Syllabus: Course Description

This is a comprehensive course on the theory and practice of derivatives securities. The course describes conceptual paradigms and their extensive applications in practice.
We will be covering the following topics:

- Buying and selling put and call options
- Understanding futures and forward contracts
- Speculating with derivatives
- Risk management: hedging with derivatives
- Analyzing structures such as equity linked CD
- Employee stock options
Syllabus: Course Description

• Pricing options with the Binomial Tree
• Understanding the B&S formula
• Pricing exotic options
• Option Greeks
• Constructing complex strategies with options
• Shortfall probability and insuring against shortfall
• Applying derivatives concepts to corporate decisions
• Credit risk modeling
• Pricing warrants and convertible bonds
• Interest rate futures such as FRAs and Eurodollars
• Hedging with swaps contracts
Syllabus: Resources

Text book:
• The textbook for this course is *Derivatives Markets* by Robert L. McDonald, Second Edition. ISBN 978-0-3-2128030-5

Case studies:
• The risk in stocks in the long run: The Barnstable College endowment
• Sara’s option
• Case Studies (30%) – See below.

• Exam (60%) - The final exam will be based on the material and examples covered in class, assignments, and assigned reading. The exam is closed books and closed notes. However, you will be allowed to bring in one piece of paper with handwritten notes (double-sided, A4 size). You are not allowed to use any other notes. I will allow the use of non-programmable calculators during the exam.
Syllabus: Assessment

• Class Participation (10%) - It is mandatory to attend the sessions. If you miss a session for good reason, make sure you catch up on all missed material, as you are fully responsible for class lectures, announcements, handouts, and discussions.

• Warning: A nontrivial fraction of the final exam questions could be based on class discussions, assigned readings, and examples which are not necessarily covered in the lecture notes.
Syllabus: Case 1 The Risk of Stocks in The Long Run - Description

• This case describes a scenario encountered frequently in practice: how to mix stocks and bonds in long run investments. The Barnstable College contemplates that stocks outperform bonds in the long run and is considering two proposals consistent with that philosophy. The long run investment with insuring against a potential shortfall event entails using the B&S option pricing formula.

• The following questions are based on figures displayed in the case and are about computing quantities typically used in risk management and measurement. The case is a bit confusing about the mean, expected return, as well as the risk free rate. Assume that all the case figures reflect gross return (not continuously compounded). That means that you need to use the approximations displayed in the class notes for recovering the mean and variance of the stock. The continuously compounded riskfree rate $rf=\ln(1+R_f)$ where $R_f$ is the figure displayed in the case.
Syllabus: Case 1 The Risk of Stocks in The Long Run - Questions

1. Compute the probability that stocks will underperform bonds for investment horizons of 1, 2, 5, 10, 20, and 30 years.
2. Compute the corresponding VaR at the 5% level for those horizons.
3. Compute the expected shortfall under those circumstances – or the expected value of the stock investment given that stocks underperform bonds.
4. Compute the value of a put option that insures against a shortfall for the same investment horizons as in part 1.
5. How would you reconcile the fact that even when the shortfall probably declines with the investment horizon the cost of insuring against a shortfall increases?
6. Assess the pros and cons of the two proposals offered to BCE. Which one would you select?
Syllabus: Case 2 Sara’s Option – Description

• Sara Becker is about to graduate with an MBA from Harvard and is faced with three employment offers, each with a different compensation package. Two of Sara’s three offers involve stock options as a significant portion of total compensation. The third offer is all cash, with large annual cash bonuses. In examining the three alternatives, this case provides an opportunity to analyze how executives and employees value their compensation packages, and in particular the option based component of their pay.

• The clear lake option package is a more traditional plan, while the WebScale package is unusual in that it uses “index options” in which the payoff is linked – in a complex way – to the performance of the stock relative to the S&P. As Sara considers and compares the three compensation packages she evaluates the value of her stock options based on different factors such as her risk tolerance, her probable length of stay at each company, and the volatility of each company’s stock.
Syllabus: Case 2 Sara’s Option – Questions

1. Use the B&S formula to value Sara’s Clear Lake annual options grant. How does the value of this option grant change with:
   - A 10% increase in the stock price?
   - A 10% increase in volatility?
   - An increase in the dividend rate from 0% to 3%?
   - A one year reduction in the “time to expiration” of the options?

2. Give an intuitive explanation why option values change in the ways indicated by your analysis.

3. Will Sara’s private value of these options generally be higher or lower than the value determined by the B&S formula? Explain!

4. What factors, not considered by the B&S formula, affect the value of Clear Lake’s options to Sara?

5. How should Sara value three years’ worth of Clear Lake option grants (i.e., how much cash should she be willing to trade off for these options?) Note: An assumption much be made about Sara’s departure date. Do the analysis under three assumptions: Sara departs after (i) three years, (ii) five years, and (iii) 20 years.

6. Repeat the analysis in Question 4 for WebScale’s options.

7. Which compensation package is worth the most to Sara – Clear Lake’s package, Davis and Rockefeller’s, or WebScale’s? Explain!
Derivative Securities:

What are Derivatives?
Why Using Derivatives?
How big is the Market for Derivatives?
What are Derivatives?

**Primary assets:**
Securities sold by firms or government to raise capital (stocks and bonds) as well as stock indexes (S&P, Nikkei), interest rates, exchange rates, credit risk, commodities (gold, coffee, corn), etc.

**Derivatives assets:**
Options, forward and futures contracts, FRAs, Eurodollars, Swaption, CDS, etc. These financial assets are *derived* from existing primary assets.
Why using derivatives?

– Risk management (e.g., hedging)
– Speculation
– Reduce market frictions, e.g., cost of default, taxes, and transaction costs
– Exploit arbitrage opportunities.
Exchange Traded Contracts

- Contracts proliferated in the last three decades

<table>
<thead>
<tr>
<th>TABLE 1.1</th>
<th>Examples of futures contracts traded on the Chicago Board of Trade (CBT), Chicago Mercantile Exchange (CME), and the New York Mercantile Exchange (NYMEX).</th>
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</thead>
<tbody>
<tr>
<td><strong>CBT</strong></td>
<td><strong>CME</strong></td>
</tr>
<tr>
<td>30-day Average Federal Funds</td>
<td>S&amp;P 500 index</td>
</tr>
<tr>
<td>10-year U.S. Treasury bonds</td>
<td>NASDAQ 100 index</td>
</tr>
<tr>
<td>Municipal Bond Index</td>
<td>Eurodollars</td>
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<tr>
<td>Corn</td>
<td>Nikkei 225</td>
</tr>
<tr>
<td>Soybeans</td>
<td>Pork bellies</td>
</tr>
<tr>
<td>Wheat</td>
<td>Heating and cooling degree-days</td>
</tr>
<tr>
<td>Oats</td>
<td>Japanese yen</td>
</tr>
</tbody>
</table>

- What were the drivers behind this proliferation?
Increased Volatility...

- Oil prices: 1951–1999
- DM/$ rate: 1951–1999

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Derivatives Securities
Led to New and Big Markets

- Exchange-traded derivatives

- Over-the-counter traded derivatives: even more!
Derivative Securities

Overview: Understanding Option and Futures Contracts
Some Basics

- Option terminology
- Call and put options
- Options as insurance
- Structures
- Employee stock options
- Forward and futures contracts
- Speculation with derivatives
- Hedging with derivatives
Option Terminology

- Buy = Long = Hold
- Sell = Short = Write
- Call - option to buy underlying asset
- Put - option to sell underlying asset
- So we have: buy call, buy put, sell call, sell put
- Key Elements
  - Exercise or Strike Price
  - Maturity or Expiration
  - Premium or Price
  - Zero Sum Game
Definition and Terminology

• A call option gives the owner the right but not the obligation to buy the underlying asset at a predetermined price during a predetermined time period.

• Strike (or exercise) price: the amount paid by the option buyer for the asset if he/she decides to exercise

• Exercise: the act of paying the strike price to buy the asset

• Expiration: the date by which the option must be exercised or become worthless

• Exercise style: specifies when the option can be exercised
  – European-style: can be exercised only at expiration date
  – American-style: can be exercised at any time before expiration
  – Bermudan-style: can be exercised during specified periods (e.g., on the first day of each month. Bermuda is located between the US and Europe.)
Examples

• Buying a call on an index
  – Today: call buyer acquires the right to pay $1,020 in six months for the index, but is not obligated to do so
  – In six months at contract expiration: if spot price is
    • $1,100, call buyer’s payoff = $1,100 – $1,020 = $80
    • $900, call buyer walks away, buyer’s payoff = $0

• Selling a call on an index
  – Today: call seller is obligated to sell the index for $1,020 in six months, if asked to do so
  – In six months at contract expiration: if spot price is
    • $1,100, call seller’s payoff = $1,020 – $1,100 = ($80)
    • $900, call buyer walks away, seller’s payoff = $0
Payoff/Profit of a Purchased Call

- Payoff = Max [0, spot price at expiration – strike price]
- Profit = Payoff – future value of option premium
- Examples:

  S&P Index 6-month Call Option
  - Strike price = $1,000, Premium = $93.81, 6-month risk-free rate = 2%
    - If index value in six months = $1100
      - Payoff = max [0, $1,100 – $1,000] = $100
      - Profit = $100 – ($93.81 x 1.02) = $4.32
    - If index value in six months = $900
      - Payoff = max [0, $900 – $1,000] = $0
      - Profit = $0 – ($93.81 x 1.02) = – $95.68
Diagrams for Purchased Call

- Payoff at expiration
  ![Payoff Diagram](image1)

- Profit at expiration
  ![Profit Diagram](image2)

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Derivatives Securities
Payoff/Profit of a Written Call

• Payoff = $-\max[0, \text{spot price at expiration} - \text{strike price}]$

• Profit = Payoff + future value of option premium

• Example
  – S&P Index 6-month Call Option
    • Strike price = $1,000, Premium = $93.81, 6-month risk-free rate = 2%
    – If index value in six months = $1100
      • Payoff = $-\max[0, 1100 - 1000] = -100$
      • Profit = $-100 + (93.81 \times 1.02) = -4.32$
    – If index value in six months = $900
      • Payoff = $-\max[0, 900 - 1000] = 0$
      • Profit = $0 + (93.81 \times 1.02) = 95.68$
Put Options

• A put option gives the owner the right but not the obligation to sell the underlying asset at a predetermined price during a predetermined time period.

• The seller of a put option is obligated to buy if asked.

• Payoff/profit of a purchased (i.e., long) put:
  – Payoff = max [0, strike price – spot price at expiration]
  – Profit = Payoff – future value of option premium

• Payoff/profit of a written (i.e., short) put:
  – Payoff = – max [0, strike price – spot price at expiration]
  – Profit = Payoff + future value of option premium
Put Option Examples

– S&P Index 6-month Put Option

  • Strike price = $1,000, Premium = $74.20, 6-month risk-free rate = 2%

– If index value in six months = $1100

  • Payoff = max [0, $1,000 – $1,100] = $0
  • Profit = $0 – ($74.20 x 1.02) = – $75.68

– If index value in six months = $900

  • Payoff = max [0, $1,000 – $900] = $100
  • Profit = $100 – ($74.20 x 1.02) = $24.32
A Few Items to Note

• A call option becomes more profitable when the underlying asset appreciates in value
• A put option becomes more profitable when the underlying asset depreciates in value
• Moneyness is an important concept in option trading.
“Moneyness”

In the Money - exercise of the option would be profitable
Call: market price > exercise price (denoted by K or X)
Put: exercise price > market price

Out of the Money - exercise of the option would not be profitable
Call: market price < exercise price
Put: exercise price < market price

At the Money - exercise price and market price are equal
# Call and Put Options on IBM

## PRICES AT CLOSE MARCH 23, 2006

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<thead>
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<th></th>
<th>IBM (IBM)</th>
<th>Underlying stock price: 83.20</th>
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<td></td>
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<td><strong>Call</strong></td>
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</tr>
<tr>
<td>Oct</td>
<td>90.00</td>
<td>2.15</td>
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</tbody>
</table>
Home’s Insurance is a Put Option

- You own a house that costs $200,000
- You buy a $15,000 insurance policy
- The deductible amount is $25,000
- Let us graph the profit from this contact.
Home’s Insurance as a Put Option

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Derivatives Securities
Call Options are also Insurance

- Banks and insurance companies offer investment products that allow investors to benefit from a rise in a stock index and provide a guaranteed return if the market falls.

- The equity linked CD provides a zero return if the index falls (refund of initial investment) and a return linked to the index if the index rises.
Equity Linked CDs

The 5.5-year CD promises to repay initial invested amount plus 70% of the gain in S&P500 index

- Assume $10,000 is invested when S&P 500 = 1300
- Final payoff = $10,000 \times \left(1 + 0.7 \times \max \left[0, \frac{S_{\text{final}}}{1300} - 1\right]\right)
- Where $S_{\text{final}}$ = the value of the S&P500 that will be in 5.5 years
The Economic Value of the Equity Linked CD

- We paid $10,000 and we get $10,000 in 5.5 years plus some extra amount if the S&P500 index level exceeds 1300.
- That payoff structure is equivalent to buying a zero coupon bond and x call options.
- Why? Assuming that the annual effective rate is 6%, the present value of $10,000 to be received in 5.5 years is $7,258.
The Economic Value of the Equity Linked CD

• Thus, we practically paid $7,258 for a zero coupon bond and $2,742 for $x$ call options.
• What is $x$? And what is the implied value of one call option?
• The component of payoff attributable to call option is

$$\frac{7000}{1300} \times \max[S_{final} - 1300, 0]$$
The Economic Value of the Equity Linked CD

- Therefore $x=5.3846$.
- One call option is priced as $2742/5.3846=509.23$
- The economic value of one call option using the B&S formula is
  \[ BSCall(1300,1300,\sigma,\ln(1.06),5.5,\delta) \]

- We will describe this formula later in this course.
- What if the economic value of the calls is smaller than the implied price of the calls?
Employee Stock Options (Warrants)

- Employee stock options are call-like options issued by a company on its own stock
- When such options are exercised the company issues new shares – thus warrants are not a zero-sum game unlike regular call options.
- The options are often at-the-money at the time of issue
- They could last as long as 10 years
Typical Features of Employee Stock Options

- There is a vesting period during which options cannot be exercised.
- When employees leave during the vesting period options are forfeited.
- When employees leave after the vesting period in-the-money options are exercised immediately and out of the money options are forfeited.
- Employees are not permitted to trade these options.
Exercise Decision

• To realize cash from an employee stock option the employee must exercise the options and sell the underlying shares
Drawbacks of Employee Stock Options

- Gain to executives from good performance is much greater than the penalty for bad performance
- Executives do very well when the stock market as a whole goes up, even if their firm does relatively poorly
- Executives are encouraged to focus on short-term performance at the expense of long-term performance
- Executives are tempted to time announcements or take other decisions that maximize the value of the options
Accounting for Employee Stock Options

- Prior to 1995 the cost of an employee stock option on the income statement was its intrinsic value on the issue date.
- After 1995 a “fair value” had to be reported in the notes (but expensing fair value on the income statement was optional).
- Since 2005 both FASB and IASB have required the fair value of options to be charged against income at the time of issue.
Nonstandard Plans

• The attraction of at-the-money call options used to be that they led to no expense on the income statement because they had zero intrinsic value on the exercise date
• Other plans were liable to lead an expense
• Now that the accounting rules have changed some companies are considering other types of plans
Possible Nonstandard Plans

- Strike price is linked to stock index so that the company’s stock price has to outperform the index for options to move in the money
- Strike price increases in a predetermined way
- Options vest only if specified profit targets are met
Dilution

• Employee stock options are liable to dilute the interests of shareholders because new shares are bought at below market price.

