two important concepts.
- Things get more complex in the context of floater, inverse floater, IO bonds, and swap contracts.
### 8% Coupon Bond

<table>
<thead>
<tr>
<th>Yield to Maturity</th>
<th>T=1 yr.</th>
<th>T=10 yr.</th>
<th>T=20 yr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>8%</td>
<td>1,000.00</td>
<td>1,000.00</td>
<td>1,000.00</td>
</tr>
<tr>
<td>9%</td>
<td>990.64</td>
<td>934.96</td>
<td>907.99</td>
</tr>
<tr>
<td>Price Change</td>
<td>0.94%</td>
<td>6.50%</td>
<td>9.20%</td>
</tr>
</tbody>
</table>

### Zero Coupon Bond

<table>
<thead>
<tr>
<th>Yield to Maturity</th>
<th>T=1 yr.</th>
<th>T=10 yr.</th>
<th>T=20 yr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>8%</td>
<td>924.56</td>
<td>456.39</td>
<td>208.29</td>
</tr>
<tr>
<td>9%</td>
<td>915.73</td>
<td>414.64</td>
<td>171.93</td>
</tr>
<tr>
<td>Price Change</td>
<td>0.96%</td>
<td>9.15%</td>
<td>17.46%</td>
</tr>
</tbody>
</table>
To understand the first order (duration) and second order (convexity) effects of changing yields on bond prices we have to resort to some useful machinery.

It is the Taylor Approximation (TA).

Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point.
Taylor approximation can be written as

\[ f(x) = f(x_0) + \frac{1}{1!} f'(x_0)(x - x_0) + \frac{1}{3!} f''(x_0)(x - x_0)^2 + \cdots \]

In our context, the bond price is

\[ P(y_0) = \sum_{i=1}^{n} \frac{CF_i}{(1 + y_0)^i} \]

where \( y_0 \) is the yield to maturity.
Assume that $y_0$ changes to $y_1$.

The delta (the change) of the yield to maturity is written as

$$\Delta y = y_1 - y_0$$

Using TA we get

$$P(y_1) \approx P(y_0) + \frac{1}{1!} P'(y_0)(y_1 - y_0) + \frac{1}{2!} P''(y_0)(y_1 - y_0)^2$$

$$\Rightarrow P(y_1) - P(y_0) \approx \frac{1}{1!} P'(y_0)(y_1 - y_0) + \frac{1}{2!} P''(y_0)(y_1 - y_0)^2$$
Duration and Convexity

- Dividing by $P(y_0)$ yields
  \[
  \frac{\Delta P}{P} \approx \frac{P'(y_0) \times \Delta y}{P(y_0)} + \frac{P''(y_0) \times \Delta y^2}{2P(y_0)}
  \]

- Instead of: $\frac{P'(y_0)}{P(y_0)}$ we can write “– MD” (Modified Duration).

- Instead of: $\frac{P''(y_0)}{P(y_0)}$ we can write “Con” (Convexity).
The approximated % change in the bond price is

$$\frac{\Delta P}{P} \approx -MD \times \Delta y + \frac{Con \times \Delta y^2}{2}$$

where

$$MD = \frac{D}{1 + y}$$

$$D = \sum_{t=1}^{T} w_t t$$

$$Con = \frac{1}{(1 + y)^2} \sum_{t=1}^{T} w_t (t^2 + t)$$
Duration and Convexity

- In these formulas – $t$ is the timing of cash flows, $T$ is the overall number of cash flows, and $w_t$ is a weight function (with sum equal to one), which is equal to the present value of the bond cash flow in time $t$ divided by the current bond price.

- Duration is due to Macaulay (1938)

- It represents a first order impact of interest rate change on bond price change.

- Here is a numerical example.
Duration can be calculated by multiplying the length of time to each cash flow by the fraction of the bond’s price paid out with each cash flow and summing up the resulting products.

A ten-year bond trading to yield a 9% yield to maturity that pays an 8% annual coupon rate semi-annually has a duration of 6.95 years.
Some Rules about Duration

- The larger the coupon rate, the lower the duration.
- Duration falls as the yield-to-maturity rises. This is the *convexity* property (which we will examine shortly).
  The intuition here is that nearby cash flows (lower $t$) become of greater importance (get higher weights) relative to more heavily discounted far-away cash flows.
- Bonds with greater duration, ceteris paribus, fluctuate more in response to changing interest rates than bonds with lower duration.
- Duration of a zero-coupon bond is equal to the bond maturity.
- The longer the maturity, the higher the duration.
## Bond Durations

(Bond durations (Initial bond yield = 8% APR; semiannual coupons))

<table>
<thead>
<tr>
<th>Years to Maturity</th>
<th>6%</th>
<th>8%</th>
<th>10%</th>
<th>12%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.985</td>
<td>0.980</td>
<td>0.976</td>
<td>0.972</td>
</tr>
<tr>
<td>5</td>
<td>4.361</td>
<td>4.218</td>
<td>4.095</td>
<td>3.990</td>
</tr>
<tr>
<td>10</td>
<td>7.454</td>
<td>7.067</td>
<td>6.772</td>
<td>6.541</td>
</tr>
<tr>
<td>Infinite (perpetuity)</td>
<td>13.000</td>
<td>13.000</td>
<td>13.000</td>
<td>13.000</td>
</tr>
</tbody>
</table>
The duration of liabilities is 2.5 years. A portfolio manager is considering two bonds to buy in order to match the duration of the liabilities: bond A has a duration of 1.0 year; bond B has a duration of 3.0 years. How shall the manager combine the bonds?

It is important to note that the duration of a portfolio of bonds is equal to the weighted average of individual bond durations, using the prices of individual bonds to calculate the weights.
Using Duration to Immunize

Solution:

- Buy $w_A$ of bond A and $(1 - w_A)$ of bond B.
- To find $w_A$, set $1.0 \times w_A + 3.0 \times (1 - w_A) = 2.5 \Rightarrow w_A = .25$
- Put 25% of the fund in bond A and 75% in B.
Immunizing: Caution!

- Immunizing using duration assumes that the term structure of interest rates is horizontal and that any shifts in the term structure is parallel.

- Even if these assumptions were to hold, duration can be used only to approximate price changes - it is a linear approximation based on the local slope of the price-yield curve, and thus it is accurate only for small changes in interest rates. You may need to match convexity as well.

- In addition, one has to worry about other risks affecting the bond price, for example, default risk, liquidity risk, and call risk. Recall, duration reflects interest rate risk only.
Calculating Convexity

Consider a 2-year 1000 face value bond with a semi-annual coupon and YTM of 4% and 5%, respectively.

The convexity (per six-month period) is:

\[ C = \frac{40(1^2+1) + 40(2^2+2) + 40(3^2+3) + 1040(4^2+4)}{(1.05)^1 + (1.05)^2 + (1.05)^3 + (1.05)^4} \]

\[ C = \frac{17,820.73}{964.54} = \frac{16,163.93}{964.54} = 16.758 \]

To make the convexity of a semi-annual bond comparable to that of an annual bond, we divide the outcome by 4.

In general, to convert convexity to an annual figure, divide by \( m^2 \), where \( m \) is the frequency of payments per year.
Bond Price Convexity

30-Year Maturity, 8% Coupon; Initial Yield to Maturity = 8%
Taking together Duration and Convexity

- Consider a 30-year bond with a face value of $1,000, a coupon rate of 8% (paid once a year), and a YTM of 8%.

- You can find that the modified duration of the bond is 11.26 years while the convexity is 212.43.

- Assume that the YTM increases to 10%.

- The new price is $811.46, a 18.85% drop.
Taking together Duration and Convexity

- Using duration only, the bond price change is

\[-11.26 \times 0.02 = -22.52\%\]

- Incorporating convexity, the change is given by

\[-11.26 \times 0.02 + \frac{1}{2} \times 212.43 (0.02)^2 = -18.27\%\]

- Much more accurate!! *Even so – duration is a first-order effect while convexity is only a second order.*
Why do Investors like Convexity?

- The more convex a bond price is -- the more valuable the bond is.
- The notion is that bonds with greater curvature gain more in price when yields fall than they lose when yields rise.
- If interest rates are volatile this asymmetric response of prices is desirable.
- The higher the convexity is the lower the bond yield is.
- Bonds that are not callable are said to have “positive convexity.”
Impact of Convexity

- Notice the **convex shape** of price-yield relationship

- Bond 1 is more convex than Bond 2
- Price falls at a slower rate as yield increases
Example: Duration and Convexity of Bond Portfolios

You observe the following default-free non-callable term structure of interest rates:

<table>
<thead>
<tr>
<th>Convexity</th>
<th>Macaulay duration</th>
<th>Modified duration</th>
<th>Price</th>
<th>Yield (APR)</th>
<th>Coupon</th>
<th>Maturity</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.417</td>
<td>1</td>
<td>0.97</td>
<td>94.434$</td>
<td>5.81%</td>
<td>0.00%</td>
<td>1</td>
<td>T bill</td>
</tr>
<tr>
<td>4.426</td>
<td>1.91</td>
<td>1.85</td>
<td>100.130$</td>
<td>6.18%</td>
<td>6.25%</td>
<td>2</td>
<td>T note</td>
</tr>
<tr>
<td>8.895</td>
<td>2.78</td>
<td>2.69</td>
<td>100.094$</td>
<td>6.34%</td>
<td>6.38%</td>
<td>3</td>
<td>T note</td>
</tr>
<tr>
<td>21.278</td>
<td>4.34</td>
<td>4.2</td>
<td>100.273$</td>
<td>6.56%</td>
<td>6.63%</td>
<td>5</td>
<td>T note</td>
</tr>
<tr>
<td>64.694</td>
<td>7.38</td>
<td>7.14</td>
<td>101.507$</td>
<td>6.79%</td>
<td>7.00%</td>
<td>10</td>
<td>T note</td>
</tr>
<tr>
<td>264.67</td>
<td>13.3</td>
<td>12.8</td>
<td>87.865$</td>
<td>6.97%</td>
<td>6.00%</td>
<td>30</td>
<td>T bond</td>
</tr>
</tbody>
</table>
Example: Continued

- You are a manager of a $100m pension fund and can only invest in the bonds listed above. You want a portfolio with Macaulay duration of 5 years.

- There are many ways using these securities to create portfolios with Macaulay durations of 5 years. Create two different portfolios both having Macaulay durations of 5 years. Which one would you prefer? Why? [Explain clearly all the steps you took to arrive at the answer.]
Solution: Bond Portfolios

Many possible combinations. Here are two:

**Portfolio A:**

<table>
<thead>
<tr>
<th>Market Value</th>
<th>Face</th>
<th>Weight</th>
<th>Bond</th>
</tr>
</thead>
<tbody>
<tr>
<td>$78.290 M</td>
<td>$78.076 M</td>
<td>78.3%</td>
<td>5 – year note</td>
</tr>
<tr>
<td>$21.710 M</td>
<td>$21.388 M</td>
<td>21.7%</td>
<td>10 – year note</td>
</tr>
</tbody>
</table>

Macaulay Duration = 5 years

Convexity = 30.704

**Portfolio B:**

<table>
<thead>
<tr>
<th>Market Value</th>
<th>Face</th>
<th>Weight</th>
<th>Bond</th>
</tr>
</thead>
<tbody>
<tr>
<td>$67.453 M</td>
<td>$71.729 M</td>
<td>67.5%</td>
<td>1 – year bill</td>
</tr>
<tr>
<td>$32.547 M</td>
<td>$37.042 M</td>
<td>32.5%</td>
<td>30 – year bond</td>
</tr>
</tbody>
</table>

Macaulay Duration = 5 years

Convexity = 87.097

You would probably prefer B, because it has higher convexity. Then, if interest rates fall, B’s value will increase more than A’s. If interest rates rise, B’s value will fall less than A’s.
Bonds with Embedded Options

- Duration need not reflect the average time to maturity.

- It still measures the bond price sensitivity to changing yields.
Callable Bond

- The buyer of a callable bond has written an option to the issuer to call the bond back.

- Rationally, this should be done when ...
  ... interest rate falls and the debt issuer can refinance at a lower rate.
Puttable Bond

- The buyer of such a bond can request the loan to be returned.

- The rational strategy is to exercise this option when interest rates are high enough to provide interesting alternative investments.
Embedded Call Option

regular bond

callable bond

strike

r
Embedded Put Option

- puttable bond
- regular bond
Floating Rate Bonds

- In the absence of embedded options as well as floating coupon rates, duration measures the average time to maturity as shown earlier.

- Let us now deal with floaters and inverse floaters – here duration has almost nothing to do with the time to maturity.
Understanding Floaters

- Floating rate bonds are medium to long term coupon bonds

- The rate for each coupon period is determined with reference to some relevant short-term money-market rate, most often LIBOR.

- So periodical coupons are not fixed.
Understanding Floaters

- The date at which the floating coupon rates are recalculated are known as resettlement dates or re-pricing dates.

- Usually, the re-pricing date is two business days before the beginning of the coupon accrual period (in line with the money market convention of $T+2$ settlement).

- Floater coupons are usually paid semiannually or quarterly, in arrears.
Pricing a LIBOR-Flat Floater at Repricing Date

- Let us start with the simple case of determining the price and duration of a floater that pays (and is expected to pay) the reference rate flat (no spread).

- We also assume that LIBOR-flat is all that the market requires and is expected to require in the future.
Pricing a LIBOR-Flat Floater at Repricing Date

- At each re-pricing date (usually semiannually) the floater should quote at par because its coupon rate is, by definition, perfectly in line with market conditions.

- Therefore, its price does not have to compensate for a discrepancy between the contractual rate and the market yield.

- This is shown in greater detail in the exhibit on the next page, which relies on the method of backward induction.
# Pricing a Floater at Re-pricing Date

<table>
<thead>
<tr>
<th></th>
<th>T-3t</th>
<th>T-2t</th>
<th>T-t</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIBOR</td>
<td>3.00%</td>
<td>3.60%</td>
<td>4.00%</td>
<td>----</td>
</tr>
<tr>
<td>Principal</td>
<td>-----</td>
<td>----</td>
<td>-----</td>
<td>100.00</td>
</tr>
<tr>
<td>Interest payment</td>
<td>----</td>
<td>1.50</td>
<td>1.80</td>
<td>2.00</td>
</tr>
<tr>
<td>FRN Price</td>
<td>100.00</td>
<td>100.00</td>
<td>100.00</td>
<td>---</td>
</tr>
</tbody>
</table>

Backward induction to determine the price of a floating rate note at LIBOR flat. Interest paid semiannually in arrears. Principal=100.
Decomposing the Floater

Notice that the floater can be decomposed into two components PO and IO:

a. PO: A zero coupon bond – easy to compute duration.
b. IO: An annuity paying Libor rate at semi-annual intervals – what is the duration of such an annuity? Nice question!!!
Duration of the Annuity

- Consider a 5 year 100 face value floater paying Libor at semi annual intervals (the discount rate is also Libor). The annual effective spot rate for a five year discounting is 4%. What is the duration of the annuity at a reset date?
- The value of annuity is $100 - pv(100,5,4\%) = 17.807$.
- The modified duration of the zero coupon bond is $5/1.04 = 4.808$
- The modified duration of annuity solves:
  $0.82193 \times 4.808 + 0.17807 \times MD = 0$
- The MD is $-22.19$ - what does negative duration mean?
Convexity of the Annuity

- Follow the same steps.
- First find the convexity of the zero coupon.
- The computed convexity is 27.74.
- Then solve:
  \[ 0.82193 \times 27.74 + 0.17807 \times \text{Convexity} = 0 \]
- The convexity of the annuity is \(-128.04\).
- This means that there is a positive (negative duration) and concave (negative convexity) relation between the price of the annuity and the yield to maturity.
Pricing between Repricing Dates

Consider a floater paying LIBOR flat every six months. Assume that:

- The time before the next coupon payment is \( t = \frac{5}{12} \). Therefore, we are one month into the coupon accrual period.
- The coupon that will be paid at the end of the interest period is \( 1.25 \) $ (annual coupon =2.5%).
- The current level of LIBOR for 5 month deposit is 0.8% (already adjusted for the days/360 day-count)

\[
P = \frac{100 \left( 1 + \frac{1}{2} \times 2.5\% \right)}{1 + 0.008 \times \frac{5}{12}} = 100.9136
\]
Duration between Re-pricing Dates

We first have to assess the price change when the interest rate rises by 0.01% = 0.0001.

The price change is
\[101.25 \times \left[ \frac{1}{1 + 0.0081 \times 5/12} - \frac{1}{1 + 0.008 \times 5/12} \right] = -0.00419\]

The price change in % terms
\[
\frac{0.00419}{100.9136} = 0.00419\%
\]

\[Duration = \frac{\% Price Change}{0.0001} = 0.4152\]

Check if this duration figure matches with the duration of a zero coupon bonds maturing in 5 months.
Accounting for Spreads

- Coupon rate: L+C
- Discount rate: L+S

- If S<C - premium; floater price is above par.
- If S>C - discount; floater price is below par.
- If S=C par.

To price floaters with spreads – it is essential to have the yield curve of LIBOR rates. As discussed later, such yield curve could be derived using Eurodollar futures.
Inverse Floater (IF)

- IF is usually created from a fixed rate security.

- Consider a $1000 face value collateral paying coupon of 5%.

- You divide that collateral into $500 Floater and $500 IF.

- Floater coupon is LIBOR + 1%

- Inverse Floater coupon = 9% - LIBOR (we will explain later)

- There is typically a floor such that 9%-LIBOR won’t be negative, perhaps even slightly positive.
Inverse Floater (IF)

- Contract rates move inversely to the reference rate.

- If the Libor increases the IF’s coupon diminishes.

- Nowadays, IFs are not as widely issued as before.

- During the 90-s many IF holders lost money on IFs because of their unexpectedly large duration. Back then interest rates sharply advanced.
Inverse Floater (IF)

- IFs provides a very interesting illustration of how one can create a highly levered financial product by combining two plain vanilla securities: a fixed rate bond and a floater.

- There are several ways one can structure an IF.

- We demonstrated a straightforward mechanism taking a plain vanilla fixed-coupon bond, known as collateral, and issuing a floater and IF for a total part amount equal to that of the fixed rate bond.

- The usual proportion is 50-50, not a must.
Finding the IF Coupon

- Consider a 5% $1,000 coupon bond maturing in five years.
- This bond establishes a collateral, from which we construct $500 Floater paying Libor and $500 IF. What is the % annual coupon paid by the IF?
- The bond pays $50 coupon. The $ coupon paid by the floater is 500L. So the $ coupon paid by the IF is 50-500L.
- In % it amounts to 10%-L.
- We next study the pricing, duration, and convexity of IF.
### Pricing IF

<table>
<thead>
<tr>
<th>Six-month floating rates</th>
<th>CF per 200$ Face-Value Divided to 2 Bonds</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Floater 100 Face Value</td>
<td>IF 100 Face Value</td>
<td>Sum 200 Face Value</td>
<td></td>
</tr>
<tr>
<td>4%</td>
<td>4</td>
<td>6</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>5%</td>
<td>5</td>
<td>5</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>6%</td>
<td>6</td>
<td>4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>7%</td>
<td>7</td>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>8%</td>
<td>108</td>
<td>102</td>
<td>210</td>
<td></td>
</tr>
</tbody>
</table>
Pricing IF

- Note: cash flows of floater plus inverse floater same as two 5% coupon bonds.
- By law of one price:
  
  price of floater + price of inverse floater = price of two 5% coupon bonds
  
  \[ P_f + P_I = 2P_5 \]
  
  \[ P_I = 2P_5 - P_f \]

- What is the IF duration?
- What is the IF convexity?
IF Coupon: Departing from 50:50

Q: Consider a 5% $1,000 coupon bond maturing in five years. This bond establishes a collateral, from which we construct the fraction $Q$ as Floater paying Libor and $1000(1-Q)$ IF. What is the % annual coupon paid by the IF?

A: The $ coupon paid by the coupon bond is 50. The $ coupon paid by the floater is 1000L. So the $ coupon paid by the IF is $50 - 1000 \times Q \times L$.

In % -- it amounts to $\frac{0.05}{1-Q} - \frac{Q}{1-Q} \times L$. 
IF Coupon: Example

- Check: if $Q=0.5$ then the % coupon is 10%-L.

- Example: if $Q=0.75$ then % coupon is 20%-3L
Why is Duration so High?

When rates increase IF gets double hit:

a. PV of future cash flows diminishes.
b. Coupon diminishes.

On the next pages - there are several important practice questions.
End of Chapter Questions

Question 1

A bond is trading at a price of 70.357 with a yield of 9%. If the yield increases by 10 bp, the price of the bond will decrease to 69.6164. If the yield decreases by 10 bp, the price will increase to 71.1105. What is the approximated convexity of this bond?
Answer Q1

Convexity = \frac{P^+ + P^- - 2P_0}{P_0 \Delta Y^2} = 183.85
Consider a 2-year 1000 face value bond with a coupon of 4% and YTM of 5%, both are for a six month period. The bond price is 965.54. What is the bond duration?
Answer Q2

\[ D = \frac{40}{(1.05)^1} + \frac{40}{(1.05)^2} (2) + \frac{40}{(1.05)^3} (3) + \frac{1040}{(1.05)^4} (4) \]

\[ \frac{\text{ }964.54}{\text{ }964.54} = 3.77 \]

Note that this is 3.77 six-month periods, which is about 1.89 years.
Consider a 10 year zero coupon bond with a semiannual yield of 3%.

- **Q3**: What is the modified duration of the bond?
- **Q4**: What is its convexity?
The bond price is

\[ P = \frac{100}{(1 + 0.03)^{20}} = 55.368 \]

The duration is 10 years, the modified duration is:

\[ D^* = \frac{10}{(1 + 0.03)} = 9.71 \]
Answer Q4

- The convexity per six months is

\[
\frac{20^2 + 20}{(1 + 0.03)^2} = 395.83
\]

- The one-year convexity is

\[
0.25 \times 395.83 = 98.97
\]
Question 5

If the annual yield of the zero coupon bond changes to 7% what are the approximated and actual price change?

\[ \Delta P = -9.71 \times 55.37 \times 0.01 + 98.97 \times 55.37 \times 0.01^2 \times 0.5 \]
\[ = -5.375 + 0.274 = -5.101 \]

\[ \Delta P = \frac{100}{(1 + 0.035)^{20}} - \frac{100}{(1 + 0.03)^{20}} = -5.111 \]
Question 6

What is the IF coupon if the coupon bond pays C% and the fraction of floater is Q?

Answer:

- Total coupon: $100 \times C$
- Floater coupon: $100 \times L \times Q$
- IF coupon: $100 \times C - 100 \times L \times Q$
- In %: $\frac{C-LQ}{1-Q}$.
- Example: if Q=0.5 and C=5% we get 10%-L.
- If Q=0.6 and C=5% we get 12.5%-1.5L.
Question 7

What is the price of a 10 bp increase in yield on a 10-year par bond with a modified duration of 7 and convexity of 50?
\[
\Delta P = -D \times P \times \Delta y + \frac{\Delta y^2}{2} C \times P
\]

\[
= -7 \times 100 \times 0.001 + \frac{0.001^2}{2} \times 50 \times 100 = -0.6975
\]
Question 8

A portfolio consists of two positions. One is long $100 of a two year bond priced at 101 per 100 face value with a duration of 1.7; the other position is short $50 of a five year bond priced at 99 for 100 face value with a duration of 4.1. What is the modified duration of the portfolio?
Answer Q8

\[ D = D_1 \frac{P_1}{P} + D_2 \frac{P_2}{P} \]

\[ = 1.7 \frac{101}{101 - 49.5} + 4.1 \frac{-49.5}{101 - 49.5} = -0.61 \]
You have a large short position in two bonds with similar credit risk. Bond A is priced at par yielding 6% with 20 years to maturity. Bond B has 20 years to maturity, coupon 6.5% and yield of 6%. Which bond contributes more to the risk of the portfolio?

a. Bond A  
b. Bond B  
c. A and B have similar risk  
d. None of the above
Construct a portfolio of two bonds: A and B to match the value and duration of a 10-year, 6% coupon bond with value $100 and modified duration of 7.44 years.

a. 1 year zero bond - price $94.26

b. 30 year zero - price $16.97
Question 11

A brokerage firm purchases a 1000 face value coupon bond paying 6% per year on a semi-annual frequency. From that bond the firm creates two bonds: Floater and IF. The face value of the floater is 700 and it pays LIBOR.

What is the annual coupon of the IF?
Questions 12, 13

A 100 face value floater is paying LIBOR on a semi-annual frequency. During the last reset LIBOR=4%. Assume that three months before the payment of next coupon the LIBOR is rising to 4.2%.

- **Q12**: What is the duration of the floater on this date?
- **Q13**: What is its convexity?
More Questions to Practice

Q: State why you would agree or disagree with the following statement: When interest rates are low, there will be little difference between the Macaulay duration and modified duration measures.

A: The Macaulay duration is equal to the modified duration times one plus the yield. Rearranging this expression gives:

$$\text{modified duration} = \frac{\text{Macaulay duration}}{1 + y}$$

It follows that the modified duration will approach equality with the Macaulay duration as yields approach zero. Thus, if by low interest rates one means rates approaching zero, then one would agree with the statement.
More Questions

Q: Given the following information about bonds A, B, C, and D

<table>
<thead>
<tr>
<th>Bond</th>
<th>Price</th>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>96</td>
<td>100</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>91</td>
<td>0</td>
<td>100</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>80</td>
<td>0</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>D</td>
<td>?</td>
<td>10</td>
<td>10</td>
<td>110</td>
</tr>
</tbody>
</table>

Assume that bond prices admit no arbitrage opportunities. What is the modified duration of bond D?
More Questions

A:

100D1 = 96
100D2 = 91
100D3 = 80

Get: D1 = 0.96, D2 = 0.91, D3 = 0.80

Price bond D = 10 \times 0.96 + 10 \times 0.91 + 110 \times 0.80 = 106.7

The modified duration of bond D is 2.58
The Yield Curve
The Yield Curve

- First step of any fixed income exercise is the analysis of the term structure of interest rates also termed - the yield curve.

- The yield curve summarizes the relation between spot rates and time to maturity for various types of bonds.

- As noted earlier, spot rates are the engine of bond pricing.
YTM vs. Spot Rates

- While YTM apply only to specific cash flow pattern of a bond, spot rates apply to any cash flows of similar bonds.

- Default free spots are basis for pricing all default free fixed income securities.

- Default free spots also constitute a reference level for pricing defaultable fixed income instruments – add the credit spread and often also the illiquidity premium.
Three Characteristics Underlying the Yield Curve

a. The change in yields of different term bonds tends to move in the same direction.

b. The yields on short-term bonds are more volatile than the yields on long-term bonds.

c. The yields on long-term bonds *tend* to be higher than those of short-term bonds. There are some exceptions in which the yield curve is either flat or inverted.
Yield Curve is Issuer Specific

- Governments
- Corporate
- Municipals
- Others ...
Current Yield Curve: Try this one

Shapes

- Upward sloping
- Downward sloping
- Hump-shaped
- Flat
Types of Yield Curve

- There is no single yield curve describing the cost of money for everybody. The most important factor in determining a yield curve is the currency in which the securities are denominated.

- From the post-Great Depression era to the present, the yield curve has usually been “normal“ - the longer the maturity, the higher the yield.

- A flat curve sends signals of uncertainty in the economy. This mixed signal can revert to a normal curve or could later result into an inverted curve.
Types of Yield Curve

- Inverted yield curve – Under unusual circumstances, long-term investors will settle for lower yields now if they think the economy will slow or even decline in the future.

- In 1986, Campbell R. Harvey showed that an inverted yield curve accurately forecasts U.S. recessions. Actually, all 7 recessions in the US since 1970 (up through 2015) have been preceded by an inverted yield curve.
The slope of the yield curve is believed to predict future economic growth, inflation, and recessions.

In addition to potentially signaling an economic decline, inverted yield curves also imply that the market believes inflation will remain low.
Curve’s Theory

Market expectations (pure expectations) hypothesis:
- If future interest rates are expected to rise, then the yield curve slopes upward, with longer term bonds paying higher yields and vice versa.

Market segmentation theory:
- The supply and demand in the markets for short-term and long-term instruments is determined largely independently.

Preferred habitat theory:
- In addition to interest rate expectations, investors have distinct investment horizons and require a meaningful premium to buy bonds with maturities outside their "preferred" maturity, or habitat.
Market Segmentation
and Preferred Habitat Theories

- The “Segmented Markets Theory” states that most investors have set preferences regarding the length of maturities that they will invest in.

- Market segmentation theory maintains that the buyers and sellers in each of the different maturity lengths cannot be easily substituted for each other.
An offshoot to this theory is that if an investor chooses to invest outside their term of preference, they must be compensated for taking on that additional risk. This is known as the Preferred Habitat Theory.

The theory implies that there is no connection between interest rates for different maturities.
Factors Affecting Bond Yields

- Base interest rate – typically rate on treasuries
- Risk Premium – due to credit risk
- Liquidity premium
- Market forces - demand and supply
- Taxation
Issues

- T bill and STRIPS data are not as smooth as we would like

Yield curve

- Reasons
  - Differences in liquidity
  - Stale quotes
  - Rounding
**Issues (cont.)**

- STRIPS data are sparse at long maturities

**Yield curve**

<table>
<thead>
<tr>
<th>Yield (%)</th>
<th>3.0</th>
<th>3.3</th>
<th>3.5</th>
<th>3.8</th>
<th>4.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Years to maturity</td>
<td>8</td>
<td>8.5</td>
<td>9</td>
<td>9.5</td>
<td>10</td>
</tr>
</tbody>
</table>

- Reasons
  - STRIPS maturities are tied to Treasury auction cycle and coupon payment dates
Solution 1 – Interpolation

- Linear interpolation, draw straight line between two points that straddle a desired maturity.

![Diagram showing yield curve with points A, B, C, and D, and a desired maturity line.](image)
Solution 1 – Interpolation

- **Linear interpolation:**
  - Resolves sparseness problem.
  - Uses little information from whole term structure (only 2 points).
  - Does not smooth the data and is sensitive to outliers.

- Higher-order interpolations (e.g., cubic splines) use more information but still fit all points exactly (i.e., no smoothing).
Solution 2 – Yield Regression

- Specify model for spot rate as function of years to maturity $t$
  $$s = f(t; \alpha)$$
  and estimate

- Two common specifications
  $$s(t) = \alpha_0 + \alpha_1 t + \alpha_2 \ln t + \varepsilon(t)$$
  or
  $$s(t) = \alpha_0 + \alpha_1 t + \alpha_2 \ln \left(\frac{1}{t}\right) + \varepsilon(t)$$

  where $t$ denotes years to maturity.
Solution 2 – Yield Regression

- Both specifications are problematic for short maturities

- Better specification

\[ s(t) = \alpha_0 + \alpha_1 t + \alpha_2 \ln(1 + t) + \alpha_2 \left( \frac{1}{1 + t} - 1 \right) + \varepsilon(t) \]

- Intercept = \( \alpha_0 \) = instantaneous spot rate, or the rate at time 0.
Solution 2 – Yield Regression

Solution 2: Yield regression (cont)

- Using Excel regression tool

<table>
<thead>
<tr>
<th>Regression Statistics</th>
<th></th>
<th>Instantaneous spot rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiple R</td>
<td>0.9978</td>
<td></td>
</tr>
<tr>
<td>R Square</td>
<td>0.9958</td>
<td></td>
</tr>
<tr>
<td>Adjusted R Square</td>
<td>0.9957</td>
<td></td>
</tr>
<tr>
<td>Standard Error</td>
<td>0.0007</td>
<td></td>
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<tr>
<td>Observations</td>
<td>157</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>Standard Error</th>
<th>t Stat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>0.0013</td>
<td>0.0003</td>
</tr>
<tr>
<td>( t )</td>
<td>-0.0005</td>
<td>0.0002</td>
</tr>
<tr>
<td>( \ln(1+t) )</td>
<td>0.0334</td>
<td>0.0016</td>
</tr>
<tr>
<td>( 1/(1+t)-1 )</td>
<td>0.0411</td>
<td>0.0026</td>
</tr>
</tbody>
</table>
Solution 2 – Yield Regression

- Assuming an adequate fit, the zero coupon yield for any maturity can be computed from the fitted value of the regression.

\[ s(t) = 0.0013 - 0.0005 \times t ... 
+ 0.0334 \times \ln(1 + t) + 0.0411 \left( \frac{1}{1 + t} - 1 \right) + \epsilon(t) \]

- But extrapolation beyond the range of the data is dangerous, especially if model is fitted only to short (<=5 yrs) maturities.
Solution 2 – Yield Regression
Solution 3 – Discount Rate Regression

- Specify model for discount factor as function of years to maturity $t$:

$$ P(t) = f(f; \alpha) $$

and estimate

- A sensible specification

$$ P(t) = 100 \times e^{\alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \epsilon(t)} $$

or in logs

$$ \ln P(t) - \ln 100 = \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \epsilon(t) $$
Solution 3 – Discount Rate Regression

### Solution 3: Discount factor regression (cont)

- **Using Excel regression tool**

<table>
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<tr>
<th>Regression Statistics</th>
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</thead>
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<td></td>
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</tr>
<tr>
<td>R Square</td>
<td>0.9997</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Adjusted R Square</td>
<td>0.9933</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Standard Error</td>
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<td></td>
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<tr>
<td>Observations</td>
<td>157</td>
<td></td>
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<table>
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<tr>
<th>Coefficients</th>
<th>Standard Error</th>
<th>t Stat</th>
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</thead>
<tbody>
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<td>na</td>
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<td>0.0044</td>
<td>0.0002</td>
</tr>
<tr>
<td>t^2</td>
<td>-0.0075</td>
<td>0.0001</td>
</tr>
<tr>
<td>t^3</td>
<td>0.0003</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
Solution 3 – Discount Rate Regression

- Assuming an adequate fit, the PV of a 100$ cashflow at any maturity can be computed from the fitted value of the regression

\[
\hat{P}(t) = 100 \times e^{0.0044t-0.0075t^2+0.0003t^3}
\]

- Again, be careful with extrapolation.
Solution 3 – Discount Rate Regression

- Fitted discount factors

<table>
<thead>
<tr>
<th>Price</th>
<th>Years to maturity</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0</td>
</tr>
<tr>
<td>90</td>
<td>1</td>
</tr>
<tr>
<td>80</td>
<td>2</td>
</tr>
<tr>
<td>70</td>
<td>3</td>
</tr>
<tr>
<td>60</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>10</td>
</tr>
</tbody>
</table>
Yield Curve from Coupon Bonds

- So far we have had the luxury of having spot rates for various maturities due to the presence of T-Bills and Strips.