• Later in the course we will show how the dilution effect enters into an option pricing formula.
Backdating

• Backdating appears to have been a widespread practice in the United States.

• A company might take the decision to issue at-the-money options on April 30 when the stock price is $50 and then backdate the grant date to April 3 (earlier) or May 5 (later) when the stock price is lower.

• Why would they do this?
Academic Research Exposed Backdating

Abnormal stock returns around ESO grants

Day relative to option grant

Forward Contract

• A **forward contract** is an agreement made **today** between a buyer and a seller who are **obligated** to complete a transaction at a pre-specified date in the future.

• The buyer and the seller know each other. The negotiation process leads to **customized** agreements: What to trade; Where to trade; When to trade; How much to trade?
A Futures contract is an agreement made today between a buyer and a seller who are obligated to complete a transaction at a pre-specified date in the future.

The buyer and the seller do not know each other. The "negotiation" occurs in an organized future exchange.

The terms of a futures contract are standardized. The contract specifies what to trade; where to trade; When to trade; How much to trade; what quality of good to trade.
Organized Futures Exchanges

• Established in 1848, the Chicago Board of Trade (CBOT) is the oldest organized futures exchange in the United States.

• Other Major Exchanges:
  – New York Mercantile Exchange (1872)
  – Chicago Mercantile Exchange (1874)
  – Kansas City Board of Trade (1882)

• Early in their history, these exchanges only traded contracts in storable agricultural commodities (i.e., oats, corn, wheat).

• Agricultural futures contracts still make up an important part of organized futures exchanges.
Futures Contracts Basics

In general, futures contracts must stipulate at least the following five terms:

1. The identity of the underlying commodity or financial instrument.
2. The futures contract size.
3. The futures maturity date, also called the expiration date.
4. The delivery or settlement procedure.
5. The futures price.
Terminology

• Open interest: the total number of contracts outstanding
  – equal to number of long positions or number of short positions
• Settlement price: the price just before the final bell each day
  – used for the daily settlement process
• Volume of trading: the number of trades in 1 day
• See next page…..
Reading Price Quotes

**FIGURE 3.3** Futures Trading

![The Wall Street Journal Futures Chart](image)

**THE WALL STREET JOURNAL.**

**FUTURES**

**TUESDAY, JULY 19, 2005**

**Interest Rate Futures**

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<tr>
<th></th>
<th>OPEN</th>
<th>HIGH</th>
<th>LOW</th>
<th>SETTLE</th>
<th>CHG</th>
<th>LIFETIME</th>
<th>LOW</th>
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<tr>
<td><em>Treasury Bonds</em> (CBT)-$100,000; pts 32nds of 100%</td>
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<td>Est vol 162,922; vol Fri 198,479; open int 576,644, +686.</td>
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Payoff on a Futures Contract

• Payoff for a contract is its value at expiration

• Payoff for
  – Long forward = Spot price at expiration – Forward price
  – Short forward = Forward price – Spot price at expiration

• Example:
  – Today: Spot price = $1,000, 6-month forward price = $1,020
  – In six months at contract expiration: Spot price = $1,050
    • Long position payoff = $1,050 – $1,020 = $30
    • Short position payoff = $1,020 – $1,050 = ($30)
Payoff Diagram for Futures

- Long and short forward positions on the S&P 500 index
Example: Speculating in Gold Futures, Long

- You believe the price of gold will go up. So,
  - You go long 100 futures contract that expires in 3 months.
  - The futures price today is $400 per ounce.
  - Assume interest rate is zero.
  - There are 100 ounces of gold in each futures contract.

- Your "position value" is: $400 \times 100 \times 100 = $4,000,000

- Suppose your belief is correct, and the price of gold is $420 when the futures contract expires.

- Your "position value" is now: $420 \times 100 \times 100 = $4,200,000

Your "long" speculation has resulted in a gain of $200,000
Speculating with Futures, Short

• Selling a futures contract (today) is often called “going short,” or establishing a short position.
• Recall: Each futures contract has an expiration date.
  – Every day before expiration, a new futures price is established.
  – If this new price is higher than the previous day’s price, the holder of a short futures contract position loses from this futures price increase.
  – If this new price is lower than the previous day’s price, the holder of a short futures contract position profits from this futures price decrease.
Example: Speculating in Gold Futures, Short

• You believe the price of gold will go down. So,
  – You go short 100 futures contract that expires in 3 months.
  – The futures price today is $400 per ounce.
  – Assume interest rate is zero.
  – There are 100 ounces of gold in each futures contract.

• Your "position value" is: $400 \times 100 \times 100 = $4,000,000

• Suppose your belief is correct, and the price of gold is $370 when the futures contract expires.

• Your “position value” is now: $370 \times 100 \times 100 = $3,700,000

Your "short" speculation has resulted in a gain of $300,000

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Derivatives Securities
Risk Management: The Producer’s Perspective

• We can also use futures contracts for hedging
• A producer selling a risky commodity has an inherent long position in this commodity
• When the price of the commodity increases, the profit typically increases
• Common strategies to hedge profit
  – Selling forward
  – Buying puts
  – Selling Calls
Producer: Hedging With a Forward Contract

• A short forward contract allows a producer to lock in a price for his output
  – Example: a gold-mining firm enters into a short forward contract, agreeing to sell gold at a price of $420/oz. in 1 year. The cost of production is $380
Producer: Hedging With a Put Option

- **Buying a put option** allows a producer to have higher profits at high output prices, while providing a floor on the price.

  - Example: a gold-mining firm purchases a 420-strike put at the premium of $8.77. Interest rate is 5%. FV=9.21.
Producer: Insuring by Selling a Call

- A **written** call reduces losses through a premium, but limits possible profits by providing a cap on the price.
  - Example: a gold-mining firm sells a 420-strike call and receives an $8.77 premium (FV=$9.21)
How much Insurance?

• Insurance is not free!…in fact, it is expensive

• There are several ways to reduce the cost of insurance

• For example, in the case of hedging against a price decline by purchasing a put option, one can reduce the insured amount by lowering the strike price of the put option. Then the put is cheaper.
The Buyer’s Perspective

• A buyer that faces price risk on an input has an inherent short position in this commodity.

• When the price of, say raw material, is up the firm’s profitability falls.

• Some strategies to hedge profit
  – Buying forward
  – Buying calls
Buyer: Hedging With a Forward Contract

- A **long forward contract** allows a buyer to lock in a price for his input
  
  - Example: a firm, using gold as an input, purchases a forward contract, agreeing to buy gold at a price of $420/oz. in 1 year. The product is selling for $460
Buyer: Hedging With a Call Option

- **Buying a call option** allows a buyer to have higher profits at low input prices, while being protected against high prices.
  - Example: a firm, which uses gold as an input, purchases a 420-strike call at the premium (future value) of $9.21/oz.
Yet Another Example: Starbucks

• Suppose Starbucks has an inventory of about 950,000 pounds of coffee, valued at $0.57 per pound.
• Starbucks fears that the price of coffee will fall in the short run, and wants to protect the value of its inventory.
• How best to do this? You know the following:
  – There is a coffee futures contract at the New York Board of Trade.
  – Each contract is for 37,500 pounds of coffee.
  – Coffee futures price with three month expiration is $0.58 per pound.
  – Selling futures contracts provides current inventory price protection.
• 25 futures contracts covers 937,500 pounds.
• 26 futures contracts covers 975,000 pounds.
Starbucks: Short Hedging with Futures Contracts

• Starbucks decides to sell 25 near-term futures contracts.

• Over the next month, the price of coffee falls. Starbucks sells its inventory for $0.51 per pound.

• The futures price also falls, to $0.52. (There are two months left in the futures contract)

• How did this short hedge perform?

• That is, how much protection did selling futures contracts provide to Starbucks?
Starbucks: The Short Hedge Performance

<table>
<thead>
<tr>
<th>Date</th>
<th>Starbucks Coffee Inventory Price</th>
<th>Starbucks Inventory Value</th>
<th>Near-Term Coffee Futures Price</th>
<th>Value of 25 Coffee Futures Contracts</th>
</tr>
</thead>
<tbody>
<tr>
<td>Now</td>
<td>$0.57</td>
<td>$541,500</td>
<td>$0.58</td>
<td>$543,750</td>
</tr>
<tr>
<td>1-Month From now</td>
<td>$0.51</td>
<td>$484,500</td>
<td>$0.52</td>
<td>$487,500</td>
</tr>
</tbody>
</table>

| Gain (Loss)   | ($0.06)                       | ($57,000)                 | $0.06                         | $56,250                             |

- The hedge was not perfect.
- But, the short hedge “threw-off” cash ($56,250) when Starbucks needed some cash to offset the decline in the value of their inventory ($57,000).

What would have happened if prices had increased by $0.06 instead?

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Long Hedge

• We define
  \( F_1 \): Initial Futures Price
  \( F_2 \): Final Futures Price
  \( S_2 \): Final Asset Price

• If you hedge the future purchase of an asset by entering into a long futures contract then
  Cost of Asset = \( S_2 - (F_2 - F_1) = F_1 + \text{Basis} \)

• Basis is the difference between the spot and futures price
Short Hedge

• Again we define
  \[ F_1 : \text{Initial Futures Price} \]
  \[ F_2 : \text{Final Futures Price} \]
  \[ S_2 : \text{Final Asset Price} \]

• If you hedge the future sale of an asset by entering into a short futures contract then
  \[ \text{Price Realized} = S_2 + (F_1 - F_2) = F_1 + \text{Basis} \]
To hedge the risk in a portfolio the number of contracts that should be shorted is

\[
\beta \frac{P}{F}
\]

where \(P\) is the value of the portfolio, \(\beta\) is its beta, and \(F\) is the value of one futures contract.
Example

S&P 500 futures price is 1,000
Value of Portfolio is $5 million
Beta of portfolio is 1.5

What position in futures contracts on the S&P 500 is necessary to hedge the portfolio?
Hedging using Index Futures

- In practice, things are more complex, because:
- You don’t know the true beta – you have a noisy estimate
- Beta could be time varying
- Beta may not be the single determinant of equity return (or the only risk exposure).
Hedging Price of an Individual Stock

- Similar to hedging a portfolio
- Does not work as well because only the systematic risk is hedged
- The unsystematic risk that is unique to the stock is not hedged
Beware of Contango: Futures Prices for Gold on Jan 8, 2007 Increase with Maturity

![Graph showing increase in futures prices with maturity](image-url)
Normal Backwardation: Futures Prices for Orange Juice on January 8, 2007 Decrease with Maturity
Convergence of Futures to Spot Prices

(a) (b)

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Why Do Firms Manage Risk?

• Hedging can be optimal for a firm when an extra dollar of income received in times of high profits is worth less than an extra dollar of income received in times of low profits.

• Profits for such a firm are concave, so that hedging (i.e., reducing uncertainty) can increase expected cash flow.

• Concave profits can arise from:
  – Taxes
  – Bankruptcy and distress costs
  – Costly external financing
  – Preservation of debt capacity
  – Managerial risk aversion
Example I: Taxes

Consider a firm that produces one unit of a good. Cost is $10. Selling price is either 11.2 or 9 with equal probabilities. So the net gain is either 1.2 or -1 and the expected net gain is 0.1.

Assume now that the company pays tax 40% on profits but gets no tax refund on losses. Then the expected net gain is $0.5 \times 1.2 \times 0.6 - 0.5 = -0.14$.

What if there is a forward market for the firm’s output and the forward price is 10.10. Then the profit is certain at 0.06.

Note that we transfer $1 from less to more valuable state.
Example II: Bankruptcy and Distress Costs

• A large loss can threaten the survival of a firm
  – A firm may be unable to meet fixed obligations (such as debt payments and wages)
  – Customers may be less willing to purchase goods of a firm in distress

• *Hedging allows a firm to reduce the probability of bankruptcy or financial distress*
Example III: Costly External Financing

• Raising funds externally can be costly
  – There are explicit costs (such as bank and underwriting fees)
  – There are implicit costs due to asymmetric information

• Costly external financing can lead a firm to forego investment projects it would have taken had cash been available to use for financing

• *Hedging can safeguard cash reserves and reduce the probability of raising funds externally*
Example IV: Increase Debt Capacity

- The amount that a firm can borrow is its **debt capacity**
- When raising funds, a firm may prefer debt to equity because interest expense is tax-deductible
- However, lenders may be unwilling to lend to a firm with a high level of debt due to a higher probability of bankruptcy
- *Hedging allows a firm to credibly reduce the riskiness of its cash flows, and thus increase its debt capacity*
Example V: Managerial Risk Aversion

- Firm managers are typically not well-diversified
  - Salary, bonus, and compensation are tied to the performance of the firm

- Poor diversification makes managers risk-averse, i.e., they are harmed by a dollar of loss more than they are helped by a dollar of gain

- Managers have incentives to reduce uncertainty through hedging
Nonfinancial Risk Management

• Risk management is not a simple matter of hedging or not hedging using financial derivatives, but rather a series of decisions that start when the business is first conceived

• Some nonfinancial risk-management decisions are
  – Entering a particular line of business
  – Choosing a geographical location for a plant
  – Deciding between leasing and buying equipment
Reasons Not to Hedge

- Reasons why firms may elect not to hedge
  - Transaction costs of dealing in derivatives (such as commissions and the bid-ask spread)
  - The requirement for costly expertise
  - The need to monitor and control the hedging process
  - Complications from tax and accounting considerations
  - Potential collateral requirements
Empirical Evidence on Hedging

• Half of nonfinancial firms report using derivatives
• Among firms that do use derivatives, less than 25% of perceived risk is hedged, with firms likelier to hedge short-term risk
• Firms with more investment opportunities are more likely to hedge
• Firms that use derivatives have a higher market value and more leverage
Option versus Futures
Contracts

- Options differ from futures in major ways:
  - Holders of call options have no obligation to buy the underlying asset.
  - Holders of put options have no obligation to sell the underlying asset.
  - To avoid this obligation, buyers of calls and puts must pay a price today.
  - Holders of futures contracts do not pay for the contract today, but they are obligated to buy or sell the underlying asset, depending upon the position.
Option and Forward Positions: A Summary

- Long forward: Profit increases with increasing stock price.
- Short forward: Profit decreases with increasing stock price.
- Long call: Profit increases with increasing stock price.
- Short call: Profit decreases with increasing stock price.
- Long put: Profit decreases with increasing stock price.
- Short put: Profit increases with increasing stock price.
Questions on Forwards and Options

1. The spot price of the market index is $900. A 3-month forward contract on this index is priced at $930. What is the profit or loss to a short position if the spot price of the market index rises to $920 by the expiration date?