- What if we don’t observe spot rates but only have information on coupon bonds – prices, coupons, and timing?

- How could we generate the yield curve?
Generating the Yield Curve

- **Objective is to minimize pricing errors when fitted rates are used for discounting.**

- **Pricing error**
  - Difference between observed and fitted coupon bond prices
  \[ c(n) = P(n) - \hat{P}(n) \]

- **Fitted coupon bond price**
  \[ \hat{P}(n) = \frac{c/m}{1 + S\left(\frac{1}{m}; \alpha\right)} + \frac{c/m}{1 + S\left(\frac{2}{m}; \alpha\right)}^2 + \cdots + \frac{F + c/m}{1 + S\left(\frac{n}{m}; \alpha\right)^n} \]
Generating the Yield Curve

With (for example)

\[ s(t; \alpha) = \alpha_0 + \alpha_1 t + \alpha_2 \ln(1 + t) + \alpha_2 \left( \frac{1}{1 + t} - 1 \right) + \varepsilon(t) \]
Compute the quantity
\[ c(n) = P(n) - \hat{P}(n) \]
for any coupon bond.

Then minimize either of the following:

a. Sum of \(|c(n)|\) across bonds.

b. Sum of squares of \(c(n)\) across bonds.

This optimization yields \(\alpha_0, \alpha_1, \alpha_2,\) and \(\alpha_3\).
The Popular Nelson Siegel Model

- Nelson-Siegel functions take the form
  \[ s(m) = \beta_0 + \beta_1 \frac{[1 - \exp(-m/\tau)]}{m/\tau} + \beta_2 \left( \frac{[1 - \exp(-m/\tau)]}{m/\tau} - \exp(-m/\tau) \right) \]

  where \(\beta_0, \beta_1, \beta_2, \tau\) are parameters to be fitted as shown above.

- \(\beta_0\) is the intercept of the long run levels of interest rate (the loading is 1, it is constant that does not decay);

- \(\beta_1\) is the short-term component (it starts at 1, and decays monotonically and quickly to 0);
The Popular Nelson Siegel Model

- $\beta_2$ is the medium-term component (it starts at 0, increases, then decays to zero);

- $\tau$ is the decay factor: small values produce slow decay and can better fit the curve at long maturities, while large values produce fast decay and can better fit the curve at short maturities; $\tau$ also governs where $\beta_2$ achieves its maximum.
The Svensson Extension

Svensson (1994) adds a “second hump” term; this is the Nelson-Siegel-Svensson (NSS) term structure model

\[ + \beta_3 \left( \frac{1 - \exp \left( -\frac{m}{\tau_2} \right)}{m/\tau_2} \right) - \exp \left( -\frac{m}{\tau_2} \right) \]
Yield Curve Arbitrage: Revisited

- Now that you know what the yield curve is – we can revisit the yield curve arbitrage strategy, noted earlier.

- Yield curve arbitrage strategies are designed to profit from shifts in the steepness of or kinks in the Treasury yield curve by taking long and short positions on various maturities.
Yield Curve Arbitrage

- This could take the form of a butterfly trade, where the investor goes long five-year bonds and shorts two and ten-year bonds. Or, it may take the form of a spread trade, where, the investor goes short the front end of the curve and long the back end of the curve.

- The strategy requires the investor to identify some points along the yield curve that are perceived to be cheap or expensive.
Forward Rates

- Spot rates apply to a zero-coupon loan which starts today and goes until a specified time in the future.

- Forward rates apply to a zero-coupon loan which starts at a specified time in the future and goes until a later time in the future.

- Forward rates are linked to spot rates by the law of one price.
Forward Rates

- In other words, we can lock now interest rate for a loan which will be taken in future.

- To specify a forward interest rate one should provide information about
  - today’s date
  - beginning date of the loan
  - end date of the loan
Example: Three-Period Investment

- $m=2$ and spot rates $s(1)=0.17\%$, $s(2)=0.36\%$, and $s(3)=0.56\%$

- Payoff #1
  - At $n=3$, $\$100 \times \left(1 + \frac{0.0056}{2}\right)^3 = \$100.8424$

- Payoff #2
  - At $n=2$, $\$100 \times \left(1 + \frac{0.0036}{2}\right)^2 = \$100.3603$
  - At $n=3$, $\$100 \times \left(1 + \frac{0.0036}{2}\right)^2 \times \left(1 + \frac{f(2,3)}{2}\right) = \$100.3603 \times \left(1 + \frac{f(2,3)}{2}\right)$
Example: Continued

- By law of one price

\[
1 + \frac{f(2,3)}{2} = \frac{100.8424}{100.3603} = 1.0048 \Rightarrow f(2,3) = 0.96\%
\]

- If \( f(2,3) > 0.96\% \) way #2 is better.

- If \( f(2,3) < 0.96\% \) way #1 is better.
What is the Zero Coupon Bond Price if…

If we knew the one-year future interest rates:

<table>
<thead>
<tr>
<th>Year</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (Today)</td>
<td>8%</td>
</tr>
<tr>
<td>1</td>
<td>10%</td>
</tr>
<tr>
<td>2</td>
<td>11%</td>
</tr>
<tr>
<td>3</td>
<td>11%</td>
</tr>
</tbody>
</table>
Zero Coupon Bond Price

The bond price is solved as

\[
P = \frac{\$1,000}{(1 + 0.08)(1 + 0.10)(1 + 0.11)(1 + 0.11)}
\]
Future versus Spot Rates

\[ f_{01} = 8\% \quad f_{12} = 10\% \quad f_{23} = 11\% \quad f_{34} = 11\% \]

\[
\frac{1}{(1+s_2)^2} = \frac{1}{1.08 \times 1.10} \quad s_2 = 8.995\%
\]

\[
\frac{1}{(1+s_3)^3} = \frac{1}{1.08 \times 1.10 \times 1.11} \quad s_3 = 9.66\%
\]

\[
\frac{1}{(1+s_4)^4} = \frac{1}{1.08 \times 1.10 \times 1.11 \times 1.11} \quad s_4 = 9.993\%
\]
Example

Q: The prices of zero coupon bonds maturing in one, two, three, and four years (per 1$ face value) are 0.9524, 0.8900, 0.8278, and 0.7629 respectively. An investor wants to borrow $20 Million 3 years from now for one year. How could the investor do it by buying and selling those zero coupon bonds? What is the effective rate on the loan?

A: Buy 20 M of three year zero coupon bond - you pay $0.8278 \times 20 = 16.556$. Sell the same dollar amount of four-year zero coupon bond, i.e., you sell $16.556 / 0.7629 = 21.701403$.

Notice that cash flows in years 0, 1, 2 and zero, while in the third and fourth years the cash flows are +20 and -21.701 respectively. Notice that the rate on the loan is 8.51%.
End of Chapter Questions

Question 1

Prices and yield to maturity on zero-coupon bonds ($1,000 face value) are given by

<table>
<thead>
<tr>
<th>Maturity (years)</th>
<th>Yield to Maturity (%)</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5%</td>
<td>$952.38 = $1,000/1.05</td>
</tr>
<tr>
<td>2</td>
<td>6%</td>
<td>$890.00 = $1,000/1.06²</td>
</tr>
<tr>
<td>3</td>
<td>7%</td>
<td>$816.30 = $1,000/1.07³</td>
</tr>
<tr>
<td>4</td>
<td>8%</td>
<td>$735.03 = $1,000/1.08⁴</td>
</tr>
</tbody>
</table>

Compute all possible forward rates based on these zero coupon bond prices.
Answer Q1

Alternative 1: Buy and hold 2-year zero

$890 \times 1.06^2 = $1000

Alternative 2: Buy a 1-year zero, and reinvest proceeds in another 1-year zero

$890 \times 1.05 = $934.50$

$934.50(1 + r_2) = $1000$

Professor Doron Avramov, Fixed Income Securities
Answer Q1

<table>
<thead>
<tr>
<th>Year</th>
<th>f_{0,1} = 5%</th>
<th>f_{1,2} = 7.01%</th>
<th>f_{2,3} = 9.025%</th>
<th>f_{3,4} = 11.06%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Short Rate in Each Year

Current Spot Rates (Yields to Maturity) for Various Maturities

1-Year Investment
2-Year Investment
3-Year Investment
4-Year Investment

\begin{array}{l}
\text{s}_1 = 5\% \\
\text{s}_2 = 6\% \\
\text{s}_3 = 7\% \\
\text{s}_4 = 8\% \\
\end{array}
Question 2

If \( s_5 = 7\% \), \( s_{10} = 8\% \), and \( f_{5,7} = 7.5\% \), what is \( f_{7,10} \)?
Questions 3, 4

The table on the next page describes yields on five non callable government bonds. Assume that we added the sixth bond which pays $5 every six months during the second and third years (overall four payments).

- **Q3**: What is the maximal price of that bond?
- **Q4**: What is the minimal price of that bond?
Questions 3, 4 - continued

<table>
<thead>
<tr>
<th>Period</th>
<th>Year</th>
<th>Coupon</th>
<th>YTM</th>
<th>Price</th>
<th>Spot</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.00%</td>
<td>8.0%</td>
<td>96.15</td>
<td>8.00%</td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
<td>0.00%</td>
<td>8.3%</td>
<td>92.19</td>
<td>8.30%</td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
<td>8.50%</td>
<td>8.9%</td>
<td>99.45</td>
<td>8.93%</td>
</tr>
<tr>
<td>4</td>
<td>2.0</td>
<td>9.00%</td>
<td>9.2%</td>
<td>99.64</td>
<td>9.247%</td>
</tr>
<tr>
<td>5</td>
<td>2.5</td>
<td>11.00%</td>
<td>9.4%</td>
<td>103.49</td>
<td>9.468%</td>
</tr>
</tbody>
</table>
Option Pricing with Application to Fixed Income Instruments
We aim...

- ... to use option pricing tools, such as the B&S formula, to price structures, straight zero coupon default-able bonds, as well as default-able bonds convertible to stocks.

- ... to explain concepts such as default premium, recovery rate, risk neutral probability of default, and hazard rate.
Debt and Equity as Options

Consider a firm that has no dividend-paying equity outstanding, and a single zero-coupon debt issue.

- The time $t$ values of the assets of the firm, the debt, and the equity are $A_t$, $B_t$, and $E_t$, respectively.
- The debt matures at time $T$. 
Debt and Equity as Options

- The value of the equity at time $T$ is
  \[ E_T = \max(0, A_T - \bar{B}) \]
  - This is the payoff to a call option.
  - What is the underlying asset?
  - What is the strike price?
Interpreting Default-able Bonds

- The value of the debt is
  \[ B_T = \min(A_T, \bar{B}) \]
  or
  \[ B_T = A_T + \min(0, \bar{B} - A_T) = A_T - \max(0, A_T - \bar{B}) \]
- This implies that corporate bondholders could be viewed as those owning the firm assets, but have written a call option on the firm assets to the equity-holders.
- Or the bondholders own a default free bond and have written a put option on the firm assets.
- You can use the call-put parity to verify that both perspectives are indeed equivalent.
Debt and Equity as Options (cont’d)

- Thus, we can compute the value of debt and equity prior to time $T$ using option pricing methods, with the value of assets taking the place of the stock price and the face value of the debt taking the place of the strike price.

- The equity value at time $t$ is the value of a call option on the firm assets. The value of the debt is then $B_t = A_t - E_t$. 
Suppose that a non dividend paying firm issues a zero coupon bond maturing in five years.

The bond’s face value is $100, the current value of the assets is $90, the risk-free rate (cc) is 6%, and the volatility of the underlying assets is 25%.

What is the equity and debt value?

What is the bond’s yield to maturity (ytm)?
The Black-Scholes Formula

- **Call Option price:**
  \[ C(S, K, \sigma, r, T, \delta) = S e^{-\delta T} N(d_1) - K e^{-r T} N(d_2) \]

- **Put Option price:**
  \[ P(S, K, \sigma, r, T, \delta) = K e^{-r T} N(-d_2) - S e^{-\delta T} N(-d_1) \]

\[ d_1 = \frac{\ln \left( \frac{S}{K} \right) + \left( r - \delta + \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \quad , \quad d_2 = d_1 - \sigma \sqrt{T} \]
Default Risk Premium

- The equity value solves the BSCall:
  \[ \text{Equity} = \text{BSCall}(90, 100, 0.25, 0.06, 5, 0) = 27.07 \]

- The debt value is \(90 - 27.07 = 62.93\).

- The debt cc ytm \(= 1/5 \times \ln(100/62.93) = 9.26\%\).

- The ytm is greater than the riskfree rate due to default risk premium.

- The default risk premium is
  \[ \exp(0.0926) - \exp(0.06) = 0.03518. \]
Default Risk Premium
when the Asset Value Increases

What if the asset value is equal to 300?
Equity=BSCall(300,100,0.25,0.06,5,0)=226.08

The debt value is 300-226.08=73.92
The debt cc ytm =1/5×ln(100/73.92)=6.04%
So the risk premium is near zero.
Not a big surprise! The debt is virtually risk free.
Default Risk Premium when Volatility Increases

- What if the asset volatility is 50%?
  \[ \text{Equity} = \text{BSCall}(90, 100, 0.5, 0.06, 5, 0) = 43.20 \]

- The debt value is 90 - 43.20 = 46.80.
- The debt c.c. ytm = \( \frac{1}{5} \times \ln(\frac{100}{46.80}) \) = 15.18%
- So the risk premium is much higher.
- Not a big surprise! The debt is much more risky.
Default Risk Premium
when Time to Maturity Increases

What if the time to maturity is 10 years?

\[ \text{Equity} = \text{BSCall}(90, 100, 0.25, 0.06, 10, 0) = 43.73 \]

The debt value is \( 90 - 43.73 = 46.27 \).

The debt cc ytm \( = \frac{1}{10} \times \ln(\frac{100}{46.26}) = 7.7\% \)

So the risk premium is lower.

Could you explain why?

Try to examine how yield to maturity changes as the face value of debt changes fixing other parameters.
Corporate Credit Risk

- Credit risk premium compensates investors for undertaking credit risk.

- We now study several concepts of corporate credit risk or default risk and credit ratings.

- Then, back to derivatives securities...
Risk Neutral Default Probability

- Suppose that the bond has an assumed recovery rate of \( f \) (e.g., \( f=0.50 \)).

- What is the risk neutral default probability \((q)\)?

- Assuming annual compounding, \( q \) solves:

\[
P = q \times \frac{f \times 100}{\left(1 + y_{risk\,free}\right)^T} + (1 - q) \times \frac{100}{\left(1 + y_{risk\,free}\right)^T}
\]
Risk Neutral Default Probability

- Of course if the compounding is semi-annual the formula becomes

\[
P = q \times \frac{f \times 100}{(1 + \frac{y_{\text{risk free}}}{2})^{2T}} + (1 - q) \times \frac{100}{(1 + \frac{y_{\text{risk free}}}{2})^{2T}}
\]

- Now, since \( P = \frac{100}{(1 + \frac{y_{\text{defaultable}}}{2})^{2T}} \), we get a general formula for risk neutral default probability

\[
q = \frac{1}{1 - f} \times \left[1 - \frac{(1 + \frac{y_{\text{risk free}}}{2})^{2T}}{(1 + \frac{y_{\text{defaultable}}}{2})^{2T}}\right]
\]
Recovery Rate

The recovery rate for a bond, defined by $f$, is usually the price of the bond immediately after default as a percent of its face value.
## Recovery Rates

(Moody’s: 1982 to 2006)

<table>
<thead>
<tr>
<th>Class</th>
<th>Mean (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Senior Secured</td>
<td>54.44</td>
</tr>
<tr>
<td>Senior Unsecured</td>
<td>38.39</td>
</tr>
<tr>
<td>Senior Subordinated</td>
<td>32.85</td>
</tr>
<tr>
<td>Subordinated</td>
<td>31.61</td>
</tr>
<tr>
<td>Junior Subordinated</td>
<td>24.47</td>
</tr>
</tbody>
</table>
Assume $f = 0.5$, default-able and risk free rates are 8% and 4% (APR), respectively. What is $q$?

- For $T=1$, $q=0.08$.
- For $T=5$, $q=0.35$.
- For $T=10$, $q=0.64$.