2. The spot price of the market index is $900. A 3-month forward contract on this index is priced at $930. The market index rises to $920 by the expiration date. The annual rate of interest on treasuries is 4.8% (0.4% per month). What is the difference in the payoffs between a long index investment and a long forward contract investment? (Assume monthly compounding.)

3. The spot price of the market index is $900. A 3-month forward contract on this index is priced at $930. The annual rate of interest on treasuries is 4.8% (0.4% per month). What annualized rate of interest makes the net payoff zero? (Assume monthly compounding.)

4. The spot price of the market index is $900. After 3 months the market index is priced at $920. An investor has a long call option on the index at a strike price of $930. After 3 months what is the investor’s profit or loss?
Questions on Forwards and Options

5. The spot price of the market index is $900. After 3 months the market index is priced at $920. The annual rate of interest on treasuries is 4.8% (0.4% per month). The premium on the long put, with an exercise price of $930, is $8.00. What is the profit or loss at expiration for the long put?

6. The spot price of the market index is $900. After 3 months the market index is priced at $920. The annual rate of interest on treasuries is 4.8% (0.4% per month). The premium on the long put, with an exercise price of $930, is $8.00. At what index price does a long put investor have the same payoff as a short index investor? Assume the short position has a breakeven price of $930.

7. The spot price of the market index is $900. The annual rate of interest on treasuries is 4.8% (0.4% per month). After 3 months the market index is priced at $920. An investor has a long call option on the index at a strike price of $930. What profit or loss will the writer of the call option earn if the option premium is $2.00?

8. The spot price of the market index is $900. After 3 months the market index is priced at $920. The annual rate of interest on treasuries is 4.8% (0.4% per month). The premium on the long put, with an exercise price of $930, is $8.00. Calculate the profit or loss to the short put position if the final index price is $915.
9 If your homeowner’s insurance premium is $1,000 and your deductible is $2000, what could be considered the strike price of the policy if the home has a value of $120,000?

10 A put option if purchased and held for 1 year. The Exercise price on the underlying asset is $40. If the current price of the asset is $36.45 and the future value of the original option premium is (– $1.62 ), what is the put profit, if any at the end of the year?

11 The premium on a long term call option on the market index with an exercise price of 950 is $12.00 when originally purchased. After 6 months the position is closed and the index spot price is 965. If interest rates are 0.5 % per month, what is the Call Payoff?

12 The premium on a call option on the market index with an exercise price of 1050 is $9.30 when originally purchased. After 2 months the position is closed and the index spot price is 1072. If interest rates are 0.5 % per month, what is the Call Profit?
Questions on Risk Management

1. To plant and harvest 20,000 bushels of corn, Farmer Jayne incurs fixed and variable costs totaling $33,000. The current spot price of corn is $1.80 per bushel. What is the profit or loss if the spot price is $1.90 per bushel when she harvests and sells her corn?

2. Farmer Jayne decides to hedge 10,000 bushels of corn by purchasing put options with a strike price of $1.80. Six-month interest rates are 4.0% and the total premium on all puts is $1,200. If her total costs are $1.65 per bushel, what is her marginal change in profits if the spot price of corn drops from $1.80 to $1.75 by the time she sells her crop in 6 months?

3. A 6-month forward contract for corn exists with a price of $1.70 per bushel. If Farmer Jayne decides to hedge her 20,000 bushels of corn with the forward contract, what is her profit or loss if spot prices are $1.65 or $1.80 when she sells her crop in 6 months? Her total costs are $33,000.

4. Corn call options with a $1.75 strike price are trading for a $0.14 premium. Farmer Jayne decides to hedge her 20,000 bushels of corn by selling short call options. Six-month interest rates are 4.0% and she plans to close her position in 6 months. What is the total premium she will earn on her short position?
Questions on Risk Management

5. Two 6-month corn put options are available. The strike prices are $1.80 and $1.75 with premiums of $0.14 and $0.12, respectively. Total costs are $1.65 per bushel and 6-month interest rates are 4.0%. Farmer Jayne wishes to hedge 20,000 bushels for 6 months. What is the highest profit or minimum loss between the two options if the spot price in 6 months is $1.70 per bushel?

6. Corn call options with a $1.70 strike price are trading for a $0.15 premium. Farmer Jayne decides to hedge her 20,000 bushels of corn by selling short call options. Six-month interest rates are 4.0% and she plans to close her position and sell her corn in 6 months. What is her profit or loss if spot prices are $1.60 per bushel when she closes her position?

7. KidCo Cereal Company sells “Sugar Corns” for $2.50 per box. The company will need to buy 20,000 bushels of corn in 6 months to produce 40,000 boxes of cereal. Non-corn costs total $60,000. What is the company’s profit if they purchase call options at $0.12 per bushel with a strike price of $1.60? Assume the 6-month interest rate is 4.0% and the spot price in 6 months is $1.65 per bushel.

8. KidCo bought forward contracts on 20,000 bushels of corn at $1.65 per bushel. Corporate tax rates are 35.00%. Revenue is $100,000 and other costs are $60,000. Spot prices on corn are $1.75 per bushel. Calculate the after-tax net income.
Derivative Securities

Option Pricing
Option Pricing

- Option and futures relationships
- Properties of option prices
- The Binomial Tree: A simplistic approach
- The Binomial Tree: A more complex treatment
- The B&S formula
- Pricing exotic options
Effect of Variables on Option Pricing

<table>
<thead>
<tr>
<th>Variable</th>
<th>$c$</th>
<th>$p$</th>
<th>$C$</th>
<th>$P$</th>
</tr>
</thead>
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<tr>
<td>$S_0$</td>
<td>+</td>
<td>−</td>
<td>+</td>
<td>−</td>
</tr>
<tr>
<td>$K$</td>
<td>−</td>
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<td>+</td>
<td>−</td>
</tr>
<tr>
<td>$D$</td>
<td>−</td>
<td>+</td>
<td>−</td>
<td>+</td>
</tr>
</tbody>
</table>
Synopsis

• Let us first entertain the put call parity which is perhaps the most important relationship among option prices.

• The put call parity is based on a basic and important rule in asset pricing: if two different investments generate the same distribution of payoffs – they must have the same cost. Otherwise, arbitrage profitability is feasible.

• Then we will directly price options using two common methods: The Binomial Tree and the famous B&S formula.

• Lastly we will describe the so-called option Greeks.
Put Call Parity for Options on Stocks

• If the underlying asset is a stock and $Div$ is the dividend stream

\[
C(K, T) = P(K, T) + [S_0 - PV_{0,T}(Div)] - e^{-rT}(K)
\]
In words . . .

- The put-call parity says that the value of a call option implies a certain fair value for the corresponding put, and vice versa.
- The argument for this pricing relationship relies on the arbitrage opportunity that results if there is divergence between the value of calls and puts with the same strike price and expiration date.
- Arbitrageurs would step in to make profitable, risk-free trades until the departure from put-call parity is eliminated. Knowing how these trades work can give you a better feel for how put options, call options, and the underlying stock are all interrelated.
- See example on the next page. For simplicity the example excludes dividend payments.
Violating the Put Call Parity: An Arbitrage Opportunity

Stock Price = 110  Call Price = 17
Put Price = 5     Effective interest rate 5%.
Maturity = .5 yr   K = 105

\[ C - P > S_0 - \frac{K}{(1 + r_f)^T} \]

\[ 17 - 5 > 110 - (105/1.05) \]
12 > 10

Since the leveraged equity is less expensive, acquire the low cost alternative and sell the high cost alternative.
# Put-Call Parity Arbitrage

(assuming no dividends)

<table>
<thead>
<tr>
<th>Position</th>
<th>Immediate Cashflow</th>
<th>Cashflow in Six Months</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$S_T &lt; 105$</td>
</tr>
<tr>
<td>Buy Stock</td>
<td>-110</td>
<td>$S_T$</td>
</tr>
<tr>
<td>Borrow $K/(1+r)^T = 100$</td>
<td>+100</td>
<td>-105</td>
</tr>
<tr>
<td>Sell Call</td>
<td>+17</td>
<td>0</td>
</tr>
<tr>
<td>Buy Put</td>
<td>-5</td>
<td>105 - $S_T$</td>
</tr>
<tr>
<td>Total</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

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Derivatives Securities
Parity for Options on Stocks

- Two more examples:
  - Price of a non-dividend paying stock: $40, \( r = 8\% \), option strike price: $40, time to expiration: 3 months, European call: $2.78, European put: $1.99.
    \[ \$2.78 = \$1.99 + \$40 - \$40e^{-0.08 \times 0.25} \]
  - Additionally, if the stock pays $5 just before expiration, call: $0.74, and put: $4.85.
    \[ \$0.74 = \$4.85 + (\$40 - \$5e^{-0.08 \times 0.25}) - \$40e^{-0.08 \times 0.25} \]

- Synthetic security creation using parity
  - Synthetic stock: buy call, sell put, lend PV of strike and dividends
  - Synthetic T-bill: buy stock, sell call, buy put (interesting tax issues)
  - Synthetic call: buy stock, buy put, borrow PV of strike and dividends
  - Synthetic put: sell stock, buy call, lend PV of strike and dividends
The Relation between Future, Strike, Call, and Put prices

- To address this relation, let us consider the following strategy. You buy call, sell put, and sell the futures contract.

<table>
<thead>
<tr>
<th></th>
<th>( S_T &lt; K )</th>
<th>( S_T \geq K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy Call</td>
<td>0</td>
<td>( S_T - K )</td>
</tr>
<tr>
<td>Sell Put</td>
<td>( S_T - K )</td>
<td>0</td>
</tr>
<tr>
<td>Sell Futures</td>
<td>( F - S_T )</td>
<td>( F - S_T )</td>
</tr>
<tr>
<td>Total</td>
<td>( F - K )</td>
<td>( F - K )</td>
</tr>
</tbody>
</table>
The General Put-Call Parity

- For European options with the same strike price and time to expiration the parity relationship is

\[ \text{Call} - \text{Put} = PV (\text{forward price} - \text{strike price}) \]

or

\[ C(K,T) - P(K,T) = PV_{0,T}(F_{0,T} - K) = e^{-rT}(F_{0,T} - K) \]
Example

- Consider buying the 6-month $1,000 strike call for a premium of $93.808 and selling a similar put for $74.201. What must the future price (set up today) be if the 6-month interest rate is 2%?

\[ 93.808 - 74.201 = PV(F - 1000) \]
Thus \( F = 1020 \), which confirms our earlier discussion.
Synthetic Forwards

• A synthetic long forward contract
  – Buying a call and selling a put on the same underlying asset, with each option having the same strike price and time to expiration
  – Example: buy the $1,000-strike S&P call and sell the $1,000-strike S&P put, each with 6 months to expiration. Call price $93.809, put price $74.201. The effective interest rate for this period is 2%.
Synthetic Forwards (cont’d)

- Differences between synthetic and actual forward contracts
  
  - The forward contract has a zero premium, while the synthetic forward requires that we pay the net option premium (in our example $19.61).
  
  - With the forward contract, we pay the forward price which is set such that the forward premium is zero, while with the synthetic forward we pay the strike price.
Synthetic Forwards (cont’d)

• Differences between synthetic and actual forward contracts

  – A forward contract with premium different from zero is often called the off-market forward. A true forward contract has, by definition, zero premium.

  – Buying a call and selling a put with the strike equal to the forward price \( F_{0,T} = K \) creates a zero-price synthetic forward contract.
Properties of Option Prices

• American versus European

- Since an American option can be exercised at anytime, whereas a European option can only be exercised at expiration, an American option must always be at least as valuable as an otherwise identical European option

\[ C_{Amer}(S, K, T) \geq C_{Eur}(S, K, T) \]

\[ P_{Amer}(S, K, T) \geq P_{Eur}(S, K, T) \]
Properties of Option Prices (cont’d)

• Option price boundaries
  
  – Call price cannot
    
    • be negative
    • exceed stock price
    • be less than the price implied by put-call parity using zero for put price:

    \[ S > C_{Amer}(S, K, T) \geq C_{Eur}(S, K, T) \geq \max[0, PV_{0,T}(F_{0,T}) - PV_{0,T}(K)] \]

  – Put price cannot
    
    • be more than the strike price
    • be less than the price implied by put-call parity using zero for put price:

    \[ K > P_{Amer}(S, K, T) \geq P_{Eur}(S, K, T) \geq \max[0, PV_{0,T}(K) - PV_{0,T}(F_{0,T})] \]
Properties of Option Prices (cont’d)

- Early exercise of American options
  - A non-dividend paying American call option should not be exercised early.
  - If there are dividends, it may be optimal to exercise early.
  - It may be optimal to exercise a non-dividend paying put option early if the underlying stock price is sufficiently low. A nice example will come up later.
Properties of Option Prices

• Time to expiration
  – An American option (both put and call) with more time to expiration is at least as valuable as an American option with less time to expiration. This is because the longer option can easily be converted into the shorter option by exercising it early.
  – A European call option on a non-dividend paying stock will be more valuable than an otherwise identical option with less time to expiration.
  – European call options on dividend-paying stock and European puts may be less valuable than an otherwise identical option with less time to expiration.
Properties of Option Prices (cont’d)

- Different strike prices \((K_1 < K_2)\), for both European and American options
  - A call with a low strike price is at least as valuable as an otherwise identical call with higher strike price
    \[ C(K_1) \geq C(K_2) \]
  - A put with a high strike price is at least as valuable as an otherwise identical call with low strike price
    \[ P(K_2) \geq P(K_1) \]
  - The premium difference between otherwise identical calls with different strike prices cannot be greater than the difference in strike prices
    \[ C(K_1) - C(K_2) \leq K_2 - K_1 \]
Panel A shows call option premiums for which the change in the option premium ($6) exceeds the change in the strike price ($5). Panel B shows how a bear spread can be used to arbitrage these prices. By lending the bear spread proceeds, we have a zero cash flow at time $0$; the cash outflow at time $T$ is always greater than $1$.

### Panel A

<table>
<thead>
<tr>
<th>Strike</th>
<th>Premium</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>18</td>
</tr>
<tr>
<td>55</td>
<td>12</td>
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### Panel B

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Time 0</th>
<th>$S_T &lt; 50$</th>
<th>$50 \leq S_T \leq 55$</th>
<th>$S_T \geq 55$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy 55-Strike Call</td>
<td>$-12$</td>
<td>$0$</td>
<td>$0$</td>
<td>$S_T - 55$</td>
</tr>
<tr>
<td>Sell 50-Strike Call</td>
<td>$18$</td>
<td>$0$</td>
<td>$50 - S_T$</td>
<td>$50 - S_T$</td>
</tr>
<tr>
<td>Total</td>
<td>$6$</td>
<td>$0$</td>
<td>$50 - S_T$</td>
<td>$-5$</td>
</tr>
</tbody>
</table>
Properties of Option Prices
(cont’d)

• Different strike prices ($K_1 < K_2 < K_3$), for both European and American options

  – The premium difference between otherwise identical puts with different strike prices cannot be greater than the difference in strike prices
    \[ P(K_1) - P(K_2) \leq K_2 - K_1 \]

  – Premiums decline at a decreasing rate for calls with progressively higher strike prices. (Convexity of option price with respect to strike price)
    \[ \frac{C(K_1) - C(K_2)}{K_2 - K_1} \leq \frac{C(K_2) - C(K_3)}{K_3 - K_2} \]
Properties of Option Prices (cont’d)

The example in Panel A violates the proposition that the rate of change of the option premium must decrease as the strike price rises. The rate of change from 50 to 59 is $5.1/9$, while the rate of change from 59 to 65 is $3.9/6$. We can arbitrage this convexity violation with an asymmetric butterfly spread. Panel B shows that we earn at least $3 plus interest at time $T$.

<table>
<thead>
<tr>
<th>Panel A</th>
<th>Strike</th>
<th>50</th>
<th>59</th>
<th>65</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call Premium</td>
<td>14</td>
<td>8.9</td>
<td>5</td>
<td></td>
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</table>

<table>
<thead>
<tr>
<th>Panel B</th>
<th>Time 0</th>
<th>$S_T &lt; 50$</th>
<th>$50 \leq S_T \leq 59$</th>
<th>$59 \leq S_T \leq 65$</th>
<th>$S_T &gt; 65$</th>
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</thead>
<tbody>
<tr>
<td>Buy Four 50-Strike Calls</td>
<td>-56</td>
<td>0</td>
<td>$4(S_T - 50)$</td>
<td>$4(S_T - 50)$</td>
<td>$4(S_T - 50)$</td>
</tr>
<tr>
<td>Sell Ten 59-Strike Calls</td>
<td>89</td>
<td>0</td>
<td>0</td>
<td>10($59 - S_T$)</td>
<td>10($59 - S_T$)</td>
</tr>
<tr>
<td>Buy Six 65-Strike Calls</td>
<td>-30</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6($S_T - 65$)</td>
</tr>
<tr>
<td>Lend $3$</td>
<td>-3</td>
<td>$3e^{rT}$</td>
<td>$3e^{rT}$</td>
<td>$3e^{rT}$</td>
<td>$3e^{rT}$</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
<td>$3e^{rT}$</td>
<td>$3e^{rT} + 4(S_T - 50)$</td>
<td>$3e^{rT} + 6(65 - S_T)$</td>
<td>$3e^{rT}$</td>
</tr>
</tbody>
</table>

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Properties of Option Prices (cont’d)

**TABLE 9.8**

Arbitrage of mispriced puts using asymmetric butterfly spread.

### Panel A

<table>
<thead>
<tr>
<th>Strike</th>
<th>50</th>
<th>55</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>Put Premium</td>
<td>4</td>
<td>8</td>
<td>16</td>
</tr>
</tbody>
</table>

### Panel B

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Time 0</th>
<th>Expiration or Exercise</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$S_T &lt; 50$</td>
</tr>
<tr>
<td>Buy Three 50-Strike Puts</td>
<td>−12</td>
<td>3($50 - S_T$)</td>
</tr>
<tr>
<td>Sell Four 55-Strike Puts</td>
<td>32</td>
<td>4($S_T - 55$)</td>
</tr>
<tr>
<td>Buy One 70-Strike Put</td>
<td>−16</td>
<td>$70 - S_T$</td>
</tr>
<tr>
<td>Lend $4</td>
<td>−4</td>
<td>$4 e^{rT}$</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
<td>$4 e^{rT}$</td>
</tr>
</tbody>
</table>
**Collection of Parity Relationships**

**TABLE 9.2**

Versions of put-call parity. Notation in the table includes the spot currency exchange rate, $x_0$; the risk-free interest rate in the foreign currency, $r_f$; and the current bond price, $B_0$.

<table>
<thead>
<tr>
<th>Underlying Asset</th>
<th>Parity Relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>Futures Contract</td>
<td>$e^{-r_T} F_{0,T} = C(K, T) - P(K, T) + e^{-r_T} K$</td>
</tr>
<tr>
<td>Stock, No-Dividend</td>
<td>$S_0 = C(K, T) - P(K, T) + e^{-r_T} K$</td>
</tr>
<tr>
<td>Stock, Discrete Dividend</td>
<td>$S_0 - PV_{0,T}(Div) = C(K, T) - P(K, T) + e^{-r_T} K$</td>
</tr>
<tr>
<td>Stock, Continuous Dividend</td>
<td>$e^{-\delta T} S_0 = C(K, T) - P(K, T) + e^{-r_T} K$</td>
</tr>
<tr>
<td>Currency</td>
<td>$e^{-r_f T} x_0 = C(K, T) - P(K, T) + e^{-r_T} K$</td>
</tr>
<tr>
<td>Bond</td>
<td>$B_0 - PV_{0,T}(Coupons) = C(K, T) - P(K, T) + e^{-r_T} K$</td>
</tr>
</tbody>
</table>
Binomial Option Pricing: The Simplistic Approach

- Suppose that we have a stock with a price of $60 today and that in one year, the price will be either $90 or $30.
- The risk-free rate is 10% (annualized effective rate).
- Consider the payoffs to a call option with a strike price of $60.
Binomial Option Pricing Example

Stock Price

Call Option with K = 60
A Replicating Portfolio

Alternative Portfolio
Buy 0.5 shares of stock for $30
Borrow $13.64 (@10% Rate)
Net outlay $16.36
Payoff to this strategy:

\[
\begin{array}{c|c|c}
S=30 & S=90 \\
\hline
\text{Value of 0.5 shares} & 15 & 45 \\
\text{Repay loan} & -15 & -15 \\
\text{Net Payoff} & 0 & 30 \\
\end{array}
\]

Payoffs are exactly the same as the Call. Therefore the call price should be $16.36. If not, then arbitrage is possible!
Another View of Replication of Payoffs and Option Values

This example also shows us how options can be used to hedge the underlying asset. What if we held one share of stock and wrote 2 calls (K = 60)?

Portfolio is perfectly hedged

<table>
<thead>
<tr>
<th>Stock Value</th>
<th>30</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two written Calls</td>
<td>0</td>
<td>-60</td>
</tr>
<tr>
<td>Net payoff</td>
<td>30</td>
<td>30</td>
</tr>
</tbody>
</table>

The combined portfolio has a riskless return (note that this portfolio would cost (60-2×16.36=27.28), so the return is 10%, i.e. the riskless rate of return.
Binomial Option Pricing: General One-period Case

Stock Price

Call Option with strike price = K
The Replicating Portfolio

Alternative Portfolio
Buy $\Delta$ shares of stock for $\Delta S$
Put $B$ into bonds (@ r% Rate)
(if B is negative, then borrow)
Net outlay $\Delta S + B$
Payoff to this strategy:

Value of $\Delta$ shares
$S = uS$  $S = dS$

Bond (R=1+r)
$\Delta uS + RB$  $\Delta dS + RB$

Net Payoff
$\Delta uS + RB$  $\Delta dS + RB$
Solving for $\Delta$ and $B$

- We want the replicating portfolio to give us the same payoff as the option in both states.

\[
\Delta uS + RB = C_u
\]
\[
\Delta dS + RB = C_d
\]

\[
\Delta = \frac{C_u - C_d}{uS - dS}
\]
\[
B = \frac{u C_d - d C_u}{(u - d) R}
\]
The One-Period Binomial Option Value Formula

• The value of the call option today must be equivalent to the value of the replicating portfolio to prevent arbitrage.

\[ C = \Delta S + B \]

\[ = \frac{C_u - C_d}{u - d} + \frac{u C_d - d C_u}{(u - d)R} \]

\[ = \frac{p C_u + (1 - p) C_d}{R}, \text{ where } p = \frac{R - d}{u - d} \]
Risk-adjusted probabilities

- Note that the valuation formula for the call is just a probability weighted average of the payoffs discounted by the risk-free rate. The adjustment for risk is taken into account in the probabilities that are used: $p$ and $(1-p)$.
- These probabilities can be backed out by considering the underlying asset.

\[
S = \frac{puS + (1-p)dS}{R} \quad \Rightarrow \quad p = \frac{R - d}{u - d}
\]
Generalizing the Binomial Tree

While it may seem unrealistic that the price of the stock can only go up or down in a given period, we can break the life of the option into as many (very short) periods as we like!

This point can be illustrated by using a two-period example (but can be generalized to as many periods as deemed necessary).

Consider a stock that is selling today for 80, and which will go up or down by 50% in each of the next two periods (each of which is one-year long). The annual risk-free rate is 10% (i.e. R=1.10).
A Two-Period Example
The binomial tree for the call option

- Consider a call option with a strike price of $80.

\[
\begin{align*}
C_{uu} &= 180 - 80 = 100 \\
C_{ud} &= 0 \\
C_{dd} &= 0
\end{align*}
\]
Valuing $C_u$ and $C_d$

- If we are in the up-state at the end of the first period, we are looking at a one-period tree.
- First, we need to calculate $p = (R-d)/(u-d) = (1.10-.5)/(1.5-.5) = 0.6$
- Using our one-period formula:
  - $C_u = [(0.6) 100 + (0.4) 0] / 1.10 = 54.54$
  - $C_d = [(0.6) 0 + (0.4) 0] / 1.10 = 0$
Valuing C

• Now, consider the first period on its own. We now know the values of the options at the end of this first period ($C_u$ and $C_d$).

• Using the same one-period valuation technique:

$$C = \frac{[(.6)(54.54) + (.4)(0)]}{1.10} = 29.75$$
The replicating portfolios

• Recall that we have formulas for \( \Delta \) and \( B \). We can use these at each point in the tree, based on the call and stock values in the up and down state at the end of each period.

• For example, at the beginning of the first period, we can calculate:

\[
\begin{align*}
\Delta &= \frac{C_u - C_d}{uS - dS} = \frac{54.54 - 0}{120 - 40} = 0.682 \\
B &= \frac{u C_d - d C_u}{(u - d)R} = \frac{0 - (0.5)(54.54)}{(1.5 - 0.5)1.1} = -24.793
\end{align*}
\]
Option Prices and Deltas

- The tree below shows the prices and deltas (in parentheses) at each node.
Self Financing Strategy

• If the stock price is up you buy more stocks.
• You pay \((0.833-0.682) \times 120=18.2\)
• You also pay interest on your loan \(0.1 \times 24.793=2.5\)
• Who brings the money?
• Note that the new \(B=-45.45\)
• So the amount of debt increases by 20.7
Self Financing Strategy

• What if the stock price drops to 40.
• Then you sell the stock for $0.682 \times 40 = 27.3$
• You pay interest 2.5
• You net inflow is 24.8
• You also repay the loan 24.8
• Thus your net inflow is zero.
Put Options

• The analysis we have just gone through works exactly the same way for put options, or for any derivative security that depends on the stock price.

• Interpret $C_u$ and $C_d$ as the payoffs to a derivative security such as a call or put.

• In replicating a put, delta will be negative and $B$ will be positive - i.e. we would short stock and lend out $B$.

• Note that $P = \Delta S + B$, or $B = P - \Delta S$, so the amount lent out is equal to the money from the short position plus the cost of the put option.
American Options

• The binomial valuation approach can also be used for valuing American Options (in fact, it must be used since formulas such as the Black-Scholes don’t allow for early exercise).

• The only difference in the procedure is that at every point in the tree we need to check to see whether it is preferable to exercise the option early.

• In the following example, we value both a European and an American put option on the stock we just looked at, where the strike price equals $80.
European Put Option

- Find the payoffs at maturity, and work backwards to find option and delta values.