Clearly risk-neutral (and actual) default probability increases with time to maturity.

Why?
Cumulative vs. Marginal Default Probabilities

- \( CP_1 = MP_1 \)
- \( CP_2 = CP_1 + (1 - CP_1) \times MP_2 \)
- \( CP_3 = CP_2 + (1 - CP_2) \times MP_3 \)

Given cumulative default probabilities we can compute marginal default probabilities.
## Cumulative Average Default Rates (%)

(1970-2006, Moody’s)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.026</td>
<td>0.099</td>
<td>0.251</td>
<td>0.521</td>
</tr>
<tr>
<td>Aa</td>
<td>0.008</td>
<td>0.019</td>
<td>0.042</td>
<td>0.106</td>
<td>0.177</td>
<td>0.343</td>
<td>0.522</td>
</tr>
<tr>
<td>A</td>
<td>0.021</td>
<td>0.095</td>
<td>0.220</td>
<td>0.344</td>
<td>0.472</td>
<td>0.759</td>
<td>1.287</td>
</tr>
<tr>
<td>Baa</td>
<td>0.181</td>
<td>0.506</td>
<td>0.930</td>
<td>1.434</td>
<td>1.938</td>
<td>2.959</td>
<td>4.637</td>
</tr>
<tr>
<td>B</td>
<td>5.236</td>
<td>11.296</td>
<td>17.043</td>
<td>22.054</td>
<td>26.794</td>
<td>34.771</td>
<td>43.343</td>
</tr>
<tr>
<td>Caa-C</td>
<td>19.476</td>
<td>30.494</td>
<td>39.717</td>
<td>46.904</td>
<td>52.622</td>
<td>59.938</td>
<td>69.178</td>
</tr>
</tbody>
</table>
Interpretation

- The table shows the probability of default for companies starting with a particular credit rating.

- A company with an initial credit rating of Baa has a probability of 0.181% of defaulting by the end of the first year, a cumulative probability of 0.506% by the end of the second year, and so on.
Default Intensities versus Unconditional Default Probabilities

- The default intensity (also called hazard rate) is the marginal probability – the probability of default for a certain time period conditional on no earlier default.

- The unconditional default probability is the probability of default for a certain time period as seen at time zero.

- What are the default intensities and unconditional default probabilities for a Caa rated company in the third year?
Default Intensity (Hazard Rate)

Let $V(t)$ be the probability of a company surviving to time $t$. Default intensity could nicely be modeled as

$$V(t + \Delta t) - V(t) = -\lambda(t)V(t)$$

Hence

$$V(t) = e^{-\int_0^t \lambda(t) dt}$$

The cumulative probability of default by time $t$ is then

$$Q(t) = 1 - e^{-\bar{\lambda}(t)t}$$

From this expression you can derive the marginal and unconditional default probabilities.

See end-of-chapter questions.
We now turn to pricing structures, warrants, and zero coupon default-able bonds that are convertible to stocks.
Equity Linked CDs

- The 5.5-year CD promises to reply initial invested amount plus 70% of the gain in S&P500 index
- Assume $10,000 is invested when S&P500 = 1300
- Final payoff = 
  \[10,000 \times \left(1 + 0.7 \times \max \left[0, \frac{S_{final}}{1300} - 1\right]\right)\]
- Where \(S_{final}\) = the value of the S&P500 that will be in 5.5 years
The Economic Value of the Equity Linked CD

- We paid $10,000 and we get $10,000 in 5.5 years plus some extra amount if the S&P500 index level exceeds 1300.

- That payoff structure is equivalent to buying a zero coupon bond and x call options.

- Why? Assuming that the annual effective rate is 6%, the present value of $10,000 to be received in 5.5 years is $7,258.
The Economic Value of the Equity Linked CD

Thus, we practically paid $7,258 for a zero coupon bond and $2,742 for $x$ call options.

What is $x$? And what is the implied value of one call option?

The component of payoff attributable to call option is

\[
\frac{7000}{1300} \times \max[S_{\text{final}} - 1300, 0]
\]
The Economic Value of the Equity Linked CD

- Therefore $x = 5.3846$.
- One call option is priced as $\frac{2742}{5.3846} = 509.23$
- The economic value of one call option is

$$BSCall(1300, 1300, \sigma, \ln(1.06), 5.5, \delta)$$
Convertible Bonds

- A convertible bond is a bond that, at the option of the bondholder, can be exchanged for shares in the issuing company.
- The valuation of a convertible bond entails valuing both default-able bond and a warrant.
- If a firm issues a call option on its own stock, it is known as a warrant.
- When a warrant is exercised, the warrant-holder receives a share worth more than the strike price. This is dilutive to other shareholders.
- The question is how to value a warrant, and how to value the equity given the existence of warrants.
Suppose

- A firm has $n$ shares outstanding
- The outstanding warrants are European, on $m$ shares, with strike price $K$
- The asset value is $A$

At expiration, if the warrant-holders exercise the warrants, they pay $K$ per share and receive $m$ shares
After the warrants are exercised, the firm has assets worth $A + mK$. Thus, at expiration the value of warrant is the max of zero and the following expression

$$\frac{A + mK}{n + m} - K = \frac{n}{n + m} \left( \frac{A}{n} - K \right)$$

**dilution correction factor**

Hence, the warrant can be valued by using

$$\frac{n}{n + m} BSCall \left( \frac{A}{n}, K, \sigma, r, t, \delta \right)$$
Terms in Convertible Bonds

Consider a three-period, 9% bond convertible into four shares of the underlying company’s stock.

a. The *conversion ratio* (CR) is the number of shares of stock that can be converted when the bond is tendered for conversion. The conversion ratio for this bond is four.

b. The *conversion value* (CV) is equal to the conversion ratio times the market price of the stock. If the current price of the stock were 92, then the bond’s conversion value would be $\text{CV} = (4)(\$92) = \$368$.

c. The *conversion premium* (CP) is equal to the strike price of the bond divided by the share price. See example below.
CB: Valuation

Convertible bond = straight bond (of the same issuing firm) + certain amount of warrants.
Convertible Bonds (cont’d)

- Suppose
  - There are $m$ bonds with maturity payment $M$, each of which is convertible into $q$ shares
  - There are $n$ original shares outstanding
  - The asset value is $A$

- If the bonds are converted, there will be $n + mq$ shares.
Convertible Bonds (cont’d)

At expiration, the bondholders will convert if the value of the assets after conversion exceeds the value of the maturity payment per share

\[
\frac{A}{n + mq} - \frac{M}{q} > 0
\]

or

\[
\frac{n}{n + mq} \left( \frac{A}{n} - \frac{M}{q} \frac{n + mq}{n} \right) > 0
\]
Convertible Bonds (cont’d)

>- Do not get it wrong: the stock price is $\frac{A}{n+mq}$ and the strike price is $\frac{M}{q}$

>- The B&S inputs for the stock and strike prices are different from the actual ones in order to make that formula applied to price the warrants component of the convertible bond.
Convertible Bonds (cont’d)

Assuming the convertible is the only debt issue, bankruptcy occurs if \( \frac{A}{m} < M \). Thus, for each bond, the payoff of the convertible at maturity, \( T \), is

\[
\frac{M}{\text{Bond}} - \max \left( 0, \frac{M - A_T}{m} \right) + q \times \frac{n}{n + mq} \times \max \left( 0, \frac{A_T}{n} - \frac{M n + mq}{q n} \right)
\]

Therefore, a single convertible bond can be valued as

- Owning a risk-free bond with maturity payment \( M \)
- Selling a put on \( \frac{1}{m} \) of the firm’s assets, and
- Buying \( q \) warrants on the firm’s assets with strike \( \frac{M n + mq}{q n} \)
Convertible Bonds (cont’d)

- It could also be useful to present the bond price as

\[
\frac{M}{\text{Bond}} - \frac{1}{m} \max(0, mM - A_T) + q \times \frac{1}{n + mq} \times \max\left(0, A_T - \frac{M}{q} n + mq\right)
\]

- Therefore, a single convertible bond can be valued as
  - Owning a risk-free bond with maturity payment \( M \)
  - Selling \( \frac{1}{m} \) put on of the firm’s assets, and
  - Buying \( q/n \) warrants on the firm’s assets with strike \( \frac{M}{q} (n + mq) \)
Example

- Suppose a firm has issued \( m = 6 \) convertible bonds, each with maturity value \( M = \$1,000 \) and convertible into \( q = 50 \) shares. The firm has \( n = 400 \) common shares outstanding. Note that the six bonds have a total promised maturity value of \$6000. Conversion occurs when assets exceed \( \$1,000 \times \frac{700}{50} = \$14,000 \), which is \( A > M \frac{(n+mq)}{q} \).

  The slope of the convertible payoff above \$14,000\ is \( \frac{mq}{n+mq} = \frac{3}{7} \).

- Whenever the assets value is below \$6,000\ the firm is in default. If the value is between \$6,000\ and \$14,000\ the warrant component is zero. The bond will be converted into shares only when the assets value exceeds \$14,000. \[\]
The figure below illustrates the convertible bond value as a function of the company’s total asset value.
Example: Convertible Bond Valuation using the B&S Formula

Suppose a firm has assets worth $10,000 \((A=10000)\) with a single debt issue consisting of six zero-coupon bonds \((m=6)\), each with a maturity value of $1,000 \((M=1000)\) and with five years to maturity. The firm asset volatility is 30% and the cc. risk-free rate is 6%. No dividend payouts.
Let us first assume that the single debt issue is nonconvertible. Then its price is the price of a riskless bond minus a put option on 1/6 of the firm assets.

In particular, the price of one defaultable bond is

\[ 1000 \times \exp(-0.06 \times 5) - \text{BSPut}(10,000/6,1,000,0.30,0.06,5,0) = 701.27 \]

Then cc ytm = \( \frac{1}{5} \times \ln(1000/701.27) = 0.07 > 0.06 \)
Example (cont’d)

Let us now assume that the debt is convertible. The price of all six bonds is

\[ 6 \times 1000 \times \exp(-0.06 \times 5) \]

\[ - \text{BSPut}(10000,6000,0.30,0.06,5,0) \]

\[ + 6 \times 50 \times \frac{400}{400 + 6 \times 50} \times \text{BSCall}\left(\frac{10000}{400}, \frac{1000}{50}\right) \times \frac{400 + 6 \times 50}{400}, 0.30,0.06,5,0) \]

\[ = 5276.5 \]
Example (cont’d)

- The price of each convertible bond is 879.39
- \( Ytm = \frac{1}{5} \times \ln \left( \frac{1000}{879.39} \right) = 0.0257 \)
- The share value
  \[
  \frac{10,000 - 5276.35}{400} = 11.809
  \]
- Bondholders will convert if the assets are worth more than \( M \frac{(n+mq)}{q} = 14000. \)
- Each bond gives the holder the right to convert into 50 shares so the strike price is \( \frac{1000}{50} = 20. \) The conversion premium is \( \frac{20}{11.809} - 1 = 69.4\% \).
Convertible Bonds (cont’d)

- Valuation of convertible bonds is usually complicated by
  - Early exercise (American options)
  - Callability
  - Floating interest rates
  - Payment of dividends
  - Payment of coupons

- Why do firms issue convertible bonds?
  - One possible explanation is that it helps to resolve a conflict between equity-holders and debt-holders
  - Plus, firms would typically like to reduce interest payments
The Risk Neutral Probability of Default Revisited - Example

- The market value of a firm assets is $1M.
- The capital structure consists of 1,000 shares and 500 convertible, zero coupon bonds, maturity is in 3 years and each bond is convertible for 10 shares.
- The Volatility of assets is 40% and the CC interest rate is 9%. What is the risk neutral probability of default?
The Risk Neutral Probability of Default Revisited - Example

We calculate the risk neutral probability as follow:

\[ P[A < (m \times M)] = P[\ln(A) < \ln(500,000)] \]

\[
P[A < (m \times M)] = P\left[ z < \frac{\ln(500,000) - \left[ \ln(1,000,000) + \left( 0.09 - \frac{0.4^2}{2} \right) \times 3 \right]}{0.4 \times \sqrt{3}} \right]
\]

\[ P[A < (m \times M)] = P[z < -1.043] \]

\[ P = 14.92\% \]
End of Chapter Questions

Question 1

Go back to the table with cumulative average default rates and compute:

a. The probability that a Baa rated firm will default in the fifth year.

b. The probability that a B rated firm will default in either the third, or the forth, or the fifth year.
A firm issues six convertible bonds with maturity of five years. The face value of each bond is $1,000 and each bond is convertible to 50 stocks. The current outstanding number of shares is 400. The market price of the firm assets is $10,200. The volatility of the assets is 30% per year and the cc risk free rate is 6%.

a. What are the annual cc as well as effective yields of the convertible bonds?

b. What is the share price?
Questions 3-5

$Q(t)$, the probability of the firm defaulting at time $t$ is given by $Q(t) = 1 - \exp(-0.02t)$.

- **Q3:** What is the time 2 cumulative default probability of the firm?
- **Q4:** What is the probability that the firm defaults between times 2 and 3?
- **Q5:** Consider the alternative default function $Q(t) = 1 - \exp(-0.04t)$
  Does it correspond to a higher or lower rated firm?
Questions 6, 7

In the equity linked CD we found out that one call price is sold for $509.23.

- **Q6**: What is the implied volatility – or the volatility that when inserted into the BS formula would deliver the $509.23 option valuation? Assume that the effective dividend yield=0.02 and the effective risk-free rate is 0.06, both are effective.

- **Q7**: What is the fraction of gain in the S&P500 index you should get if the economic value of one call option is $450 rather than $509.23?
The market value of a firm’s assets is $20,000. The capital structure consists of 200 shares and 12 convertible zero coupon bonds. Each bond has face value of $1000, matures is in five years, and is convertible to 10 shares. The volatility of assets is 30% and the cc risk free rate is 6%. Using BS you find out that the market value of each bond is $635.

Q8: What is the conversion premium?
Q9: What is the risk neutral probability of default?
Interest Rates Futures and Swaps
Overview

- There are futures/forward contracts on many types of underlying assets including stocks, commodities, foreign exchange, interest rates, and interest rate based instruments.

- Here, we analyze interest rate contracts, including FRAs, Eurodollar futures, T-bills futures, T-bonds futures, and Swap contracts.
Interest Rate Futures:

Forward Rate Agreements (FRA)

- Let us consider a firm intending to borrow $100m for 91 days beginning 120 days from today, in June. Today is February.
- Suppose that the effective quarterly interest rate in June could be either 1.5% or 2% and the today’s implied June-91 day forward rate (the rate from June to September) is 1.8%.
- Note that there is a nontrivial difference in the interest rate expense of $100m \times (0.020 - 0.015) = $0.5m.
- The borrowing firm would like to guarantee the borrowing rate to hedge against increasing interest rates.
- Forward rate agreements (FRA) will do the job!
Forward Rate Agreements (FRA) are financial contracts that allow counterparties to lock in a forward interest rate.

Don’t get it wrong: FRA is a forward contract on an interest rate - not on a bond, or a loan.

The buyer of an FRA contract locks in a fixed borrowing rate, the seller locks in a fixed lending rate.

Thus, the buyer profits from increasing interest rates, while the seller benefits from diminishing rates.

Cash settlement is made either through the maturity of the contract (arrears) or at the time of borrowing.
FRA “Jargon”

- A $t_1 \times t_2$ FRA: The start date, or delivery date, is in $t_1$ months. The end of the forward period is in $t_2$ months.

- Thus, the loan period is $t_2 - t_1$ months long.

- E.g., a 9 X 12 FRA has a contract period beginning in nine months and ending in 12 months.
Example 1: settlement in arrears

(when the contract matures)

- If the borrowing rate in June is 1.5% - the borrower (the FRA buyer) will pay the FRA seller in September
  \((0.018-0.015) \times 100\text{m} = 300,000\).

- Unhedged interest expense is $1,500,000 \([100\text{m} \times 0.015]\].
  Total cost of borrowing is $1,800,000.

- If the borrowing rate in June is 2% the borrower will receive in September
  \((0.02-0.018) \times 100\text{m} = 200,000\).
  Unhedged interest expense is $2,000,000 \([100\text{m} \times 0.02]\].
  Total cost of borrowing is $1,800,000.

- The locked-in borrowing rate is 1.8% in both cases.
Example 2: settlement at the borrowing time

- Assume that $r_{qrtly}=1.5\%$ then the payment made by the buyer to the seller in June (the time of borrowing) is $300,000/1.015=295,566$.

- If $r_{qrtly}=2\\%$ then the payment made by the seller to the buyer in June is $200,000/1.02=196,078$. 
European Dollar Futures

- When foreign banks receive dollar deposits, those dollars are called Eurodollars.
- Like FRA, the Eurodollar contract is used to hedge interest rate risk.
- Similar to FRA, Eurodollar contracts involve 3-month forward rates.
- The Eurodollar is linked to a LIBOR rate.
- Libor (the London Interbank Offer Rate) is the average borrowing rate faced by large international London banks.
Eurodollar Futures

- The contract exists on many currencies (dollar, yen, euro, sterling, Swiss franc).
- Traded at the Chicago Mercantile Exchange (CME).
- Every contract has a face value of $1 million.
- A borrower, who would like to hedge against increasing interest rates, would take a short position (sell the contract).
- This is different from FRA wherein the borrower is the contract buyer (long) - why?
Eurodollar Futures

- Because FRA is a contract on interest rate, while Eurodollar is a contract on the loan itself.