Diagram:
- Payoff at maturity: 0, 20, 60
- node values: 7.27, 15.87, 32.73
- delta values: -.167, -.318, -1.0

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American Put Option

• In this case, check at each node to see whether it would be preferable to exercise (it is when the stock price goes down to $40 in the first year). Note that the American option is worth $2.64 more than the European option.
The Binomial Solution: More Complex, yet Appealing, Approach

- Here we consider continuously compounded figures, allow for dividend payments, and use the time interval $h$.
- In the end we show convergence to the B&S formula.
- The replicating portfolio satisfies

$$(\Delta \times uS \times e^{\delta h}) + (B \times e^{rh}) = C_u$$

$$(\Delta \times dS \times e^{\delta h}) + (B \times e^{rh}) = C_d$$

$$B = e^{-rh} \frac{uC_d - dC_u}{u - d} \quad \Delta = e^{-\delta h} \frac{C_u - C_d}{S(u - d)}$$
The Binomial Solution

• The cost of creating the option is the cash flow required to buy the shares and bonds. Thus, the cost of the option is $\Delta S+B$

$$\Delta S + B = e^{-rh}\left( C_u \frac{e^{(r-\delta)h} - d}{u-d} + C_d \frac{u - e^{(r-\delta)h}}{u-d} \right)$$

• The no-arbitrage condition is

$$u > e^{(r-\delta)h} > d$$
Arbitraging a Mispriced Option

• If the observed option price differs from its theoretical price, arbitrage is possible
  – If an option is overpriced, we can sell the option and buy a synthetic option at the same time.
  – If an option is underpriced, we buy the option and sell a synthetic option at the same time.
Risk-Neutral Pricing

• We can interpret the terms \((e^{(r-\delta)h} - d)/(u - d)\) and \((u - e^{(r-\delta)h})/(u - d)\) as probabilities

• Let

\[
p^* = \frac{e^{(r-\delta)h} - d}{u - d}
\]

• Then equation can then be written as

\[
C = e^{-rh} \left[ p^* C_u + (1 - p^*) C_d \right]
\]

– Where \(p^*\) is the risk-neutral probability of an increase in the stock price
Constructing a Binomial Tree

• In the absence of uncertainty, a stock must appreciate at the risk-free rate less the dividend yield. Thus, from time $t$ to time $t+h$, we have

$$F_{t,t+h} = S_{t+h} = S_t e^{(r-\delta)h}$$

• This is also the forward price of the stock.
Constructing a Binomial Tree
(cont’d)

• With uncertainty (discrete distribution), we have
  \[ uS_t = F_{t,t+h} e^{\sigma \sqrt{h}} \]
  \[ dS_t = F_{t,t+h} e^{-\sigma \sqrt{h}} \]

• Where \( \sigma \) is the annualized standard deviation of the continuously compounded return, and \( \sigma \sqrt{h} \) is standard deviation over a period of length \( h \)

• Those relations can be rewritten as
  \[ u = e^{(r-\delta)h+\sigma \sqrt{h}} \]
  \[ d = e^{(r-\delta)h-\sigma \sqrt{h}} \]

We refer to a tree constructed based on these up and down moves as a “forward tree.”
Summary

• In order to price an option, we need to know
  – Stock price
  – Strike price
  – Standard deviation of stock returns
  – Dividend yield
  – Risk-free rate
  – Time to expiration

• Using the risk-free rate, the dividend yield, and $\sigma$, we can approximate the future distribution of the stock by creating a binomial tree as outlined above.

• Once we have the binomial tree, it is possible to price the option using the derived equations.
A Two-Period European Call

- We can extend the previous example to price a 2-year option, assuming all inputs are the same as before.

\[
\begin{align*}
\text{S}_{uu} &= 87.669 \\
\text{S}_{u} &= 59.954 \\
\text{S}_{d} &= 32.903 \\
\text{S}_{dd} &= 26.405 \\
\end{align*}
\]

\[
\begin{align*}
\Delta_u &= 1.000 \\
\Delta_d &= 0.734 \\
\Delta &= 0.374 \\
B &= -36.925 \\
B &= -19.337 \\
B &= -9.111 \\
B &= -9.111 \\
\end{align*}
\]
A Two-period European Call (cont’d)

• Note that an up move by the stock followed by a down move ($S_{ud}$) generates the same stock price as a down move followed by an up move ($S_{du}$).

• This is called a recombining tree.

• Otherwise, we would have a non-recombining tree.
Pricing the Call Option

• To price an option with two binomial periods, we work *backward* through the tree
  
  – *Year 2, Stock Price*=$87.669:
    since we are at expiration, the option value is \( \text{max}(0, S - K) = $47.669 \)
  
  – *Year 2, Stock Price*=$48.114:
    similarly, the option value is $8.114
  
  – *Year 2, Stock Price*=$26.405:
    since the option is out of the money, the value is 0
Pricing the Call Option (cont’d)

• **Year 1, Stock Price=$59.954:**
  at this node, we compute the option value
given that \(uS\) is $87.669 and \(dS\) is $48.114

\[
e^{-0.08} \left( \frac{e^{0.08} - 0.803}{1.462 - 0.803} \times 47.669 + \frac{1.462 - e^{0.08}}{1.462 - 0.803} \times 8.114 \right) = 23.029
\]

• **Year 1, Stock Price=$32.903:** the option value is $3.187
• **Year 0, Stock Price = $41:** the option value is $10.737
Pricing the Call Option (cont’d)

Notice that

- The option was priced by working backward through the binomial tree
- The option price is greater for the 2-year than for the 1-year option ($10.737 versus $7.839)
- The option’s $\Delta$ and $\mathbf{B}$ are different at different nodes. At a given point in time, $\Delta$ increases to 1 as we go further into the money
- Permitting early exercise would make no difference. At every node prior to expiration, the option price is greater than $S - K$; thus, we would not exercise even if the option was American
Many Binomial Periods

• Dividing the time to expiration into more periods allows us to generate a more realistic tree with a larger number of different values at expiration
  – Consider the previous example of the 1-year European call option
  – Let there be three binomial periods. Since it is a 1-year call, this means that the length of a period is $h = 1/3$
  – Assume that other inputs are the same as before (so, $r = 0.08$ and $\sigma = 0.3$)
Many Binomial Periods (cont’d)

- The stock price and option price tree:
Many Binomial Periods (cont’d)

• Note that since the length of the binomial period is shorter, $u$ and $d$ are smaller than before: $u = 1.2212$ and $d = 0.8637$ (as opposed to 1.462 and 0.803 with $h = 1$)

  – The second-period nodes are computed as follows

    $$S_u = 41e^{0.08\times1/3+0.3\sqrt{1/3}} = 50.071$$

    $$S_d = 41e^{0.08\times1/3-0.3\sqrt{1/3}} = 35.411$$

  – The remaining nodes are computed similarly

• Analogous to the procedure for pricing the 2-year option, the price of the three-period option is computed by working backward.

• The option price is $7.074
Put Options

• We compute put option prices using the same stock price tree and in the same way as call option prices.

• The only difference with a European put option occurs at expiration:
  – Instead of computing the price as max \((0, S - K)\), we use max \((0, K - S)\).
Put Options (cont’d)

• A binomial tree for a European put option with 1-year to expiration

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American Options

- The value of the option if it is left “alive” (i.e., unexercised) is given by the value of holding it for another period.

- The value of the option if it is exercised is given by $max(0, S - K)$ if it is a call and $max(0, K - S)$ if it is a put.

- For an American call, the value of the option at a node is given by $Max(continuation\ value, exercised\ value)$. 
American Options (cont’d)

• The valuation of American options proceeds as follows
  – At each node, we check for early exercise
  – If the value of the option is greater when exercised, we assign that value to the node. Otherwise, we assign the value of the option unexercised
  – We work backward through the three as usual
American Options (cont’d)

- Consider an American version of the put option valued in the previous example.
American Options (cont’d)

• The only difference in the binomial tree occurs at the $S_{dd}$ node, where the stock price is $30.585$. The American option at that point is worth $40 - $30.585 = $9.415$, its early-exercise value (as opposed to $8.363$ if unexercised). The greater value of the option at that node ripples back through the tree.

• Thus, an American option is more valuable than the otherwise equivalent European option.
To get a feel …

• To get a feel of how the binomial tree work you can use the on-line binomial tree calculators:

• http://www.hoadley.net/options/binomialtree.aspx?tree=B
Nice features

- The big advantage the binomial model is that it can be used to accurately price American options.
- This is because it's possible to check at every point in an option's life (at every step of the binomial tree) for the possibility of early exercise.
- Where an early exercise point is found it is assumed that the option holder would elect to exercise, and the option price can be adjusted to equal the intrinsic value at that point. This then flows into the calculations higher up the tree and so on.
- The online binomial tree graphical option calculator highlights those points in the tree structure where early exercise would have caused an American price to differ from a European price.
- The binomial model basically solves the same equation, using a computational procedure that the Black-Scholes model solves using an analytic approach and in doing so provides opportunities along the way to check for early exercise for American options.
Options on Currency

• With a currency with spot price (the current exchange rate) $x_0$, the forward price is

$$F_{0,t} = x_0 e^{(r-r_f)t}$$

– Where $r_f$ is the foreign interest rate

• Thus, we construct the binomial tree using

$$ux = xe^{(r-r_f)h + \delta \sqrt{h}}$$

$$dx = xe^{(r-r_f)h - \delta \sqrt{h}}$$
Options on Currency (cont’d)

• Investing in a currency means investing in a money-market fund denominated in that currency. The dividend yield is the foreign currency denominated rate (the forward contract does not earn that rate).

• Taking into account interest on the foreign-currency denominated obligation, the two equations are

\[ \Delta \times u x e^{r_f h} + e^{r_h} \times B = C_u \]
\[ \Delta \times d x e^{r_f h} + e^{r_h} \times B = C_d \]

• The risk-neutral probability of an up move is

\[ p^* = \frac{e^{(r-r_f)h} - d}{u - d} \]
Options on Currency: Example

• Consider a dollar-denominated American put option on the euro, where
  – The current exchange rate is $1.05/€
  – The strike is $1.10/€
  – The euro-denominated interest rate is 3.1%
  – The dollar-denominated rate is 5.5%
Options on Currency (cont’d)

• The binomial tree for the American put option on the euro
Options on Futures Contracts

• Assume the forward price is the same as the futures price

• The nodes are constructed as

\[ u = e^{\sigma \sqrt{h}} \]
\[ d = e^{-\sigma \sqrt{h}} \]

• We need to find the number of futures contracts, \( \Delta \), and the lending, \( B \), that replicates the option

  – A replicating portfolio must satisfy

\[ \Delta \times (uF - F) + e^{rh} \times B = C_u \]
\[ \Delta \times (dF - F) + e^{rh} \times B = C_d \]
Options on Futures Contracts (cont’d)

• Solving for $\Delta$ and $B$ gives

$$\Delta = \frac{C_u - C_d}{F(u - d)}$$

$$B = e^{-rh} \left( C_u \frac{1 - d}{u - d} + C_d \frac{u - 1}{u - d} \right)$$

- $\Delta$ tells us how many futures contracts to hold to hedge the option, and $B$ is simply the value of the option.

• We can now price the option.

• The risk-neutral probability of an up move is given by

$$p^* = \frac{1 - d}{u - d}$$
Options on Futures Contracts (cont’d)

• The motive for early-exercise of an option on a futures contract is the ability to earn interest on the mark-to-market proceeds

  – When an option is exercised, the option holder pays nothing, is entered into a futures contract, and receives mark-to-market proceeds of the difference between the strike price and the futures price
Options on Futures Contracts (cont’d)

• A tree for an American call option on a gold futures contract
Stock Paying Discrete Dividends

• Suppose that a dividend will be paid between times $t$ and $t+h$ and that its future value at time $t+h$ is $D$

• The time $t$ forward price for delivery at $t+h$ is

$$F_{t,t+h} = S_t e^{rh} - D$$

• Since the stock price at time $t+h$ will be ex-dividend, we create the up and down moves based on the ex-dividend stock price

$$S_t^u = (S_t e^{rh} - D)e^{\sigma \sqrt{h}}$$

$$S_t^d = (S_t e^{rh} - D)e^{-\sigma \sqrt{h}}$$
Stock Paying Discrete Dividends (cont’d)

• When a dividend is paid, we have to account for the fact that the stock earns the dividend

\[(S^u_t + D)\Delta + e^{rh} B = C_u\]
\[(S^d_t + D)\Delta + e^{rh} B = C_d\]

• The solution is

\[\Delta = \frac{C_u - C_d}{S^u_t - S^d_t}\]

\[B = e^{-rh}\left[\frac{S^u_t C_d - S^d_t C_u}{S^u_t - S^d_t}\right] - \Delta De^{-rh}\]

– Because the dividend is known, we decrease the bond position by the PV of the certain dividend
Problems with the Discrete Dividend Tree

• The conceptual problem is that the stock price could in principle become negative if there have been large downward moves in the stock prior to the dividend.

• The practical problem is that the tree does not completely recombine after a discrete dividend.