- Four months prior to the delivery date, a Eurodollar futures contract is equivalent to a $4 \times 7$ FRA.

- 12 months prior to delivery a Eurodollar futures contract is equivalent to $12 \times 15$ FRA.
Suppose the current LIBOR is 1.5% per 3 months. By convention this is annualized by multiplying by 4, so the quoted LIBOR rate is 6%. (APR - not effective!)

The Eurodollar future price at any time is

\[ 100 - \text{Annualized 3\text{-}month\ LIBOR} \]

Thus, if LIBOR is 6% at maturity of the Eurodollar futures contract, the final futures price will be 100-6=94.

Three-month Eurodollar contracts have maturities up to 10 years, which means that it is possible to use the contract to lock in a 3-month rate for 10 years.
Eurodollar Futures: The Contract Price

- The contract price is calculated by
  \[ P_t = 10,000 \times [100 - 0.25(100 - Fut_t)] \]
  where \( Fut_t \) is the quoted Eurodollar futures rate, or
  \[ Fut_t = 100 - \text{Annualized 3 – month LIBOR} \]
  and 0.25 represents 3 months to maturity out of 12 months per year.

- You can also express the contract price as
  \[ P_t = 1,000,000 - 2500 \times \text{Annualized 3-month LIBOR} \]
Eurodollar Futures: The Contract Price

- For example, if the market quotes $Fut=94.47$, the contract price is then
  \[ P = 10,000 \times [100 - 0.25 \times 5.53] = 986,175 \]
- What if the annual Libor increases by 1 bp?
- The contract price diminishes by
  \[ 1,000,000 \times 0.01\% \times 0.25 = 25 \]
Eurodollar Futures: Hedging

- Suppose that in 7 months you plan to borrow $1 million for 90 days when the borrowing rate is LIBOR.
- The Eurodollar price for 7 months from today is 94, implying a 90-day rate of $(100-94)\times0.25=1.5\%$.
- As noted earlier, the borrower is shorting the contract.
- Suppose now that after 7 months the three-month LIBOR is 8\%, which implies a Eurodollar futures price of 92.
Eurodollar Futures: Hedging

- Your extra borrowing expense will be
  \[(0.02-0.015) \times 1,000,000 = $5,000.\]

- But you gain on the Eurodollar contract $5,000, which is the difference between
  \[P = 10,000 \times [100-0.25 \times 6] = $985,000\]
  and
  \[P = 10,000 \times [100-0.25 \times 8] = $980,000\]

- The short position fully compensates for the increasing borrowing cost.
Eurodollar Futures: An Impressive Success

- Eurodollar futures have taken over T-bill futures as the preferred contract to manage interest rate risk.
- LIBOR tracks the corporate borrowing rates better than the T-bill rate.

**Figure 7.3**

Three-month LIBOR rate, 1982-2004, and the difference between 3-month LIBOR and the yield on the 3-month Treasury bill.

Source: Datastream.
# Eurodollar Futures: Listing and Specifications

## WSJ listing

<table>
<thead>
<tr>
<th>Open</th>
<th>High</th>
<th>Low</th>
<th>Settle</th>
<th>Change</th>
<th>Yield</th>
<th>Change</th>
<th>Open Int.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treasury Bills (CME)-$1 mil.; pts of 100%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mar</td>
<td>...</td>
<td>...</td>
<td>98.26</td>
<td>+1.74</td>
<td>.01</td>
<td>+1.74</td>
<td>749</td>
</tr>
</tbody>
</table>

**Eurodollar (CME)-$1 Million; pts of 100%**

| Feb | 98.08 | 98.08 | 98.08 | 98.09 | +0.01 | 1.91 | -0.01 | 37,483 |
| Mar | 98.02 | 98.05 | 98.01 | 98.04 | +0.02 | 1.96 | -0.02 | 755,781 |
| Apr | 97.94 | 97.96 | 97.94 | 97.96 | +0.04 | 2.04 | -0.04 | 4,565 |
| June | 97.67 | 97.74 | 97.65 | 97.73 | +0.06 | 2.27 | -0.06 | 684,063 |
| Sept | 97.17 | 97.30 | 97.17 | 97.28 | +0.09 | 2.72 | -0.09 | 829,125 |
| Dec | 96.58 | 97.64 | 96.58 | 96.72 | +0.12 | 3.28 | -0.12 | 1,073,233 |
| Mar03 | 96.05 | 97.15 | 96.05 | 96.14 | +0.12 | 3.86 | -0.12 | 1,399,549 |
| June | 95.57 | 96.65 | 95.57 | 95.64 | +0.11 | 4.36 | -0.11 | 2,603,328 |
| Sept | 95.19 | 96.26 | 95.19 | 95.26 | +0.11 | 4.71 | -0.11 | 3,232,148 |
| Dec | 94.65 | 95.70 | 94.64 | 94.80 | +0.10 | 5.10 | -0.10 | 3,739,398 |
| Mar’04 | 94.54 | 94.70 | 94.53 | 94.68 | +0.09 | 5.31 | -0.09 | 3,777,125 |
| June | 94.22 | 94.47 | 94.22 | 94.45 | +0.09 | 5.54 | -0.09 | 1,165,946 |
| Sept | 94.22 | 94.28 | 94.22 | 94.27 | +0.09 | 5.73 | -0.09 | 1,197,477 |
| Dec | 94.03 | 94.08 | 94.02 | 94.07 | +0.09 | 5.83 | -0.09 | 7,803,657 |
| Mar’05 | 93.97 | 94.02 | 93.97 | 94.01 | +0.08 | 5.99 | -0.08 | 79,902,902 |
| June | 93.86 | 93.90 | 93.85 | 93.90 | +0.08 | 6.10 | -0.08 | 67,057,902 |
| Sept | 93.77 | 93.82 | 93.76 | 93.81 | +0.08 | 6.19 | -0.08 | 79,848,002 |
| Dec | 93.64 | 93.69 | 93.63 | 93.67 | +0.08 | 6.33 | -0.08 | 54,328,002 |
| Mar’06 | 93.53 | 93.60 | 93.52 | 93.67 | +0.07 | 6.53 | -0.07 | 46,771,002 |
| June | 93.59 | 93.61 | 93.58 | 93.61 | +0.07 | 6.59 | -0.07 | 39,319,002 |
| Sept | 93.54 | 93.56 | 93.54 | 93.55 | +0.06 | 6.45 | -0.06 | 44,724,002 |
| Dec | 93.43 | 93.45 | 93.42 | 93.44 | +0.06 | 6.56 | -0.06 | 32,917,002 |
| Jul’07 | 93.38 | 93.42 | 93.38 | 93.42 | +0.05 | 6.59 | -0.05 | 18,263,002 |
| Sept | 93.35 | 93.39 | 93.35 | 93.38 | +0.05 | 6.62 | -0.05 | 14,107,002 |
| Dec | 93.24 | 93.28 | 93.24 | 93.27 | +0.04 | 6.73 | -0.04 | 13,227,002 |
| Jul’08 | 93.23 | 93.26 | 93.23 | 93.25 | +0.04 | 6.75 | -0.04 | 11,239,002 |
| Jul’09 | 93.04 | 93.08 | 93.04 | 93.08 | +0.02 | 6.92 | -0.02 | 2,406,002 |

Est vol 568,932; vol Fri 1,193,069; open int 4,884,582; +95,457.

## Contract Specifications

**Specifications for the Eurodollar futures contract.**

- **Where traded**: Chicago Mercantile Exchange
- **Size**: 3-month Eurodollar time deposit, $1 million principal
- **Months**: Mar, Jun, Sep, Dec, out 10 years, plus 2 serial months and spot month
- **Trading ends**: 5 A.M. (11 A.M. London) on the second London bank business day immediately preceding the third Wednesday of the contract month.
- **Delivery**: Cash settlement
- **Settlement**: 100 – British Banker’s Association Futures Interest Settlement Rate for 3-Month Eurodollar Interbank Time Deposits. (This is a 3-month rate annualized by multiplying by 360/90.)
Swaps

- Forward contracts deal with a single transaction that will occur on a specific date in the future.

- What if many transactions occur repeatedly? E.g.,
  - Firms that issue bonds make periodical payments.
  - Multinational firms frequently exchange currencies.

- What is the easiest way to hedge risk in the presence of a risky payment stream – as opposed to a single risky payment?

- Perhaps we can enter into multiple future contracts for each payment we wish to hedge.

- Better solution – use a swap contract!
Introduction to Swaps

- A swap is a contract calling for an exchange of one or multiple payments, on one or more dates, determined by the difference in two prices.
- A swap provides a means to hedge a stream of risky payments.
- A forward contract is essentially a single-payment swap.
- There are commodity, exchange rate, as well as interest rate swaps (IRS).
- IRS is our focus here.
Interest Rate Swaps

- Companies use interest rate swaps to modify their interest rate exposure.

- The notional principle (NP) of the swap is the amount on which the interest payments are based.

- The life of the swap is the **swap term** or **swap tenor**

- If swap payments are made at the end of the period (when interest is due), the swap is said to be settled in arrears.
Basic terms of IRS

- Payer and receiver - quoted relative to fixed interest (i.e., payer = payer of fixed rate)

- Short party = payer of fixed interest.

- Long party = receiver of fixed interest.
An Example of an IRS

XYZ Corp. has $200M of floating-rate debt at LIBOR, i.e., every year it pays that year’s current LIBOR

XYZ would prefer to have fixed-rate debt with 3 years to maturity

One possibility is to retire the floating rate debt and issue fixed rate debt in its place. That could be too costly.

XYZ could enter a swap, in which they receive a floating rate and pay the fixed rate, which is, say, 6.9548%
On net, XYZ pays 6.9548%

\[ XYZ \text{ net payment} = - \frac{LIBOR}{\text{Floating Payment}} + \frac{LIBOR - 6.9548\%}{\text{Swap Payment}} = -6.9548\% \]
Computing the Fixed Swap Rate

Suppose there are \( n \) swap settlements, occurring on dates \( t_i, \ i = 1, \ldots, n \)

The implied forward interest rate from date \( t_{i-1} \) to date \( t_i \), known at date 0, is \( r_0(t_{i-1}, t_i) \)

The price of a zero-coupon bond maturing on date \( t_i \) is \( P(0, t_i) \)

The fixed swap rate is \( R \)

The market-maker is a counterparty to the swap in order to earn fees, not to take on interest rate risk. Therefore, the market-maker will hedge the floating rate payments by using, for example, forward rate agreements.
Computing the Fixed Swap Rate

The requirement that the hedged swap have zero net PV is

$$\sum_{i=1}^{n} P(0, t_i) [R - r_0(t_{i-1}, t_i)] = 0$$

which can be rewritten as

$$R = \frac{\sum_{i=1}^{n} P(0, t_i) r(t_{i-1}, t_i)}{\sum_{i=1}^{n} P(0, t_i)}$$
Computing the Fixed Swap Rate

- We can further rewrite the equation to make it easier to interpret

\[
R = \sum_{i=1}^{n} \left[ \frac{P(0, t_i)}{\sum_{j=1}^{n} P(0, t_j)} \right] r(t_{i-1}, t_i)
\]

- Thus, the fixed swap rate is as a weighted average of the implied forward rates, where the corresponding zero-coupon bond prices are used to determine the weights.
Computing the Fixed Swap Rate

- Alternative way to express the swap rate is

\[ R = \frac{1 - P_0(0, t_n)}{\sum_{i=1}^{n} P_0(0, t_i)} \]

- This equation is equivalent to a formula for the coupon on a par coupon bond.

- Thus, the swap rate is the coupon rate on a par coupon bond.
Example: Computing the Swap Rate

- Define $s(0,t)$ as the spot interest rate for a zero coupon bond maturing at time $t$.
- Define $f(t_1,t_2)$ as the forward interest rate from time $t_1$ until time $t_2$.
- Assume the zero (spot) term structure is:
  
  $s(0,1) = 5\%$, $s(0,2) = 6\%$, $s(0,3) = 7.5\%$.

Therefore the forward rates are:

$f(1,2) = 7.01\%$; and $f(2,3) = 10.564\%$.

These forward rates should exist in the FRA and futures markets.
Example: Computing the Swap Rate

- Now consider a swap with a tenor of 3 years.
- The floating rate is the one-year LIBOR.
- Settlement is yearly.
- What is the “fair” fixed rate?
- *Let the forward rates be the expected future spot rates.*
- Set the loan size to be $100.
- Thus, the “expected” floating rate cash flows are:
  - \( CF_1 = (0.05)(100) = 5 \)
  - \( CF_2 = (0.0701)(100) = 7.01 \)
  - \( CF_3 = (0.1056)(100) + 100 = 110.564 \)
Example: Computing the Swap Rate

- Value these expected cash flows at the appropriate discount rates: the spot zero coupon interest rates:

\[
\frac{5}{1.05} + \frac{7.01}{(1.06)^2} + \frac{10.564}{(1.075)^3} + \frac{100}{(1.075)^3} = 100
\]

- An important lesson: The value of the floating rate side of the swap equals the value of the fixed side of the swap immediately after a floating payment has been made.
Example: Computing the Spot Rate

Because the value of a swap at origination is set to zero, the fixed rate payments must satisfy:

\[
\frac{100A}{1.05} + \frac{100A}{(1.06)^2} + \frac{100A}{(1.075)^3} + \frac{100}{(1.075)^3} = 100
\]

\[
A \left\{ \frac{1}{1.05} + \frac{1}{(1.06)^2} + \frac{1}{(1.075)^3} \right\} = 1 - 0.80496
\]

\[
A\{2.6473\} = 0.19504
\]

\[
A = 7.367\%
\]
Valuing a Swap after Origination

Consider the previous interest rate swap example.

Suppose that 3 months after the origination date, the yield curve flattens at 7%.

The next floating cash flow is known to be 5.

Immediately after this payment is made, \( PV(\text{remaining floating payments}) = 100 \). Recall from the analysis of floaters, at the reset date the contract is valued at par.
Valuing a Swap after Origination.

\[ V_{floating} = \frac{5}{1.07^{0.75}} + \frac{100}{1.07^{0.75}} = 99.805 \]

\[ (V_{floating} = PV(\text{Next Payment}) + PV(\text{NP})) \]

Hence, after 3-months, \( V_{floating} \) has decreased.

\[ V_{fixed} = \frac{7.367}{1.07^{0.75}} + \frac{7.367}{1.07^{1.75}} + \frac{107.367}{1.07^{2.75}} = 102.685 \]

Thus, after 3-months, \( V_{fixed} \) has increased.

- The swap’s value is $2.88, per $100 of NP. (102.685 – 99.805).
- That is, one would have to pay $2.88 today to eliminate this swap. Who pays whom?
The Swap Curve

- A set of swap rates at different maturities is called the *swap curve*

- The swap curve should be consistent with the interest rate curve implied by the Eurodollar futures contract, which is used to hedge swaps

- Recall that the Eurodollar futures contract provides a set of 3-month forward LIBOR rates. In turn, zero-coupon bond prices can be constructed from implied forward rates. Therefore, we can use this information to compute swap rates
# The Swap Curve

## Table 8.4

Three-month LIBOR forward rates implied by Eurodollar futures prices with maturity dates given in the first column. Prices are from June 2, 2004.

<table>
<thead>
<tr>
<th>Maturity Date, $t_i$</th>
<th>Eurodollar Futures Price</th>
<th>Implied Quarterly Rate, $r(t_i, t_{i+1})$</th>
<th>Implied June 2004 Price of $1$ Paid on Maturity Date, $t_i$, $P(0, t_i)$</th>
<th>Swap Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jun-04</td>
<td>98.555</td>
<td>0.0037</td>
<td>—</td>
<td>1.4611%</td>
</tr>
<tr>
<td>Sep-04</td>
<td>98.010</td>
<td>0.0050</td>
<td>0.9964</td>
<td>1.7359%</td>
</tr>
<tr>
<td>Dec-04</td>
<td>97.495</td>
<td>0.0063</td>
<td>0.9914</td>
<td>2.0000%</td>
</tr>
<tr>
<td>Mar-05</td>
<td>97.025</td>
<td>0.0075</td>
<td>0.9851</td>
<td>2.2495%</td>
</tr>
<tr>
<td>Jun-05</td>
<td>96.600</td>
<td>0.0086</td>
<td>0.9778</td>
<td>2.4836%</td>
</tr>
<tr>
<td>Sep-05</td>
<td>96.235</td>
<td>0.0095</td>
<td>0.9695</td>
<td>2.6997%</td>
</tr>
<tr>
<td>Dec-05</td>
<td>95.910</td>
<td>0.0103</td>
<td>0.9603</td>
<td>2.8995%</td>
</tr>
<tr>
<td>Mar-06</td>
<td>95.650</td>
<td>0.0110</td>
<td>0.9505</td>
<td>3.0808%</td>
</tr>
</tbody>
</table>

*Source: Eurodollars futures prices from Datastream.*
Computing the Implied 3M Future Rate

- Of course, the future prices are set by the market.
- The implied quarterly rate is simply computed as the difference between 100 and the Futures price divided by 4 and multiplied by 1%:

\[ \text{Implied Rate} = \frac{100 - \text{Future Price}}{4} \times \frac{1}{100} \]

- For instance, the Jun-04 rate is computed as:

\[ \text{Implied Rate} = \frac{100 - 98.555}{4} \times \frac{1}{100} = 0.003625 \]
Computing the Price of Zeros

Next, the Zero Coupon Price is computed by dividing one by the product of future rates (which is the spot rate). For instance

\[
\frac{1}{1.0037} = 0.9964
\]

\[
\frac{1}{1.0037 \times 1.005} = 0.9914
\]

\[
\frac{1}{1.0037 \times 1.005 \times 1.0063} = 0.9851
\]
Computing the Fixed Swap Rate

- Finally, the Swap Rate is based upon the formulas exhibited earlier.
- For Jun-04, the Swap Rate is computed as

\[
\frac{100 \times R + 100}{1.0037} = 100
\]

which gives a quarterly rate \( R = 0.0037 \) and an annual rate of 1.4579\%.
Computing the Fixed Swap Rate (cont.)

- For Sep-2004, the Swap Rate computed as

\[
\frac{100R}{1.0037} + \frac{100R + 100}{1.0037 \times 1.005} = 100
\]

- It is important to note that the futures rate reported establish the term structure of futures rates. From this term structure we could generate the term structure of spot rates taught in the term structure section.
Unwinding an Existing Swap

- Enter into an offsetting swap at the prevailing market rate.

- If we are between two reset dates the offsetting swap will have a short first period to account for accrued interest.

- It is important that floating payment dates match!!
Risks of Swaps

- Price risk - value of fixed side may change due to changing interest rates.

- Credit risk - default or change of rating of counterparty

- Mismatch risk - payment dates of fixed and floating side are not necessarily the same

- Basis risk and Settlement risk
Credit Risk of a Swap Contract

- Default of counterparty (change of rating).

- Exists when the value of swap is positive

- Frequency of payments reduces the credit risk, similar to mark to market.

- Credit exposure changes during the life of a swap.
Deferred Swap

- A **deferred swap** is a swap that begins at some date in the future, but its swap rate is agreed upon today.

- The fixed rate on a deferred swap beginning in $k$ periods is computed as

$$R = \frac{\sum_{i=k}^{T} P_0(0, t_i)r_0(t_{i-1}, t_i)}{\sum_{i=k}^{T} P(0, t_i)}$$

- Previously we dealt with $k = 1$
Amortizing and Accreting Swaps

- An amortizing swap is a swap where the notional value is declining over time (e.g., floating rate mortgage)
- An accreting swap is a swap where the notional value is growing over time
- The fixed swap rate is still a weighted average of implied forward rates, but now the weights also involve changing notional principle, $Q_t$

$$
R = \frac{\sum_{i=1}^{n} Q_{t_i} P(0, t_i)r(t_{i-1}, t_i)}{\sum_{i=1}^{n} Q_{t_i} P(0, t_i)}
$$
Swaptions

- **A swaption** is an *option* to enter into a swap with specified terms. This contract will have an upfront premium.

- A swaption is analogous to an ordinary option, with the PV of the swap obligations (the price of the prepaid swap) as the underlying asset.

- Swaptions can be American or European.
Swaptions

- A **payer swaption** gives its holder the right, but not the obligation, to pay the fixed price and receive the floating price.
  - The holder of a receiver swaption would exercise when the fixed swap price is above the strike.

- A **receiver swaption** gives its holder the right to pay the floating price and receive the fixed strike price.
  - The holder of a receiver swaption would exercise when the fixed swap price is below the strike.
**T-bill Futures**

- One can use T-bill and Eurodollar futures to speculate on, or hedge against changes in, short-term (3-months to a year) interest rates.

- The longs profit when interest rates fall; the shorts profit when interest rates rise (and fixed income instrument prices fall).

- The T-bill futures market is thinly traded (illiquid).

- LIBOR futures contracts are quite liquid.
T-bill Futures: Example

- Then, someone who goes long a T-bill futures contract has “essentially” agreed to buy $1 million face value of 3-month T-bills on June 12, 1995, at a forward discount yield (APR) of 6.1% [100-93.99].
- The price of the futures contract is

\[
1,000,000 \left[ 1 - 0.061 \times \frac{91}{360} \right] = 984,808
\]

- The implied effective YTM is 6.33%. Check!
T-bill Futures: The Underlying Index

- Define: index price = 100 \( (1-Y_d) \), where \( Y_d \) is the yield on a bank discount basis. \( Y_d \) is also the quoted price of the T-bill in the market.

- The corresponding dollar discount is

\[
D = Y_d \times 1,000,000 \times \frac{t}{360}
\]
T-bill futures, example

- Assume that the index price is 92.52.
- The corresponding yield on a bank discount basis is 7.48% \((100-92.52)/100\)
- Assume there are 91 days to maturity, then the discount is

\[
D = 0.0748 \times 1,000,000 \times \frac{91}{360} = 18,907.78
\]

- The invoice price is then $981,092.22 (nominal–D).
- Here, one tick change in the value of futures contract 0.01 will change the value by $25.28 (since 91/360).
T-bond Futures

- T-bond Futures are futures contracts tied to a pool of Treasury bonds, with a remaining maturity greater than 15 years (and non callable within 15 years).

- Similar contracts exists of 2, 5, and 10 year Notes.

- Physical Delivery.
T-bond Futures

- Futures contracts are quoted like T-bonds, e.g. 97-02, in percent plus 1/32, with a notional of $100,000.

- Thus the price of the contract will be

  \[
  \$100,000 \times \frac{97 + \frac{2}{32}}{100} = \$97,062.50
  \]

- Assume that the next day the yield goes up and the price drops to 95-0, the new price is $95,000 and the loss of a long side is $2,062.50.
Treasury Bond/Note Futures

WSJ listings for T-bond and T-note futures

### INTEREST RATE

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**Treasury Bonds (CBT)-$100,000; pts 32nds of 100%**
- Mar 103-11 104-09 103-11 104-05 + 26 111-16 97-11 457,263
- June 102-17 103-02 102-14 103-00 + 26 110-00 96-30 41,817
- Est vol 122,000; vol Fri 193,638; open int 498,310, +10,880.

**Treasury Notes (CBT)-$100,000; pts 32nds of 100%**
- June 105-07 05-205 105-04 105-19 + 20.5 107-10 101-10 61,297
- Est vol 220,000; vol Fri 272,886; open int 593,529, +1,178.

**10 Yr Agency Notes (CBT)-$100,000; pts 32nds of 100%**
- Mar 102-04 102-14 102-01 102-13 + 18.5 106-04 96-27 36,659
- Est vol 1,000; vol Fri 1,519; open int 36,659, +69.

**5 Yr Treasury Notes (CBT)-$100,000; pts 32nds of 100%**
- Mar 08-145 106-27 06-115 105-25 + 12.5 109-06 04-015 531,946
- June 105-23 105-29 05-205 105-29 + 14.0 06-105 103-30 30,569
- Est vol 107,000; vol Fri 132,108; open int 552,515, +2,338.

**2 Yr Treasury Notes (CBT)-$200,000; pts 32nds of 100%**
- Mar 04-282 105-02 04-282 105-02 + 6.5 105-27 03-255 104,180
- Est vol 6,500; vol Fri 11,566; open int 104,180, +858.

---

### Figure 7.6

Specifications for the Treasury-note futures contract.

- **Where traded**: CBOT
- **Underlying**: 6% 10-year Treasury note
- **Size**: $100,000 Treasury note
- **Months**: Mar, Jun, Sep, Dec, out 15 months
- **Trading ends**: Seventh business day preceding last business day of month. Delivery until last business day of month.
- **Delivery**: Physical T-note with at least 6.5 years to maturity and not more than 10 years to maturity. Price paid to the short for notes with other than 6% coupon is determined by multiplying futures price by a conversion factor. The conversion factor is the price of the delivered note ($1 par value) to yield 6%. Settlement until last business day of the month.
Treasury Bond/Note Futures (cont’d)

- Long T-note futures position is an obligation to buy a 6% bond with maturity between 6.5 and 10 years to maturity.
- The short party is able to choose from various maturities and coupons: the “cheapest-to-deliver” bond.
- In exchange for the delivery the long pays the short the “invoice price.”

\[
\text{Invoice price} = (\text{Futures price} \times \text{conversion factor}) + \text{accrued interest}
\]

**Table 7.5**

<table>
<thead>
<tr>
<th>Description</th>
<th>8-Year 7% Coupon, 6.4% Yield</th>
<th>7-Year 5%, 6.3% Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market Price</td>
<td>103.71</td>
<td>92.73</td>
</tr>
<tr>
<td>Price at 6% (Conversion Factor)</td>
<td>106.28</td>
<td>94.35</td>
</tr>
<tr>
<td>Invoice Price (Futures ×)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Conversion Factor</td>
<td>103.71</td>
<td>92.09</td>
</tr>
<tr>
<td>Invoice – Market</td>
<td>0</td>
<td>-0.66</td>
</tr>
</tbody>
</table>

Prices, yields, and the conversion factor for two bonds. The futures price is 97.583. The short would break even delivering the 8-year 7% bond, and lose money delivering the 7-year 5% bond. Both bonds make semiannual coupon payments.
Repurchase Agreements

- A repurchase agreement or a repo entails selling a security with an agreement to buy it back at a fixed price.

- The underlying security is held as collateral by the counterparty ⇒ A repo is collateralized borrowing.

- Frequently used by securities dealers to finance inventory.

- Speculators and hedge funds also use repos to finance their speculative positions.

- A "haircut" is charged by the counterparty to account for credit risk.
A company will receive $100M in 6 months to be invested for the following six months. The CFO is concerned that the investment rates will fall. The CFO considers selling a $6 \times 12$ FRA on $100M at a rate 5%. This effectively locks in an annual investment rate of 5% starting in 6 months for the following 6 months. Assume further that after six months the interest rate indeed falls to 4%.

How much will the company get/pay due to the FRA contract?
Answer Q1

- The company (the FRA seller) will receive in the end of the investment period:
  \[(5\%-4\%) \times $100M \times 0.5 = $500,000\]
  which will fully compensate for the lower investment rate.
- If the settlement is at the time of investment the company will receive \(\frac{500,000}{1.02} = 490,196\)
Question 2

- A firm sells a 5X8 FRA, with a NP (notional principal) of $300MM, and a contract (three-month forward LIBOR) rate of 5.8% in annual terms.

- On the settlement date (five months hence) the spot LIBOR is 5.1%. There are 91 days in the contract period (8-5=3 months), and a year is defined to be 360 days.

- The settlement is at the time of borrowing.

- How much does the firm receive in that time?
Answer Q2

$524,077.11, which is calculated as the PV of the expected future payment:

\[
\frac{(300,000,000) \times (0.058 - 0.051) \times \left(\frac{91}{360}\right)}{1 + (0.051) \times \left(\frac{91}{360}\right)}
\]
Question 3

A firm sells a 5X8 FRA, with a NP of $300MM, and a contract rate of 5.8% (3-month forward LIBOR). Settlement is at maturity. Suppose that one month after the FRA origination 4X7 FRAs are priced at 5.5% and 5X8 FRAs are priced at 5.6%. Also assume that after that month, the annual spot LIBOR rate corresponding to seven month discounting is 4.9% (adjusted for seven month discounting).

What is the value of the FRA?
Answer Q3

- 4X7 FRAs are now priced at 5.5%. The firm’s original 5X8 FRA is now equivalent to a 4X7 FRA. The original FRA has a positive value for the firm because the interest rate has declined from 5.8% to 5.5%.

- The FRA value is the present value (seven month discounting) of the expected future payment:

\[
\frac{(300\, MM) \times (0.058 - 0.055) \times \left(\frac{3}{12}\right)}{1 + (0.049) \times \left(\frac{7}{12}\right)} = 218,747.5
\]
Question 4

You want to price a SWAP contract whose size is $100 and maturity of two years. The floating rate is the LIBOR. The effective spot rates are 5% per year and 6% per two years. Assume that three months after the SWAP initiation interest rates are flatten at 6%

What is the value of the SWAP contract?
Questions 5, 6, 7

The next three questions are based upon the following table of effective annual LIBOR rates for the next five quarters:

<table>
<thead>
<tr>
<th>Quarter</th>
<th>Effective Annual Spot Rate (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.4</td>
</tr>
<tr>
<td>2</td>
<td>3.2</td>
</tr>
<tr>
<td>3</td>
<td>3.0</td>
</tr>
<tr>
<td>4</td>
<td>2.9</td>
</tr>
<tr>
<td>5</td>
<td>2.8</td>
</tr>
</tbody>
</table>
Questions 5, 6, 7

- **Q5**: The current value of FRA(t₁,t₂) is $5.9 million, where t₁ is the beginning of the fourth quarter and t₂ is the end of the forth quarter – or it is an FRA(9,12) contract. The contract gets LIBOR on a NP of $1,000 million and pays fixed rate. What is the fixed rate the contract is paying?

- **Q6**: what is the effective future rate f(6,12)?

- **Q7**: what is the minimal spot rate for the next seven quarters?
Questions 8, 9

Here is a table describing futures price on Eurodollar contracts:

<table>
<thead>
<tr>
<th>Maturity Date</th>
<th>Future Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>June 2004</td>
<td>98.555</td>
</tr>
<tr>
<td>September 2004</td>
<td>98.010</td>
</tr>
<tr>
<td>December 2004</td>
<td>97.495</td>
</tr>
<tr>
<td>March 2005</td>
<td>97.025</td>
</tr>
<tr>
<td>June 2005</td>
<td>96.600</td>
</tr>
</tbody>
</table>

- **Q8**: What is the forward rate for the forth quarter ending Dec 2004? That is the effective forward rate from the end of Sep 2004 to the end of Dec 2004.

- **Q9**: What is the price of a zero coupon bond with face value of $1000 maturing in December 2004?
Interest Rate Models and Bond Pricing
Definitions

- An interest rate model describes the dynamics of interest rates.

- Spot (and therefore future) rates vary over time while variation is attributable to either one or more sources of risk.