• The following tree, where a $5 dividend is paid between periods 1 and 2, demonstrates that with a discrete dividend, the order of up and down movements affects the price.
  – In the third binomial period, there are six rather than four possible stock prices.
Problems With the Discrete Dividend Tree (cont’d)

<table>
<thead>
<tr>
<th>Binomial Period:</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dividend:</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$67.417

$55.203

$50.071

$39.041

$41.000

$33.720

$39.041

$50.071

$55.203

$67.417
A Binomial Tree Using the Prepaid Forward

- Hull (1997) presents a method of constructing a tree for a dividend-paying stock that solves both problems

- If we know for certain that a stock will pay a fixed dividend, then we can view the stock price as being the sum of two components
  - The dividend, which is like a zero-coupon bond with zero volatility, and
  - The PV of the ex-dividend value of the stock, i.e., the prepaid forward price
A Binomial Tree Using the Prepaid Forward (cont’d)

• Suppose we know that a stock will pay a dividend $D$ at time $T_D < T$, where $T$ is the expiration date of the option
  – We base stock price movements on the prepaid forward price
    $$F^P_{t,T} = S_t - De^{-r(T_D-t)}$$
  – The one-period forward price for the prepaid forward is
    $$F^P_{t,t+h} = F^P_{t,T} e^{rh}$$
  – This gives us up and down movements of
    $$u = e^{rh+\sigma\sqrt{h}}$$
    $$d = e^{rh-\sigma\sqrt{h}}$$
  – However, the actual stock price at each node is given by
    $$S_t = F^P_{t,T} + De^{-r(T_D-t)}$$
A Binomial Tree Using the Prepaid Forward (cont’d)

• A binomial tree constructed using the prepaid-forward method
The Black-Scholes Model

• What happens if the number of periods in the binomial tree are greatly increased, and the periods become extremely short?
• If the volatility and interest rate in each period is the same, then we arrive at the model developed by Fischer Black and Myron Scholes.
Black-Scholes Formula (cont’d)

• Call Option price

\[ C(S, K, \sigma, r, T, \delta) = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) \]

• Put Option price

\[ P(S, K, \sigma, r, T, \delta) = Ke^{-rT} N( -d_2) - Se^{-\delta T} N( -d_1) \]

where

\[ d_1 = \frac{ln(S / K) + (r - \delta + \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma \sqrt{T} \]
The Put Call Parity Revisits

- Note that the put price easily follows by the implementing the put call parity and using the relations

\[ N(-d1) = 1 - N(d1) \]
\[ N(-d2) = 1 - N(d2) \]
Option Pricing Parameters

• The formula requires six parameters:
  o Stock price, dividend, and volatility (stock level)
  o Strike price and time to expiration (option level)
  o Risky free rate (economy level)
Black-Scholes (BS) Assumptions

• Assumptions about stock return distribution
  – Continuously compounded returns on the stock are normally distributed and independent over time (no “jumps”)
  – The volatility of continuously compounded returns is known and constant
  – Future dividends are known, either as dollar amount or as a fixed dividend yield
Black-Scholes (BS) Assumptions (cont’d)

• Assumptions about the economic environment
  – The risk-free rate is known and constant
  – There are no transaction costs or taxes
  – It is possible to short-sell costlessly and to borrow at the risk-free rate
Other Underlying Assets

- So far we have been dealing with the underlying asset being a stock that pays continuous dividends.
- Next, we would like to implement the B&S formula to price:
  a. Options on futures contracts
  b. Options on stocks paying discrete dividends
  c. Options on currencies.
Option on Future Contract

• Call Price

\[
C(F_{0,T}^{P}(S), F_{0,T}^{P}(K), \sigma, T) = F_{0,T}^{P}(S)N(d_1) - F_{0,T}^{P}(K)N(d_2)
\]

where

\[
d_1 = \frac{\ln \left( \frac{F_{0,T}^{P}(S)}{F_{0,T}^{P}(K)} \right) + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}}, \quad d_2 = d_1 - \sigma \sqrt{T}
\]

and

\[
F_{0,T}^{P}(K) = Ke^{-rT}
\]
Applying BS to Stocks with Discrete Dividends

The prepaid forward price for stock with discrete dividends is

\[ F_{0,T}^P(S) = S_0 - PV_{0,T}(Div) \]

- Examples
  - \( S = $41, K = $40, \sigma = 0.3, r = 8\%, t = 0.25, Div = $3 \) in one month
  - \( PV(Div) = 3e^{-0.08/12} = $2.98 \)
  - Use $41 – $2.98 = $38.02 as the stock price in BS formula
  - The BS European call price is $1.763
  - Compare this to European call on stock without dividends: $3.399
Applying BS to Options on currencies

- The prepaid forward price for the currency is

\[ F_{0,T}^P(x) = x_0e^{-r_f T} \]

- Where \( x \) is domestic spot rate and \( r_f \) is foreign interest rate

- Example
  - \( x = \$0.92/€ \), \( K = \$0.9 \), \( \sigma = 0.10 \), \( r = 6\% \), \( T = 1 \), and \( \delta = 3.2\% \)
  - The dollar-denominated euro call price is \$0.0606
  - The dollar-denominated euro put price is \$0.0172
Applying BS to Options on futures

- The prepaid forward price for a futures contract is the PV of the futures price. Therefore

\[ C(F, K, \sigma, r, t) = Fe^{-rt}N(d_1) - Ke^{-rt}N(d_2) \]

where

\[ d_1 = \frac{ln(F/K) + \frac{1}{2}\sigma^2T}{\sigma\sqrt{T}} \]

and

\[ d_2 = d_1 - \sigma\sqrt{T} \]

- Example

  - Suppose 1-yr. futures price for natural gas is $2.10/MMBtu, \( r = 5.5\% \)
  - Therefore, \( F = 2.10, K = 2.10, \) and \( \delta = 5.5\% \)
  - If \( \sigma = 0.25, T = 1, \) call price = put price = $0.197721
Pricing Exotic Options

• Below we display the price of four exotic options while the starting point is the B&S formula $C=BS(S,K,\sigma,r,T,\delta)$.

• That is, we will use the BS formula but will change one or more of the six underlying parameters.
Option Payoff: \( \max[S^2 - K, 0] \)

- Here, at the time of expiration it is not the price of the underlying stock that matters – rather it is the squared stock price.
- The option price is then given by

\[ \text{Price} = BS\left(S^2, K, 2\sigma, r, T, -(r + \sigma^2) \right) \]
**Option Payoff:** \[ \max[S_1S_2 - K, 0] \]

- Here, at the time of expiration it is not the price of the underlying stock that matters – rather it is the product of two stock prices.

- The option price is then given by:
  \[
  \text{Price} = BS\left( S_1S_2, K, \sigma^*, r, T, -(r + \rho \sigma_1 \sigma_2) \right)
  \]

  \[
  \sigma^* = \sqrt{\sigma_1^2 + \sigma_2^2 + 2 \rho \sigma_1 \sigma_2}
  \]
Option Payoff: \( \max \left[ \frac{S_1}{S_2} - K, 0 \right] \)

- At the time of expiration it is not the price of the underlying stock that matters – rather it is the ratio of two stock prices.

- The option price is then given by:

\[
\text{Price} = BS \left( \frac{S_1}{S_2}, K, \sigma^*, r, T, r - \sigma_2 (\sigma_2 - \rho \sigma_1) \right)
\]

\[
\sigma^* = \sqrt{\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2}
\]
Payoff: \[ \max \left[ \left( S_1 S_2 \right)^{0.5} - K , 0 \right] \]

- At the time of expiration it is not the price of the underlying stock that matters – rather it is the geometric average of two stock prices.

- The option price is then given by:

\[
\text{Price} = BS \left( \sqrt{S_1 S_2}, K, \sigma^*, r, T, -\frac{1}{8} \left( \sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2 \right) \right) \\
\sigma^* = \frac{1}{2} \sqrt{\sigma_1^2 + \sigma_2^2 + 2 \rho \sigma_1 \sigma_2}
\]
Option Pricing:

The Log Normal Distribution
Risk Neutral vs. Physical
Probability
Implied Volatility
Lognormality and the Binomial Model

• The binomial tree approximates a lognormal distribution and the B&S explicitly assumes that stock returns follow the lognormal distribution.

• The lognormal distribution is the probability distribution that arises from the assumption that continuously compounded returns on the stock are normally distributed.

• With the lognormal distribution, the stock price is positive, and the distribution is skewed to the right, that is, there is a chance of extremely high stock prices.
Lognormality and the Binomial Model (cont’d)

- The binomial model implicitly assigns probabilities to the various nodes.
Lognormality and the Binomial Model (cont’d)

- The following graph compares the probability distribution for a 25-period binomial tree with the corresponding lognormal distribution.
Are the Binomial Model and B&S Realistic?

• Both models are a form of the random walk model, adapted to modeling stock prices. The lognormal random walk model here assumes that
  – Volatility is constant
  – “Large” stock price movements do not occur
  – Returns are independent over time

• All of these assumptions appear to be violated in the data
Estimated Volatility

- We need to decide what value to assign to $\sigma$, which we cannot observe directly

- One possibility is to measure $\sigma$ by computing the standard deviation of continuously compounded historical returns
  - Volatility computed from historical stock returns is historical volatility
  - This is a reasonable way to estimate volatility when continuously compounded returns are independent and identically distributed
  - If returns are not independent—as with some commodities—volatility estimation becomes more complicated
The Log Normal Distribution

• Next, we study the log normal distribution in depth.

• We will start with explaining properties of the normal distribution then move to the lognormal distribution.
The Normal Distribution

- Normal distribution (or density)

\[ \Phi(x; \mu, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2} \]
The Normal Distribution (cont’d)

• Normal density is symmetric: \( \Phi(\mu + x; \mu, \sigma) = \Phi(\mu - x; \mu, \sigma) \)

• If a random variable \( x \) is normally distributed with mean \( \mu \) and standard deviation, \( \sigma \)

\[ x \sim N(\mu, \sigma^2) \]

• \( z \) is a random variable distributed standard normal: \( z \sim N(0,1) \)

• The value of the cumulative normal distribution function \( N(a) \) equals to the probability \( P \) of a number \( z \) drawn from the normal distribution to be less than \( a \).

\[ N(a) \equiv \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx \]
The Normal Distribution (cont’d)

Standard Normal Density, $\phi(z)$

Area = 0.6179

$z = 0.3$

Standard Normal Distribution, $N(z)$

Height = $N(0.3) = 0.6179$

$z = 0.3$
The Normal Distribution (cont’d)

• The probability of a number drawn from the standard normal distribution will be between $a$ and $-a$:

\[
\text{Prob} (z < -a) = N(-a) \\
\text{Prob} (z < a) = N(a)
\]

therefore

\[
\text{Prob} (-a < z < a) = N(a) - N(-a) = N(a) - [1 - N(a)] = 2 \cdot N(a) - 1
\]

• Example: \( \text{Prob} (-0.3 < z < 0.3) = 2 \cdot 0.6179 - 1 = 0.2358 \)
The Normal Distribution (cont’d)

• Converting a normal random variable to standard normal:
  – If \( x \sim N(\mu, \sigma^2) \), then \( z \sim N(0,1) \) if \( z = \frac{x-\mu}{\sigma} \)

• And vice versa:
  – If \( z \sim N(0,1) \), then \( x \sim N(\mu, \sigma^2) \) if \( x = \mu + \sigma z \)

• Example 18.2: Suppose \( x \sim N(3,5) \) and \( z \sim N(0,1) \) then \( \frac{x-3}{5} \sim N(0,1) \), and \( 3 + 5 \times z \sim N(3,25) \)
The Normal Distribution (cont’d)

- The sum of normal random variables is also

\[
\sum_{i=1}^{n} \omega_i x_i \sim N\left(\sum_{i=1}^{n} \omega_i \mu_i, \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_i \omega_j \sigma_{ij}\right)
\]

- where \(x_i, i = 1, \ldots, n\), are \(n\) random variables,
  with mean \(E(x_i) = \mu_i\), variance \(Var(x_i) = \sigma_i^2\),
  covariance \(Cov(x_i, x_j) = \sigma_{ij} = \rho_{ij} \sigma_i \sigma_j\)
The Lognormal Distribution

• A random variable $x$ is lognormally distributed if $\ln(x)$ is normally distributed
  – If $x$ is normal, and $\ln(y) = x$ (or $y = e^x$), then $y$ is lognormal
  – If continuously compounded stock returns are normal then the stock price is lognormally distributed

• Product of lognormal variables is lognormal
  – If $x_1$ and $x_2$ are normal, then $y_1 = e^{x_1}$ and $y_2 = e^{x_2}$ are lognormal
  – The product of $y_1$ and $y_2$: $y_1 \times y_2 = e^{x_1} \times e^{x_2} = e^{x_1 + x_2}$
  – Since $x_1 + x_2$ is normal, $e^{x_1 + x_2}$ is lognormal
The Lognormal Distribution (cont’d)

• The lognormal density function

\[
g(S; m, \nu, S_0) = \frac{1}{S\nu\sqrt{2\pi}} \ e^{-\frac{1}{2} \left( \frac{\ln(S) - [\ln(S_0) + m - 0.5\nu^2]}{\nu} \right)^2}
\]

where \( S_0 \) is initial stock price, and \( \ln(S/S_0) \sim N(m, \nu^2) \), \( S \) is future stock price, \( m \) is mean, and \( \nu \) is standard deviation of continuously compounded return.

• If \( x \sim N(m, \nu^2) \), then

\[
E(e^x) = e^{m + \frac{1}{2} \nu^2}
\]
The Lognormal Distribution
(cont’d)
A Lognormal Model of Stock Prices

- If the stock price $S_t$ is lognormal, $S_t / S_0 = e^x$, where $x$, the continuously compounded return from 0 to $t$ is normal

- If $R(t, s)$ is the continuously compounded return from $t$ to $s$, and, $t_0 < t_1 < t_2$, then $R(t_0, t_2) = R(t_0, t_1) + R(t_1, t_2)$

- From 0 to $T$, $E[R(0, T)] = n\alpha_h$, and $Var[R(0, T)] = n\sigma_h^2$

- If returns are iid, the mean and variance of the continuously compounded returns are proportional to time
A Lognormal Model of Stock Prices (cont’d)

• If we assume that

\[ \ln(S_t / S_0) \sim N[(\alpha - \delta - 0.5\sigma^2)t, \sigma^2 t] \]

then

\[ \ln(S_t / S_0) = (\alpha - \delta - 0.5\sigma^2)t + \sigma\sqrt{t\zeta} \]

and therefore

\[ S_t = S_0 e^{(\alpha - \delta - 0.5\sigma^2)t + \sigma\sqrt{t\zeta}} \]

• If current stock price is \( S_0 \), the probability that the option will expire in the money, i.e.

\[ \text{Prob}(S_t > K) = N(d_2) \]

– where the expression contains \( \alpha \), the true expected return on the stock in place of \( r \), the risk-free rate
Lognormal Probability Calculations

- Prices $S_t^L$ and $S_t^U$ such that $\text{Prob} (S_t^L < S_t) = p/2$ and $\text{Prob} (S_t^U > S_t) = p/2$

$$S_t^L = S_0 e^{\left(\alpha - \frac{1}{2} \sigma^2\right)t + \sigma \sqrt{t} N^{-1}(p/2)}$$

$$S_t^U = S_0 e^{\left(\alpha - \frac{1}{2} \sigma^2\right)t - \sigma \sqrt{t} N^{-1}(p/2)}$$

<table>
<thead>
<tr>
<th>Horizon</th>
<th>Fraction of a Year</th>
<th>$-2\sigma$</th>
<th>$-1\sigma$</th>
<th>$+1\sigma$</th>
<th>$+2\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Day</td>
<td>0.0027</td>
<td>48.47</td>
<td>49.24</td>
<td>50.81</td>
<td>51.61</td>
</tr>
<tr>
<td>1 Month</td>
<td>0.0849</td>
<td>42.35</td>
<td>46.22</td>
<td>55.06</td>
<td>60.09</td>
</tr>
<tr>
<td>1 Year</td>
<td>1</td>
<td>30.48</td>
<td>41.14</td>
<td>74.97</td>
<td>101.19</td>
</tr>
<tr>
<td>2 Years</td>
<td>2</td>
<td>26.40</td>
<td>40.36</td>
<td>94.28</td>
<td>144.11</td>
</tr>
<tr>
<td>5 Years</td>
<td>5</td>
<td>22.10</td>
<td>43.22</td>
<td>165.31</td>
<td>323.33</td>
</tr>
</tbody>
</table>
Lognormal Probability Calculations (cont’d)

• Given the option expires in the money, what is the expected stock price? The conditional expected price

\[ E(S_t | S_t > K) = S e^{(\alpha - \delta)t} \frac{N(\hat{d}_1)}{N(\hat{d}_2)} \]

  where the expression contains \( \alpha \), the true expected return on the stock in place of \( r \), the risk-free rate.
Estimating the Parameters of a Lognormal Distribution

• The lognormality assumption has two implications
  – Over any time horizon continuously compounded return is normal
  – The mean and variance of returns grow proportionally with time

<table>
<thead>
<tr>
<th>Week</th>
<th>Price ($)</th>
<th>$\ln(S_t/S_{t-1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>—</td>
</tr>
<tr>
<td>2</td>
<td>105.04</td>
<td>0.0492</td>
</tr>
<tr>
<td>3</td>
<td>105.76</td>
<td>0.0068</td>
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<tr>
<td>4</td>
<td>108.93</td>
<td>0.0295</td>
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<td>5</td>
<td>102.50</td>
<td>-0.0608</td>
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<tr>
<td>6</td>
<td>104.80</td>
<td>0.0222</td>
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<tr>
<td>7</td>
<td>104.13</td>
<td>-0.0064</td>
</tr>
</tbody>
</table>
Estimating the Parameters of a Lognormal Distribution (cont’d)

• The mean of the second column is 0.006745 and the standard deviation is 0.038208

• Annualized standard deviation

\[ = 0.038208 \times \sqrt{52} = 0.2755 \]

• Annualized expected return

\[ = 0.006745 \times 52 + 0.5 \times 0.2755 \times 0.2755 = 0.3877 \]
Understanding the Risk-Neutral Concept

• Both the B&S and Binomial tree are using the concept of risk-neutral probability.