- Indeed, we will consider both single factor and multifactor models to describe the dynamics of interest rates.
Model Uses

- Characterize term structure of spot rates to price bonds.
- Price interest rate and bond derivatives
  - Exchange traded (e.g., Treasury bond or Eurodollar options).
  - OTC (e.g., caps, floors, collars, swaps, swaptions, exotics).
- Price fixed income securities with embedded options.
  - Callable or putable bonds.
- Computing price sensitivities to underlying risk factor(s).
- Describe risk reward trade-off.
Spot rate tree

1-period spot rates (m-period compounded APR)

- Notation
  - $r_{i,j}(n) = n$-period spot rate $i$ periods in the future after $j$ up-moves
  - $\Delta t =$ length of a binomial step in units of years
- Set $\Delta t = 1/m$ and $m=2$
- Assume $q_{ij} = 0.5$ for all steps $i$ and nodes $j$
1-period zero-coupon bond prices

At time 0

\[ P_{0,0}(1) = \frac{q_{0,0} \times P_{1,1}(0) + (1 - q_{0,0}) \times P_{1,0}(0)}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}} \]

\[ = \frac{$100}{(1 + 0.1/2)^1} = $95.24 \]
1-period zero-coupon bond prices (cont)

At time 1

\[ P_{1,1}(1) = \frac{q_{1,1} \times P_{2,2}(0) + (1 - q_{1,1}) \times P_{2,1}(0)}{(1 + r_{1,1}(1)/m)^{1 \times \Delta t \times m}} \]

\[ = \frac{$100}{(1 + 0.11/2)^1} = $94.79 \]

\[ P_{1,0}(1) = \frac{q_{1,0} \times P_{2,1}(0) + (1 - q_{1,0}) \times P_{2,0}(0)}{(1 + r_{1,0}(1)/m)^{1 \times \Delta t \times m}} \]

\[ = \frac{$100}{(1 + 0.09/2)^1} = $95.69 \]
1-period zero-coupon bond prices (cont)

At time 2

\[ P_{2,2}(1) = \frac{q_{2,2} \times P_{3,3}(0) + (1 - q_{2,2}) \times P_{3,2}(0)}{(1 + r_{2,2}(1)/m)^{1 \times \Delta t \times m}} = \frac{$100}{(1 + 0.12/2)^1} = $94.34 \]

\[ P_{2,1}(1) = \frac{q_{2,1} \times P_{3,2}(0) + (1 - q_{2,1}) \times P_{3,1}(0)}{(1 + r_{2,1}(1)/m)^{1 \times \Delta t \times m}} = $95.24 \]

\[ P_{2,0}(1) = \frac{q_{2,0} \times P_{3,1}(0) + (1 - q_{2,0}) \times P_{3,0}(0)}{(1 + r_{2,0}(1)/m)^{1 \times \Delta t \times m}} = \frac{$100}{(1 + 0.08/2)^1} = $96.15 \]
1-period zero-coupon bond prices (cont)

<table>
<thead>
<tr>
<th>m</th>
<th>n</th>
<th>P_m,n(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$95.24</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$95.69</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$96.15</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$94.79</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>$95.25</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>$94.34</td>
</tr>
</tbody>
</table>

where \( r_{2,1} = 10\% \)
2-period zero-coupon bond prices

- At time 0

\[ P_{0,0}(2) = \frac{q_{0,0} \times P_{1,1}(1) + (1 - q_{0,0}) \times P_{1,0}(1)}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}} = \frac{1}{2} \times \frac{\$94.79 + \frac{1}{2} \times \$95.69}{(1 + 0.1/2)^1} = \$90.71 \]

- Implied 2-period spot rate

\[ P_{0,0}(2) = \frac{\$100}{(1 + r_{0,0}(2)/m)^{2 \times \Delta t \times m}} \Rightarrow r_{0,0}(2) = 9.9976\% \]
2-period zero-coupon bond prices

- At time 1

\[ P_{1,1}(2) = \frac{q_{1,1} \times P_{2,2}(1) + (1 - q_{1,1}) \times P_{2,1}(1)}{(1 + r_{1,1}(1)/m)^{1 \times \Delta t \times m}} \]

\[ = \frac{1}{2} \times 94.34 + \frac{1}{2} \times 95.24 \]

\[ = \frac{1}{2} \times 94.34 + \frac{1}{2} \times 95.24 \]

\[ = \frac{94.34}{2} + \frac{95.24}{2} \]

\[ = \frac{189.58}{2} \]

\[ = 94.79 \]

\[ = (1 + 0.11/2)^{1} \]

\[ = 94.79 \]

\[ = 94.79 \]

\[ \Rightarrow r_{1,1}(2) = 10.9976\% \]

\[ P_{1,0}(2) = \frac{q_{1,0} \times P_{2,1}(1) + (1 - q_{1,0}) \times P_{2,0}(1)}{(1 + r_{1,0}(1)/m)^{1 \times \Delta t \times m}} \]

\[ = \frac{1}{2} \times 95.24 + \frac{1}{2} \times 96.15 \]

\[ = \frac{95.24}{2} + \frac{96.15}{2} \]

\[ = \frac{191.39}{2} \]

\[ = 95.695 \]

\[ = (1 + 0.09/2)^{1} \]

\[ = 95.695 \]

\[ = 95.695 \]

\[ \Rightarrow r_{1,0}(2) = 8.9976\% \]
3-period zero-coupon bond prices

- At time 0

\[ P_{0,0}(3) = \frac{q_{0,0} \times P_{1,1}(2) + (1 - q_{0,0}) \times P_{1,0}(2)}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}} \]

\[ = \frac{1}{2} \times 89.85 + \frac{1}{2} \times 91.58 \]

\[ = \frac{1}{(1 + 0.1/2)^1} \]

\[ = 86.39 \]

- Implied 3-period spot rate

\[ P_{0,0}(3) = \frac{100}{(1 + r_{0,0}(3)/m)^{3 \times \Delta t \times m}} \Rightarrow r_{0,0}(3) = 9.9937\% \]
**Implied spot rate curve**

- Current spot rate curve is slightly downward sloping
  
  \[ r_{0,0}(1) = 10.0000\% \]
  \[ r_{0,0}(2) = 9.9976\% \]
  \[ r_{0,0}(3) = 9.9973\% \]

- From one period to the next, the spot rate curve shifts in parallel

\[
\begin{align*}
  r_{0,0}(1) &= 10.0000\% \\
  r_{0,0}(2) &= 9.9976\% \\
  r_{1,1}(1) &= 11.0000\% \\
  r_{1,1}(2) &= 10.9976\% \\
  r_{1,0}(1) &= 9.0000\% \\
  r_{1,0}(2) &= 8.9976\%
\end{align*}
\]
Coupon bond price

8% 1.5-year (3-period) coupon bond with cashflow
Coupon bond price (cont)

- Discounting terminal payoffs by 1 period

\[ P_{0,0} = ? \]

\[ P_{1,0} = ? \]

\[ P_{1,1} = ? \]

\[ P_{2,0} = \frac{104.00}{1.04} = 100 \]

\[ P_{2,1} = \frac{104.00}{1.05} = 99.05 \]

\[ P_{2,2} = \frac{104.00}{1.06} = 98.11 \]

\[ P_{3,0} = 104.00 \]

\[ P_{3,1} = 104.00 \]

\[ P_{3,2} = 104.00 \]

\[ P_{3,3} = 104.00 \]
Coupon bond price (cont)

- By risk-neutral pricing

\[
P_{1,1} = \frac{c + q_{1,1}P_{2,2} + (1-q_{1,1})P_{2,1}}{(1+r_{1,1}(1)/m)^{1\times\Delta t\times m}} = \frac{4.00 + \frac{1}{2}\times 98.11 + \frac{1}{2}\times 99.05}{1.055}
\]

\[
P_{2,2} = \frac{104.00}{1.06} = 98.11
\]

\[
P_{2,1} = \frac{104.00}{1.05} = 99.05
\]

\[
P_{2,0} = \frac{104.00}{1.04} = 100
\]

\[
P_{3,3} = 104.00
\]

\[
P_{3,2} = 104.00
\]

\[
P_{3,1} = 104.00
\]

\[
P_{3,0} = 104.00
\]

\[
P_{0,0} = ?
\]

\[
P_{1,0} = ?
\]

\[
P_{1,1} = 97.23
\]

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Coupon bond price (cont)

- By risk-neutral pricing

\[ P_{1,0} = \frac{c + q_{1,0} \times P_{2,1} + (1-q_{1,0}) \times P_{2,0}}{1+r_{1,0}(1/m)^{1\times\Delta t \times m}} = \frac{4.00 + \frac{1}{2} \times 99.05 + \frac{1}{2} \times 100.00}{1.045} \]

\[ P_{1,0} = \$99.07 \]

\[ P_{1,1} = \$97.23 \]

\[ P_{1,2} = \frac{\$104.00}{1.06} = \$98.11 \]

\[ P_{1,3} = \frac{\$104.00}{1.05} = \$99.05 \]

\[ P_{2,0} = \frac{\$104.00}{1.04} = \$100 \]

\[ P_{2,1} = \frac{\$104.00}{1.05} = \$99.05 \]

\[ P_{2,2} = \frac{\$104.00}{1.06} = \$98.11 \]

\[ P_{2,3} = \frac{\$104.00}{1.05} = \$99.05 \]

\[ P_{3,0} = \frac{\$104.00}{1.04} = \$100 \]

\[ P_{3,1} = \frac{\$104.00}{1.05} = \$99.05 \]

\[ P_{3,2} = \frac{\$104.00}{1.06} = \$98.11 \]

\[ P_{3,3} = \frac{\$104.00}{1.06} = \$98.11 \]
Coupon bond price (cont)

- By risk-neutral pricing

\[ P_{0,0} = \frac{c + q_{0,0} \times P_{1,1} + (1 - q_{0,0}) \times P_{1,0}}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}} = \frac{4.00 + \frac{1}{2} \times 97.23 + \frac{1}{2} \times 99.07}{1.05} = \frac{104.00}{1.04} = 99.05 \]

\[ P_{2,2} = \frac{104.00}{1.06} = 98.11 \]

\[ P_{3,3} = \frac{104.00}{1.04} = 100 \]

\[ P_{3,1} = \frac{104.00}{1.05} = 99.07 \]

\[ P_{3,2} = 104.00 \]

\[ P_{3,0} = 104.00 \]
European call on coupon bond

1-yr European style call option on 8% 1.5-yr coupon bond with strike price $K=99.00$ pays max[$0, P_2,?-K$]
European call on coupon bond

1-yr European style call option on 8% 1.5-yr coupon bond with strike price $K=99.00$ pays $\max[0, P_2, -K]$

\[
V_{0,0} = ? \\
V_{1,1} = 0.0237 \\
V_{1,0} = ? \\
V_{2,2} = \max[0, 98.11-99.00] = 0 \\
V_{2,1} = \max[0, 99.05-99.00] = 0.05 \\
V_{2,0} = \max[0, 100.00-99.00] = 1.00
\]

\[
V_{1,1} = \frac{q_{1,1} \times V_{2,2} + (1-q_{1,1}) \times V_{2,1}}{(1+r_{1,1}(1)/m)^{1 \times \Delta t \times m}} = \frac{\frac{1}{2} \times 0.00 + \frac{1}{2} \times 0.05}{1.055}
\]
European call on coupon bond

1-yr European style call option on 8% 1.5-yr coupon bond with strike price $K=$99.00 pays max[0, $P_{2,0}$-$K$]

\[ V_{0,0} = \ ? \]

\[ V_{1,0} = $0.5024 \]

\[ V_{1,1} = $0.0237 \]

\[ V_{2,0} = \max[0, 100.00-99.00] = $1.00 \]

\[ V_{2,1} = \max[0, 99.05-99.00] = $0.05 \]

\[ V_{2,2} = \max[0, 98.11-99.00] = 0 \]

\[ V_{1,0} = \frac{q_{1,0} \times V_{2,1} + (1-q_{1,0}) \times V_{2,0}}{(1+r_{1,0}(1)/m)^{1 \times \Delta t \times m}} = \frac{\frac{1}{2} \times $0.05 + \frac{1}{2} \times $1.00}{1.045} = $0.5024 \]
1-yr European style call option on 8% 1.5-yr coupon bond with strike price $K=$99.00 pays max[0, $P_2,K$]

\[
\begin{align*}
V_{0,0} &= 0.2505 \\
V_{1,1} &= 0.0237 \\
V_{1,0} &= 0.5024 \\
V_{2,2} &= \max(0, 98.11-99.00) = 0 \\
V_{2,1} &= \max(0, 99.05-99.00) = 0.05 \\
V_{2,0} &= \max(0, 100.00-99.00) = 1.00
\end{align*}
\]

\[
V_{0,0} = \frac{q_{0,0} \times V_{1,1} + (1-q_{0,0}) \times V_{1,0}}{(1+r_{1,0}(1)/m)^{1 \times \Delta t \times m}} = \frac{\frac{1}{2} \times 0.0237 + \frac{1}{2} \times 0.5024}{1.05} = 0.05
\]
1-yr American style put option on 8% 1.5-yr coupon bond with strike price $K = 99.00$ pays $\max[0, K-P_1]$ where

\[
V_{0,0} = ?
\]

\[
V_{1,1} = ?
\]

\[
V_{1,0} = ?
\]

\[
V_{2,2} = \max[0, 99.00 - 98.11] = 0.89
\]

\[
V_{2,1} = \max[0, 99.00 - 99.05] = 0.00
\]

\[
V_{2,0} = \max[0, 99.00 - 100.00] = 0.00
\]
American put on coupon bond

1-yr American style put option on 8% 1.5-yr coupon bond with strike price $K=99.00$ pays $\max[0, K-P_{1,?}]$

\[ V_{1,1} = \max \left[ q_{1,1} V_{2,2} + \frac{1-q_{1,1}}{1+r_{1,1}(1)/m} V_{2,1}, K - P_{1,1} \right] = \max \left[ \frac{1}{2} \times 0.89 + \frac{1}{2} \times 0.00, 99.00 - 97.23 \right] \]

\[ V_{2,2} = \max[0, 99.00 - 98.11] = 0.89 \]

\[ V_{2,1} = \max[0, 99.00 - 99.05] = 0.00 \]

\[ V_{2,0} = \max[0, 99.00 - 100.00] = 0.00 \]
1-yr American style put option on 8% 1.5-yr coupon bond with strike price $K=99.00$ pays max[0, $K-P_{1,?}$]

\[ V_{0,0} = ? \]

\[ V_{1,0} = 0.00 \]

\[ V_{1,1} = 1.7674 \]

\[ V_{2,2} = \max [0, 99.00 - 98.11] = 0.89 \]

\[ V_{2,1} = \max [0, 99.00 - 99.05] = 0.00 \]

\[ V_{2,0} = \max [0, 99.00 - 100.00] = 0.00 \]

\[ V_{1,0} = \max \left[ \frac{q_{1,0}V_{2,2} + (1-q_{2,1})V_{2,0}}{1+r_{1,0}(1)/m}^{1\times\Delta t\times m}, K - P_{1,0} \right] = \max \left[ \frac{1}{2} \times 0.00 + \frac{1}{2} \times 0.00 \div 1.045, 99.00 - 99.07 \right] \]
American put on coupon bond

1-yr American style put option on 8% 1.5-yr coupon bond with strike price $K=99.00$ pays max[$0, K-P_{1,?}]
Pricing Callable Bonds

- A 10-year, 10% callable bond issued when interest rates are relatively high may be more like a 3-year bond given that a likely interest rate decrease would lead the issuer to buy the bond back.

- Determining the value of such a bond requires taking into account not only the value of the bond's cash flow, but also the value of the call option embedded in the bond.
Use the Binomial Tree – General Formulation

- \( S_0 = uS_0 \)
- \( S_u = uS_0 \)
- \( S_{uu} = u^2S_0 \)
- \( S_{uu} = u^3S_0 \)
- \( S_{ud} = udS_0 \)
- \( S_{ud} = u^2dS_0 \)
- \( S_{dd} = d^2S_0 \)
- \( S_{dd} = ud^2S_0 \)
- \( S_{ddd} = d^3S_0 \)

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Use the Binomial Tree – General Formulation

$S_0 = 10.00\%$

$S_u = uS_0 = 11.00\%$

$S_d = dS_0 = 9.5\%$

$S_{uu} = u^2S_0 = 12.10\%$

$S_{uu} = u^3S_0 = 13.31\%$

$S_{ud} = udS_0 = 10.45\%$

$S_{ud} = u^2dS_0 = 11.495\%$

$S_{dd} = d^2S_0 = 9.025\%$

$S_{dd} = d^3S_0 = 9.927\%$

$S_{udd} = ud^2S_0 = 9.927\%$

$S_{udd} = u^2ddS_0 = 9.574\%$
First Step: Valuing a 2-Period Option-Free Bond

- Given the possible one-period spot rates, suppose we want to value a bond with the following features:
  - The bond matures in two periods
  - The bond has no default risk
  - The bond has no embedded option features
  - The bond pays an 8% coupon each period
  - The bond pays a $100 principal at maturity
Coupon=8, F=100, S_0=0.10, u=1.1, d=0.95

\[ B_0 = \frac{0.5(97.297 + 8) + 0.5(98.630 + 8)}{(1.10)} = 96.330 \]

\[ S_0 = 10\% \]

\[ B_u = \frac{108}{(1.11)} = 97.297 \]

\[ S_u = 11\% \]

\[ B_u = \frac{108}{(1.095)} = 98.630 \]

\[ S_d = 9.5\% \]

\[ B_d = \frac{108}{(1.095)} = 98.630 \]

\[ B_{uu} = 108 \]

\[ B_{ud} = 108 \]

\[ B_{dd} = 108 \]
Valuing a 2-Period Callable Bond

- Suppose that the two-period, 8% bond has a call feature that allows the issuer to buy back the bond at a call price (CP) of 98.

- Using the binomial tree approach, this call option can be incorporated into the valuation of the bond by determining at each node in Period 1 whether or not the issuer would exercise his right to call.

- The issuer will find it profitable to exercise whenever the bond price is above the call price (assuming no transaction or holding costs).
Coupon=8, F=100, CP=98, $S_0=0.10$, $u=1.1$, $d=0.95$

\[ B_0 = \frac{0.5(97.297 + 8) + 0.5(98 + 8)}{1.10} = 96.044 \]

\[ S_0 = 10\% \]

\[ B_u = \frac{108}{(1.11)} = 97.297 \]

\[ B^c_u = \text{Min}[B_u, CP] \]

\[ B^c_u = \text{Min}[97.297, 98] = 97.297 \]

\[ B_d = \frac{108}{(1.095)} = 98.630 \]

\[ S_d = 9.5\% \]

\[ B^c_d = \text{Min}[B_d, CP] \]

\[ B^c_d = \text{Min}[98.630, 98] = 98 \]

$B_{uu}=108$

$B_{ud}=108$

$B_{dd}=108$
So what is the call option price?

It is the difference between an option free bond and a callable bond.