• A risk-neutral investor is indifferent between a sure thing and a risky bet with an expected payoff equal to the value of the sure thing.

• $p^*$ is the risk-neutral probability that the stock price will go up.
Understanding Risk-Neutral Pricing (Cont’d)

• The option pricing formula can be said to price options as if investors are risk-neutral
  – Note that we are not assuming that investors are actually risk-neutral, and that risky assets are actually expected to earn the risk-free rate of return
Pricing an Option Using Real Probabilities

• Is option pricing consistent with standard discounted cash flow calculations?
  – Yes!
  – Let us formalize the concept.
Pricing an Option Using Real Probabilities (cont’d)

• Suppose that the continuously compounded expected return on the stock is $\alpha$ and that the stock does not pay dividends.

• If $p$ is the true probability of the stock going up, $p$ must be consistent with $u$, $d$, and $\alpha$

$$puS + (1 - p)dS = e^{\alpha h} S$$

• Solving for $p$ gives us

$$p = \frac{e^{\alpha h} - d}{u - d}$$
Pricing an Option Using Real Probabilities (cont’d)

• Using $p$, the actual expected payoff to the option one period hence is

$$pC_u + (1 - p)C_d = \frac{e^{\alpha h} - d}{u - d} C_u + \frac{u - e^{\alpha h}}{u - d} C_d$$

• At what rate do we discount this expected payoff?
  – It is not correct to discount the option at the expected return on the stock, $\alpha$, because the option is equivalent to a leveraged investment in the stock and hence is riskier than the stock.
  – Discount rate should be higher than $\alpha$!
Pricing an Option Using Real Probabilities (cont’d)

• Denote the appropriate per-period discount rate for the option as $\gamma$

• Since an option is equivalent to holding a portfolio consisting of $\Delta$ shares of stock and $B$ bonds, the expected return on this portfolio is

$$e^{\gamma h} = \frac{S\Delta}{S\Delta + B} e^{\alpha h} + \frac{B}{S\Delta + B} e^{r h}$$
Pricing an Option Using Real Probabilities (cont’d)

• We can now compute the option price as the expected option payoff, discounted at the appropriate discount rate, which gives

\[
C = e^{-\gamma h} \left[ \frac{e^{\alpha h} - d}{u - d} C_u + \frac{u - e^{\alpha h}}{u - d} C_d \right]
\]
Pricing an Option Using Real Probabilities (cont’d)

• It turns out that this gives us the same option price as performing the risk-neutral calculation
  – Note that it does not matter whether we have the “correct” value of $\alpha$ to start with
  – Any consistent pair of $\alpha$ and $\gamma$ will give the same option price
  – Risk-neutral pricing is valuable because setting $\alpha = r$ (and thus $\gamma = r$ as well) results in the simplest pricing procedure.
Implied Volatility

- Volatility is unobservable

- Choosing a volatility to use in pricing an option is difficult but important

- One approach to obtaining a volatility is to use history of returns

- However history is not a reliable guide to the future

- Alternatively, we can invert the Black-Scholes formula to obtain option implied volatility
Volatility Index – VIX

- It provides investors with market estimates of expected volatility
- It is computed by using near-term S&P100 index options
VIX Options

• A type of non-equity option that uses the VIX as the underlying asset.

• This is the first exchange-traded option that gives individual investors the ability to trade market volatility.

• A trader who believes that market volatility will increase can purchase VIX call options.

• Sharp increases in volatility generally coincide with a falling market, so this type of option can be used as a natural hedge.
Is Implied Volatility Constant?

Two interesting empirical observations are about the relation between IV and the strike price as well as between IV and the time to expiation.

The first relation is the volatility smile, the second is the volatility term structure.
The Volatility Smile:

• It is the relationship between implied volatility and strike price for options with a certain maturity

• The volatility smile for European call options should be exactly the same as that for European put options (put-call parity)

• The same is at least approximately true for American options
The Volatility Smile for Foreign Currency Options

Implied Volatility vs. Strike Price

Prof. Doron Avramov
Derivatives Securities
Implied Distribution for Foreign Currency Options

• Both tails are heavier than the lognormal distribution
• It is also “more peaked” than the lognormal distribution
The Volatility Smile for Equity Options
Implied Distribution for Equity Options

- The left tail is heavier and the right tail is less heavy than the lognormal distribution
Ways of Characterizing the Volatility Smiles

- Plot implied volatility against $K/S_0$ (The volatility smile is then more stable)
- Plot implied volatility against $K/F_0$ (Traders usually define an option as at-the-money when $K$ equals the forward price, $F_0$, not when it equals the spot price $S_0$)
- Plot implied volatility against delta of the option (This approach allows the volatility smile to be applied to some non-standard options. At-the-money is defined as a call with a delta of 0.5 or a put with a delta of $-0.5$. These are referred to as 50-delta options)
Possible Causes of Volatility Smile

• Asset price exhibits jumps rather than continuous changes
• Volatility for asset price is stochastic
  – In the case of an exchange rate volatility is not heavily correlated with the exchange rate. The effect of a stochastic volatility is to create a symmetrical smile
  – In the case of equities volatility is negatively related to stock prices because of the impact of leverage. This is consistent with the skew that is observed in practice
Volatility Term Structure

• In addition to calculating a volatility smile, traders also calculate a volatility term structure

• This shows the variation of implied volatility with the time to maturity of the option
Volatility Term Structure

The volatility term structure tends to be downward sloping when volatility is high and upward sloping when it is low.
## Example of a Volatility Surface

<table>
<thead>
<tr>
<th>$K/S_0$</th>
<th>0.90</th>
<th>0.95</th>
<th>1.00</th>
<th>1.05</th>
<th>1.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 mnth</td>
<td>14.2</td>
<td>13.0</td>
<td>12.0</td>
<td>13.1</td>
<td>14.5</td>
</tr>
<tr>
<td>3 mnth</td>
<td>14.0</td>
<td>13.0</td>
<td>12.0</td>
<td>13.1</td>
<td>14.2</td>
</tr>
<tr>
<td>6 mnth</td>
<td>14.1</td>
<td>13.3</td>
<td>12.5</td>
<td>13.4</td>
<td>14.3</td>
</tr>
<tr>
<td>1 year</td>
<td>14.7</td>
<td>14.0</td>
<td>13.5</td>
<td>14.0</td>
<td>14.8</td>
</tr>
<tr>
<td>2 year</td>
<td>15.0</td>
<td>14.4</td>
<td>14.0</td>
<td>14.5</td>
<td>15.1</td>
</tr>
<tr>
<td>5 year</td>
<td>14.8</td>
<td>14.6</td>
<td>14.4</td>
<td>14.7</td>
<td>15.0</td>
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Implied Volatility

TABLE 12.4


<table>
<thead>
<tr>
<th>Strike ($)</th>
<th>Expiration</th>
<th>Call Price ($)</th>
<th>Implied Volatility</th>
<th>Put Price ($)</th>
<th>Implied Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>1100</td>
<td>11/20/2004</td>
<td>34.80</td>
<td>0.1630</td>
<td>6.80</td>
<td>0.1575</td>
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<td>0.1363</td>
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</table>
Derivative Securities: Option Greeks
Option Greeks

• What happens to option price when one input changes?
  – Delta ($\Delta$): change in option price when stock price increases by $1$
  – Gamma ($\Gamma$): change in delta when the stock price increases by $1$
  – Vega: change in option price when volatility increases by 1%
  – Theta ($\Theta$): change in option price when time to maturity decreases by 1 day
  – Rho ($\rho$): change in option price when interest rate increases by 1%

• Greek measures for portfolios
  – The Greek measure of a portfolio is weighted average of Greeks of individual portfolio components. For example,

\[
\Delta_{portfolio} = \sum_{i=1}^{n} \omega_i \Delta_i
\]
Option Greeks

- The following illustrations are based on strike price=$40, sigma=30%, riskfree rate=8%, and no dividend payments
Delta - Call

![Graph showing Delta for different holding periods (3 months, 1 year, 3 years). The x-axis represents Stock Price ($) and the y-axis represents Delta. The graph illustrates how Delta changes with varying stock prices and holding periods.]
Gamma - call

Graph showing Gamma for different time periods:
- 3 months
- 1 year
- 3 years

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Option Greeks (cont’d)

![Graph showing Vega against Stock Price with different time periods: 3 months, 1 year, and 3 years.](image-url)
Option Greeks (cont’d)

Stock price = $40
Option Greeks (cont’d)

![Graph of Option Greeks](image)

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Option Greeks (cont’d)

![Graph showing the relationship between Rho and Stock Price for different time periods (3 months, 1 year, and 3 years).](image)
Option Greeks (cont’d)

<table>
<thead>
<tr>
<th>Option</th>
<th>Option 1</th>
<th>Option 2</th>
<th>Combined</th>
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<tbody>
<tr>
<td>$\omega_i$</td>
<td>1</td>
<td>-1</td>
<td>—</td>
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<tr>
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<td>Rho</td>
<td>0.0511</td>
<td>0.0257</td>
<td>0.0255</td>
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</table>

Greeks for the bull spread examined in Chapter 3, where $S = $40, $\sigma = 0.3$, $r = 0.08$, and $T = 91$ days, with a purchased 40-strike call and a written 45-strike call. The column titled “combined” is the difference between column 1 and column 2.
Option Greeks (cont’d)

• Option elasticity ($\Omega$)

$\Omega$ describes the risk of the option relative to the risk of the stock in percentage terms: If stock price ($S$) changes by 1%, what is the percent change in the value of the option ($C$)?

$$\Omega \equiv \frac{\% \text{ change in } C}{\% \text{ change in } S} = \frac{\frac{\text{ change in } C}{C}}{\frac{\text{ change in } S}{S}} = \frac{\frac{\text{ change in } S \times \Delta}{S}}{\frac{\text{ change in } C}{C}} = \frac{S \Delta}{C}$$

– Example 12.8: $S = $41, $K = $40, $\sigma = 0.30$, $r = 0.08$, $T = 1$, $\delta = 0$

• Elasticity for call: $\Omega = S \Delta / C = \$41 \times 0.6911 / \$6.961 = 4.071$
• Elasticity for put: $\Omega = S \Delta / C = \$41 \times -0.3089 / \$2.886 = -4.389
Option Greeks (cont’d)

• Option elasticity ($\Omega$) (cont’d)
  
  – The volatility of an option  
    \[ \sigma_{option} = \sigma_{stock} \times |\Omega| \]
  
  – The risk premium of an option  
    \[ \gamma - r = (\alpha - r) \times \Omega \]
  
  – The Sharpe ratio of an option

    Sharpe ratio for call = $\frac{\alpha - r}{\sigma}$ = Sharpe ratio for stock

• where $|.|$ is the absolute value, $\gamma$: required return on option, $\alpha$: expected return on stock, and $r$: risk-free rate
Questions on Parity and other Option Relationships

1. Jafee Corp. common stock is priced at $36.50 per share. The company just paid its $0.50 quarterly dividend. Interest rates are 6.0%. A $35.00 strike European call, maturing in 6 months, sells for $3.20. What is the price of a 6-month, $35.00 strike put option?

2. Rankin Corp. common stock is priced at $74.20 per share. The company just paid its $1.10 quarterly dividend. Interest rates are 6.0%. A $70.00 strike European call, maturing in 6 months, sells for $6.50. How much arbitrage profit/loss is made by shorting the European call, which is priced at $2.50?

3. A company is forecasted to pay dividends of $0.90, $1.20, and $1.45 in 3, 6, and 9 months, respectively. Given interest rates of 5.5%, how much dollar impact will dividends have on option prices? (Assume a 9-month option.)

4. The price of a stock is $52.00. Lacking additional information, what is your forecasted difference between a put option and a call option on this stock? Assume 38 days to expiration and 6.0% interest.

5. Jillo, Inc. stock is selling for $54.70 per share. Calls and puts with a $55 strike and 40 days until expiration are selling for $1.65 and $1.23, respectively. What potential arbitrage profit exists?
Questions on Parity and other Option Relationships

6. The spot exchange rate of dollars per euro is 0.95. Dollar and euro interest rates are 7.0% and 6.0%, respectively. The price of a $0.93 strike 6-month call option is $0.08. What is the price of the put?

7. The 6-month call and put premiums are $0.114 and $0.098, respectively, with a $0.94 strike. Dollar and euro interest rates are 7.0% and 6.0%, respectively. What spot exchange rate is implied by this data?

8. Call options with strikes of $30, $35, and $40 have option premiums of $1.50, $1.70, and $2.00, respectively. Using strike price convexity is this possible?