That is, the call is worth

$$96.330 - 96.0444 = 0.2886$$
Valuing a 3-Period Option-Free Bond

- The binomial approach to valuing a two-period bond requires only a one-period binomial tree of one-period spot rates.
- If we want to value a three-period bond, we in turn need a two-period interest rate tree.
- Example: suppose we want to value a three-period, 9% coupon bond with no default risk or option features.
- In this case, market risk exist in two periods: Period 3, where there are three possible spot rates, and Period 2, where there are two possible rates.
Coupon = 9, F = $100, S_0 = 0.10, u = 1.1, d = 0.95

\[ S_0 = 10\% \]
\[ B_0 = \frac{0.5(96.3612 + 9) + 0.5(98.9335 + 9)}{(1.10)} \]
\[ = 96.9521 \]

\[ S_u = uS_0 = 11\% \]
\[ B_u = \frac{0.5(97.2346 + 9) + 0.5(98.6872 + 9)}{(1.11)} \]
\[ = 96.3612 \]

\[ S_d = dS_0 = 9.5\% \]
\[ B_d = \frac{0.5(96.6872 + 9) + 0.5(99.9771 + 9)}{(1.095)} \]
\[ = 98.9335 \]

\[ S_{uu} = u^2S_0 = 12.1\% \]
\[ B_{uu} = \frac{109}{1.121} \]
\[ = 97.2346 \]

\[ S_{ud} = udS_0 = 10.45\% \]
\[ B_{uu} = \frac{109}{1.1045} \]
\[ = 98.6872 \]

\[ S_{dd} = d^2S_0 = 9.025\% \]
\[ B_{dd} = \frac{109}{1.09025} \]
\[ = 99.9771 \]
Valuing a 3-Period Callable Bond

To determine the value of the bond given its callable:

a. First compare each of the noncallable bond values with the call price in Period 2 (one period from maturity); then take the minimum of the two as the callable bond value.

b. Next roll the callable bond values from Period 2 to Period 1 and determine the two bond values at each node as the present value of the expected cash flows, and then for each case select the minimum of the value calculated or the call price.

c. Finally, roll those two callable bond values to the current period and determine the callable bond’s price as the present value of Period 1's expected cash flows.
Coupon=9, F=$100, CP=98, S_0=0.10, u=1.1, d=0.95

\[ S_u = uS_0 = 11\% \]
\[ B_u = \frac{0.5(97.2346 + 9) + 0.5(98 + 9)}{(1.11)} = 96.0516 \]
\[ B_u^C = \text{Min}[B_u, CP] \]
\[ B_u^C = \text{Min}[96.0516, 98] = 96.0516 \]

\[ S_0 = 10\% \]
\[ B_u^C = \text{Min}[96.0516 + 9, 0.5(97.7169 + 9)] \]
\[ B_u^C = \text{Min}[96.258] \]

\[ S_d = dS_0 = 9.5\% \]
\[ B_u = \frac{0.5(98+9) + 0.5(98+9)}{(1.095)} = 97.7169 \]
\[ B_u^C = \text{Min}[B_u, CP] \]
\[ B_u^C = \text{Min}[97.7169, 98] = 97.7169 \]

\[ S_{ud} = u^2S_0 = 12.1\% \]
\[ B_{uu} = \frac{10^9}{1.121} = 97.2346 \]
\[ B_{uu}^C = \text{Min}[B_{uu}, CP] \]
\[ B_{uu}^C = \text{Min}[97.2346, 98] = 97.2346 \]

\[ S_{ud} = udS_0 = 10.45\% \]
\[ B_{ud} = \frac{10^9}{1.1045} = 98.6872 \]
\[ B_{ud}^C = \text{Min}[B_{ud}, CP] \]
\[ B_{ud}^C = \text{Min}[98.6872, 98] = 98 \]

\[ S_{dd} = d^2S_0 = 9.025\% \]
\[ B_{da} = \frac{10^9}{1.09025} = 99.9771 \]
\[ B_{da}^C = \text{Min}[B_{da}, CP] \]
\[ B_{da}^C = \text{Min}[99.9771, 98] = 98 \]
Putable Bond

From the bondholder’s perspective, a put option provides a hedge against a decrease in the bond price.

- If rates decrease in the market, then the bondholder benefits from the resulting higher bond prices.
- If rates increase, then the bondholder can exercise, giving her downside protection.

Given that the bondholder has the right to exercise, the price of a putable bond will be equal to the price of an otherwise identical nonputable bond plus the value of the put option ($V_0^P$):

$$B_0^P = B_0^{NP} + V_0^P$$
Valuing a 3-Period Putable Bond

Suppose the three-period, 9% option-free bond in our previous example had a put option giving the bondholder the right to sell the bond back to the issuer at an exercise price of $PP = 97$ in Periods 1 or 2.
Coupon=9, F=$100, PP=97, S_0=0.10, u=1.1, d=0.95

\[
B_0^P = \frac{S_0}{(1.10)} = 97.2425
\]

\[
B_u = \frac{S_u = uS_0}{0.5(97.2346 + 9) + 0.5(98.6872 + 9) = 96.3612}\]
\[
B_u^P = \text{Max}[B_u, PP] = 97
\]

\[
B_d = \frac{S_d = dS_0}{0.5(98.6872 + 9) + 0.5(99.9771 + 9) = 97.7169}\]
\[
B_d^P = \text{Max}[B_d, PP] = 98.9335
\]

\[
B_{uu} = \text{Max}[B_{uu}, PP] = 97.2346
\]

\[
B_{ud} = \text{Max}[B_{ud}, PP] = 98.6872
\]

\[
B_{dd} = \text{Max}[B_{dd}, PP] = 99.9771
\]

\[
S_u = u^2S_0 = 12.1%
\]
\[
B_{uu} = \frac{109}{1.121} = 97.2346
\]

\[
B_{ud} = \frac{109}{1.1045} = 98.6872
\]

\[
B_{dd} = \frac{109}{1.09025} = 99.9771
\]
Interest Rate Models

- Thus far we have taken interest rates as given and have then priced bonds – either straight bonds or bonds with embedded options.

- But the interest rates are not given – we have to find them out ourselves.

- We will consider several high profile interest rate models to deliver the spot rate along the Binomial Tree.
Spot rate process

- Binomial trees are based on spot rate values $r_{i,j}(1)$ and risk-neutral probabilities $q_{i,j}$

- In single-factor models, these values are determined by a risk-neutral spot rate process of the form

$$r_{t+\Delta t}(1) - r_{t}(1) = \mu[r_t(1), t] \times \Delta t + \sigma[r_t(1), t] \times \sqrt{\Delta t} \times \epsilon_t$$

with

$$\text{Mean}[\epsilon_t] = 0 \quad \text{Var}[\epsilon_t] = 1$$

such that

$$\text{Mean}[r_{t+\Delta t}(1) - r_{t}(1)] = \mu[r_t(1), t] \times \Delta t$$

$$\text{Var}[r_{t+\Delta t}(1) - r_{t}(1)] = \sigma[r_t(1), t]^2 \times \Delta t$$
Spot rate process (cont)

- In an $N$-factor models, these values are determined by a risk-neutral spot rate process of the form

$$r_t(1) = z_{1,t} + z_{2,t} + \cdots + z_{N,t}$$

with

$$z_{i,t+\Delta t} - z_{i,t} = \mu[z_{1,t}, z_{2,t}, \ldots, z_{N,t}, t] \times \Delta t + \underbrace{\sigma[z_{1,t}, z_{2,t}, \ldots, z_{N,t}, t]}_{\text{volatility fct}} \times \sqrt{\Delta t} \times \epsilon_{1,t}$$

and

$$\text{Mean} [\epsilon_{i,t}] = 0 \quad \text{Var} [\epsilon_{i,t}] = 1$$
Drift function

Case 1: Constant drift

\[ r_{t+\Delta t} - r_t = \lambda \times \Delta t + \sigma \times \sqrt{\Delta t} \times \epsilon_t \]

with

\[ \epsilon_t \sim N[0,1] \]

Implied distribution of 1-period spot rate

\[ r_{t+\Delta t} \sim N \left[ r_t + \lambda \times \Delta t, \sigma^2 \times \Delta t \right] \]
Drift function (cont)

- Binomial tree representation

- Properties
  - No mean reversion
  - No heteroskedasticity
  - Spot rates can become negative, but not if we model ln[r(1)] ⇒ “Rendleman-Barter model”
  - 2 parameters
  - ⇒ fit only 2 spot rates
Drift function (cont)

Example

- $r_{0,0} = 5\%$
- $\lambda = 1\%$
- $\sigma = 2.5\%$
- $\Delta t = 1/m$ with $m = 2$

$q = 1/2$

$r_{0,0} = 5\%$

$q = 1/2$

$r_{1,1} = 7.27\%$

$r_{1,0} = 3.73\%$

$q = 1/2$

$r_{2,2} = 9.54\%$

$r_{2,1} = 6.00\%$

$r_{2,0} = 2.47\%$
Drift function (cont)

- Binomial tree representation

- Properties
  - No heteroskedasticity
  - Spot rates can become negative, but not if we model $\ln[r(1)]$ ⇒ “Salomon Brothers model”
  - Arbitrary many parameters ⇒ fit term structure of spot rates but not necessarily spot rate volatilities (i.e., derivative prices)
Drift function (cont)

- **Case 3: Mean reversion**
  
  \[ r_{t+\Delta t} - r_t = \kappa \times [\theta - r_t(1)] \times \Delta t + \sigma \times \sqrt{\Delta t} \times \varepsilon_t \]

  with

  \[ \varepsilon_t \sim N[0,1] \]

- **Implied distribution of 1-period spot rate**
  
  \[ r_{t+\Delta t} \sim N \left[ r_t + \kappa \times [\theta - r_t(1)] \times \Delta t, \sigma^2 \times \Delta t \right] \]

Drift function (cont)

- Binomial tree representation

- Properties
  - Non-recombining, but can be fixed
  - No heteroskedasticity
  - Spot rates can become negative, but not if we model $\ln[r(1)]$
  - 3 parameters
    $\Rightarrow$ fit only 3 spot rates
Drift function (cont)

Example

- \( r_{0,0} = 5\% \)
- \( \theta = 1\% \)
- \( \kappa = 0.25 \)
- \( \sigma = 2.5\% \)
- \( \Delta t = 1/m \) with \( m = 2 \)

\[ q = \frac{1}{2} \]

\[ r_{0,0} = 5\% \]
\[ r_{1,0} = 3.73\% \]
\[ r_{1,1} = 7.39\% \]

\[ r_{2,0} = 2.86\%, \text{ with } \kappa = 0 \]
\[ r_{2,1} = 6.39\%, \text{ with } \kappa = 0 \]
\[ r_{2,2} = 9.49\%, \text{ with } \kappa = 0 \]

\[ r_{2,1} = 5.00\% \]
\[ r_{2,0} = 1.46\% \]
\[ r_{2,2} = 8.54\% \]
Example

- $r_{0,0} = 15\%$
- $\theta = 10\%$
- $\kappa = 0.25$
- $\sigma = 2.5\%$
- $\Delta t = 1/m$ with $m = 2$

Drift function (cont)
Volatility function

- **Case 1: Square-root volatility**

\[
 r_{t+\Delta t} - r_t = \lambda \times \Delta t + \sigma \times \sqrt{r_t} \times \Delta t \times \epsilon_t \\
\]

with

\[
 \epsilon_t \sim N[0,1] 
\]

- **Implied distribution of 1-period spot rate**

\[
 r_{t+\Delta t} \sim N \left[ r_t + \lambda \times \Delta t, \sigma^2 \times r_t \times \Delta t \right] \\
\]

- **Cox, Ingresoll, and Ross (1985, *Econometrics*) \Rightarrow “CIR model”**
Volatility function (cont)

- **Binomial tree representation**

  - $r_{0,0} + \lambda \times \Delta t + \sigma \times \sqrt{r_{0,0}} \times \sqrt{\Delta t}$
  - $r_{0,0} + \lambda \times \Delta t - \sigma \times \sqrt{r_{0,0}} \times \sqrt{\Delta t}$

  - $r_{1,1} + \lambda \times \Delta t + \sigma \times \sqrt{r_{1,1}} \times \sqrt{\Delta t}$
  - $r_{1,1} + \lambda \times \Delta t - \sigma \times \sqrt{r_{1,1}} \times \sqrt{\Delta t}$
  - $r_{1,0} + \lambda \times \Delta t + \sigma \times \sqrt{r_{1,0}} \times \sqrt{\Delta t}$
  - $r_{1,0} + \lambda \times \Delta t - \sigma \times \sqrt{r_{1,0}} \times \sqrt{\Delta t}$

  - $q = \frac{1}{2}$

- **Properties**

  - Non-recombining, but can be fixed
  - No mean-reversion, but can be fixed by using different drift function
  - Spot rates can become negative, but not as $\Delta t \to 0$
  - 1 volatility parameter (and arbitrarily many drift parameters)
    $\Rightarrow$ fit term structures of spot rates but only 1 spot rate volatility
Volatility function (cont)

Example

- $r_{0,0} = 5\%$
- $\lambda = 1\%$
- $\sigma = 11.18\% \Rightarrow \sigma \times \sqrt{r_{0,0}} = 2.5\%$
- $\Delta t = 1/m$ with $m = 2$

$q = 1/2$

<table>
<thead>
<tr>
<th>$r_{0,0}$</th>
<th>$r_{1,0}$</th>
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$q = 1/2$

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$q = 1/2$

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Volatility function (cont)

- **Binomial tree representation**

  $r_{0,0} = r_{0,0} + \lambda \times \Delta t + \sigma(1) \times \sqrt{\Delta t}$

  $q = 1/2$

  $r_{1,1} = r_{1,1} + \lambda \times \Delta t + \sigma(2) \times \sqrt{\Delta t}$

  $r_{1,0} = r_{1,0} + \lambda \times \Delta t - \sigma(2) \times \sqrt{\Delta t}$

- **Properties**

  - Non-recombining, but can be fixed
  - No mean-reversion, but can be fixed by using different drift function
  - Spot rates can become negative, but not if we model $\ln[r(1)]$
    $\Rightarrow$ “Black-Karasinski model” and “Black-Derman-Toy model”
  - Arbitrarily many volatility and drift parameters)
    $\Rightarrow$ fit term structures of spot rates and volatility
Calibration

- To calibrate parameters of a factor model to bonds prices
  - Step 1: Pick arbitrary parameter values
  - Step 2: Calculate implied 1-period spot rate tree
  - Step 3: Calculate model prices for liquid securities
  - Step 4: Calculate model pricing errors given market prices
  - Step 5: Use solver to find parameters values which minimize the sum of the squared or absolute pricing errors
Constant drift example

- Step 1: Pick arbitrary parameter values

Parameters

\[
\begin{align*}
\lambda & \quad 0.00\% \\
\sigma & \quad 0.10\%
\end{align*}
\]

Observed 1-period spot rate

\[
\begin{align*}
 r(1) & \quad 0.54\%
\end{align*}
\]
### Step 2: Calculate implied 1-period spot rate tree

<table>
<thead>
<tr>
<th>0.54%</th>
<th>0.61%</th>
<th>0.68%</th>
<th>0.75%</th>
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</tbody>
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### Constant drift example (cont)

- **Step 3:** Calculate model prices for liquid securities
  - E.g., for a 2.5-yr STRIPS

<table>
<thead>
<tr>
<th></th>
<th>98.66</th>
<th>98.98</th>
<th>99.25</th>
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</tbody>
</table>
Constant drift example (cont)

Step 3: Calculate model prices for liquid securities (cont)

Parameters

\[ \lambda \quad 0.0000\% \]

\[ \sigma \quad 0.1000\% \]

<table>
<thead>
<tr>
<th>Years to Maturity</th>
<th>Model implied spot rates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.54%</td>
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<tr>
<td>2</td>
<td>0.54%</td>
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<td>3</td>
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<td>7</td>
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<td>8</td>
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<td>9</td>
<td>0.54%</td>
</tr>
<tr>
<td>10</td>
<td>0.54%</td>
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</table>

Observed 1-period spot rate

\[ r(1) \quad 0.54\% \]
Constant drift example (cont)

- **Step 4: Use solver**

<table>
<thead>
<tr>
<th>Parameters</th>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.0000%</td>
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<tr>
<td>$\sigma$</td>
<td>0.1000%</td>
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</table>

- **Observed 1-period spot rate**

| $r(1)$      | 0.54%    |          |          |

Minimize sum of squared errors by choice of parameters

<table>
<thead>
<tr>
<th>Years to Maturity</th>
<th>Model implied spot rates</th>
<th>Observed rates</th>
<th>Pricing errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.54%</td>
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<td>2</td>
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<td>5</td>
<td>0.54%</td>
<td>1.42%</td>
<td>0.88%</td>
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<td>7</td>
<td>0.54%</td>
<td>1.71%</td>
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<td>8</td>
<td>0.54%</td>
<td>1.89%</td>
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<td>10</td>
<td>0.54%</td>
<td>2.24%</td>
<td>1.70%</td>
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</tbody>
</table>

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Constant drift example (cont)

Solution

Parameters

\[ \lambda = 0.4896\% \]
\[ \sigma = 2.2970\% \]

Observed 1-period spot rate

\[ r(1) = 0.54\% \]

<table>
<thead>
<tr>
<th>Years to Maturity</th>
<th>Model implied spot rates</th>
<th>Observed rates</th>
<th>Pricing errors</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.54%</td>
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<td>2</td>
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### Constant drift example (cont)

#### Resulting model

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