9. Put options with strikes of $70, $75, and $85 have option premiums of $6.00, $8.50, and $11.00, respectively. Using strike price convexity, which option premium, if any, is not possible?
Questions on Parity and other Option Relationships

6. Consider the case of an exchange option in which the underlying stock is Eli Lilly and Company with a current price of $56.00 per share. The strike asset is Merck, with a per share price of $52.00. Interest rates are 5% and the 3 month call option is trading for $7.00. What is the price of the put?

7. The spot exchange rate in dollars per euro is $1.31. Dollar denominated interest rates are 4.0% and euro denominated interest rates are 3.0%. What is the difference in call and put option prices given a two year option and a $1.34 strike price?

8. The price of a non-dividend paying stock is $55 per share. A 6 month, at the money call option is trading for $1.89. If the interest rate is 6.5%, what is the likely price of a European put at the same strike and expiration?
Questions on the Black-Scholes Formula

1. What is the price of a $35 strike call? Assume $S = $38.50, $\sigma = 0.25$, $r = 0.06$, the stock pays no dividend and the option expires in 45 days?

2. What is the price of a $60 strike put? Assume $S = $63.75, $\sigma = 0.20$, $r = 0.055$, the stock pays no dividend and the option expires in 50 days?

3. What is the price of a $25 strike call? Assume $S = $23.50, $\sigma = 0.24$, $r = 0.055$, the stock pays a 2.5% continuous dividend and the option expires in 45 days?

4. What is the price of a $30 strike put? Assume $S = $28.50, $\sigma = 0.32$, $r = 0.04$, the stock pays a 1.0% continuous dividend and the option expires in 110 days?

5. What is the delta on a $20 strike call? Assume $S = $22.00, $\sigma = 0.30$, $r = 0.05$, the stock pays a 1.0% continuous dividend and the option expires in 80 days?

6. What is the delta on a $25 strike put? Assume $S = $24.00, $\sigma = 0.35$, $r = 0.06$, the stock pays a 2.0% continuous dividend and the option expires in 40 days?

7. Assume that a $50 strike call has a 3.0% continuous dividend, $\sigma = 0.27$, $r = 0.06$ and 60 days from expiration. What is the gamma for a stock price movement from $48.00 to $49.00?

8. Assume that a $55 strike call has a 1.5% continuous dividend, $r = 0.05$ and the stock price is $50.00. If the option has 45 days until expiration, what is the vega given a shift in volatility from 33.0% to 34.0%?
Questions on the Black-Scholes Formula

9. Suppose the spot exchange rate is $1.43 per British pound and the strike on a dollar
denominated pound call is $1.30. Assume \( r = 0.045, r_f = 0.06, \sigma = 0.15 \) and the option
expires in 180 days. What is the call option price?

10. Suppose the spot exchange rate is $1.22 per British pound and the strike on a dollar
denominated pound put is $1.20. Assume \( r = 0.04, r_f = 0.05, \sigma = 0.20 \) and the option
expires in 270 days. What is the put option price?

11. Suppose the 180-day futures price on gold is $110.00 per ounce and the volatility is
20.0%. Assume interest rates are 3.5%. What is the price of a $120 strike call futures
option that expires in 180 days?

12. Suppose the 120-day futures price on gold is $115.00 per ounce and the volatility is
20.0%. Assume interest rates are 3.5%. What is the price of a $110 strike call futures
option that expires in 120 days?

13. Assume that a $60 strike call has a 2.0% continuous dividend, \( r = 0.05 \), and the stock
price is $61.00. What is the theta of the option as the expiration time declines from 60 to
50 days?

14. Assume that a $75 strike call has a 1.0% continuous dividend, 90 days until expiration
and stock price of $72.00. What is the rho of the option as the interest rate changes from
6.0% to 5.0%?

15. Suppose a $60 strike call has 45 days until expiration and pays a 1.5% continuous
dividend. Assume \( S = $58.50, \sigma = 0.25 \), and \( r = 0.06 \). What is the option elasticity given
an immediate price increase?
Derivative Securities: Downside Risk and Insuring against Shortfall
Shortfall Probability in Long Horizon Asset Management

• Let us denote by $R$ the cumulative return on the investment over several years (say $T$ years).
• Rather than finding the distribution of $R$ we analyze the distribution of

$$r = \ln \left(1 + R \right)$$

which is the continuously compounded return over the investment horizon.
• The investment value after $T$ years is

$$V_T = V_0 \left(1 + R_1 \right)\left(1 + R_2 \right)\ldots \left(1 + R_T \right)$$
Dividing both sides of the equation by $V_0$ we get

$$\frac{V_T}{V_0} = (1 + R_1)(1 + R_2)\ldots (1 + R_T)$$

Thus

$$1 + R = (1 + R_1)(1 + R_2)\ldots (1 + R_T)$$
Shortfall Probability in Long Horizon Asset Management

• Taking natural log from both sides we get

\[ r = r_1 + r_2 + \ldots + r_T \]

• Assuming that log returns are IID normal (as in the B&S economy) we get

\[ r \sim N\left(T\mu, T\sigma^2\right) \]
Shortfall Probability and Long Horizon

- Let us now understand the concept of shortfall probability.

- We ask: what is the probability that the investment yields a return smaller than the riskfree rate, or any other threshold level?

- To answer this question we need to compute the value of a riskfree investment over the $T$ year period.
Shortfall Probability and Long Horizon

- The value of such a riskfree investment is

\[ V_{rf} = V_0 \left( 1 + R_f \right)^T \]

\[ = V_0 \exp \left( T r_f \right) \]

where \( r_f \) is the continuously compounded risk free rate.
Shortfall Probability and Long Horizon

• Essentially we ask: what is the probability that
  \[ V_T < V_{rf} \]

• This is equivalent to asking what is the probability that
  \[ \frac{V_T}{V_0} < \frac{V_{rf}}{V_0} \]

• This, in turn, is equivalent to asking what is the probability that
  \[ \ln \left( \frac{V_T}{V_0} \right) < \ln \left( \frac{V_{rf}}{V_0} \right) \]
Shortfall Probability and Long Horizon

- So we need to work out

\[ p(r < Tr_f) \]

- Subtracting \( T\mu \) and dividing by \( \sqrt{T} \sigma \) both sides of the inequality we get

\[ P \left( z < \sqrt{T} \left( \frac{r_f - \mu}{\sigma} \right) \right) \]

- We can denote this probability by

\[ \text{Shortfall probability} = N \left( \sqrt{T} \left( \frac{r_f - \mu}{\sigma} \right) \right) \]
Shortfall Probability and long horizon

- Typically \( r_f < \mu \) which means the probability diminishes with increasing \( T \).
Example

• Take $r=0.04$, $\mu=0.08$, and $\sigma=0.2$ per year. What is the Shortfall Probability for investment horizons of 1, 2, 5, 10, and 20 years?
• Use the excel normdist function.
• If $T=1$ $SP=0.42$
• If $T=2$ $SP=0.39$
• If $T=5$ $SP=0.33$
• If $T=10$ $SP=0.26$
• If $T=20$ $SP=0.19$
Cost of Insuring against Shortfall

• Let us now understand the mathematics of insuring against shortfall.

• Without loss of generality let us assume that

\[ V_0 = 1 \]

• The investment value at time \( T \) is a given by the random variable \( V_T \)
Cost of Insuring against Shortfall

- Once we insure against shortfall the investment value after $T$ years becomes
  
  - If $V_T > \exp(TR_f)$ you get $V_T$
  - If $V_T < \exp(TR_f)$ you get $\exp(Tr_f)$
Cost of Insuring against Shortfall

- So you essentially buy an insurance policy that pays 0 if \( V_T > \exp(Tr_f) \)
  Pays \( \exp(Tr_f) - V_T \) if \( V_T < \exp(Tr_f) \)
- You ultimately need to price a contract with terminal payoff given by
  \[
  \max \left\{ 0, \exp(Tr_f) - V_T \right\}
  \]
Cost of Insuring against Shortfall

- This is a European put option expiring in $T$ years with
  
a. $S=1$
  
b. $K = \exp(T \cdot r_f)$
  
c. Log riskfree rate given by $r_f$
  
d. Volatility given by $\sigma$
  
e. Dividend yield given by $\delta = 0$
Cost of Insuring against Shortfall

- From the B&S formula we know that

\[\text{Put} = K \exp\left(-Tr_f\right)N\left(-d_2\right) - S \exp\left(-\delta T\right)N\left(-d_1\right)\]

- Which becomes

\[\text{Put} = N\left(\frac{1}{2} \sigma \sqrt{T}\right) - N\left(-\frac{1}{2} \sigma \sqrt{T}\right)\]
Cost of Insuring against Shortfall

• The B&S option-pricing model gives the current put price $P$ as

$$Put = N(d_1) - N(d_2)$$

where

$$d_1 = \frac{\sigma \sqrt{T}}{2}$$

and

$$d_2 = -d_1$$

and $N(d)$ is $\text{prob}\{z < d\}$
Cost of Insuring against Shortfall

• For $\sigma = 0.2$ (per year)

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<tr>
<th>T (years)</th>
<th>P</th>
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<tbody>
<tr>
<td>1</td>
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<td>30</td>
<td>0.42</td>
</tr>
<tr>
<td>50</td>
<td>0.52</td>
</tr>
</tbody>
</table>

• The cost of the insurance increases in $T$, even though the probability of needing it decreases in $T$ (if $\mu > r$).
Derivative Securities: Insurance, Collars, and other Strategies
Basic Insurance Strategies

• Options can be
  – used to insure long positions (floors)
  – used to insure short positions (caps)
  – selling insurance
Insuring a Long Position: Floors

• An investor employing the protective put strategy owns the underlying stock.
• Due to concerns about downside market risks in the near term the investor wants some protection against diminishing share value, thus purchasing a put option.
• So the investor is long on both stock and put
• The protective put investor retains all benefits of continuing stock ownership (dividends, voting rights, etc.) during the lifetime of the put contract. At the same time, the protective put serves to limit downside loss.
Example: Insuring a Long Position

- Buy the index ($1,000) and a put option ($74.201) with a strike price of $1,000. Interest rate 2%.

Buying an asset and a put generates a position that looks like a call!
Profit and Loss

- The maximum profit is unlimited
- The maximum loss is limited: the strike price – FV( share price + premium paid).
- Considered bullish strategy.
- What is the break even point?
Insuring a Short Position: Caps

• A call option is combined with a short position in the underlying asset

• Goal: to insure against an increase in the price of the underlying asset
Example: Insuring a Short Position

- Short-selling the index ($1,000) and holding a call option ($93.809) with a strike price of $1,000. Interest rate 2%.

An insured short position looks like a put!
Selling Insurance

• For every insurance buyer there must be an insurance seller

• Strategies used to sell insurance
  – **Covered writing** (option overwriting or selling a covered call) is writing an option when there is a corresponding long position in the underlying asset is called covered writing
  – **Naked writing** is writing an option when the writer does not have a position in the asset
Covered Writing: Covered Calls

- Example: holding the index ($1,000) and writing a call option ($93.809) with a strike price of $1,000

Writing a covered call generates the same profit as selling a put!
Covered Call

• The covered call is a strategy in which an investor writes a call option while at the same time owns a share of the underlying stock.

• If this stock is purchased simultaneously with writing the call contract, the strategy is commonly referred to as a "buy-write."

• If the shares are already held from a previous purchase, it is commonly referred to an "overwrite."

• In either case, the stock is generally held in the same brokerage account from which the investor writes the call, and fully collateralizes, or "covers," the obligation conveyed by writing a call option contract.
Covered Call: when to use?

- Though the covered call can be utilized in any market condition, it is most often employed when the investor, while bullish on the underlying stock, feels that its market value will experience little range (low volatility) over the lifetime of the call contract.
- The investor desires to either generate additional income (over dividends) from shares of the underlying stock, and provide a limited amount of protection against a decline in underlying stock value.
Covered Call: Profit and Loss

- Maximum profit occurs if the price of the underlying stock is at or above the call option's strike price.
- Financial loss can become substantial if the stock price continues to decline in price as the written call expires.
- Any loss accrued from a decline in stock price is offset by the premium you receive from the initial sale of the call option.
Covered Writing: Covered Puts

- Example: shorting the index ($1,000) and writing a put option ($93.809) with a strike price of $1,000

"Writing a covered put generates the same profit as writing a call!"
Spreads and Collars

- An option **spread** is a position consisting of only calls or only puts, in which some options are purchased and some are written
  - Examples: bull spread, bear spread, box spread

- A **collar** is the purchase of a put option and the sale of a call option with a higher strike price, with both options having the same underlying asset and having the same expiration date
  - Example: zero-cost collar
Spreads

- A **bull spread** is a position, in which you buy a call and sell an otherwise identical call with a higher strike price
  - It is a bet that the price of the underlying asset will increase
  - Use the put-call parity to show that the spread can be constructed using puts
Bull Spreads

• Consider buying a 40-strike call and selling a 45-strike call – both are with 3 months to expiration. The premiums are $2.78 and $0.97. So the initial net cost is $1.81.
Profit and Loss

- Maximum Loss: Limited to premium paid for the long option minus the premium received for the short option.

- Maximum Gain: Limited to the difference between the two strike prices minus the net premium paid for the spread.
Characteristics of Bull Spread

• Use when you are mildly bullish on market price.

• The strategy can yield as much as the difference between the two strike prices. So, when putting on a bull spread remember that the wider the strikes the more you can make. But the downside to this is that you will end up paying more for the spread. So, the deeper in the money calls you buy relative to the call options that you sell means a greater maximum loss if the market sells off.
Characteristics of Bull Spreads

• This strategy is a very cost effective way to take a position when you are bullish on market direction. The cost of the bought call option is partially offset by the premium received by the sold call option. This does, however, limit your potential gain if the market rallies but also reduces the cost of entering into this position.

• This type of strategy is suited to investors who want to go long on market direction and also have an upside target in mind. The sold call acts as a profit target for the position. So, if the trader sees a short term move in an underlying but doesn't see the market going past $X, then a bull spread is ideal. With a bull spread he can easily go long without the added expenditure of an outright long stock and can even reduce the cost by selling the additional call option.
Box Spread

• A box spread in an alternative to buying a bond.
• Option market makers have low transaction costs and can sell a box spread which is equivalent to borrowing.
• Example: Suppose you buy a 950-strike call ($120.405), sell a 1000-strike call (93.809), sell a 950-strike put (51.777), and buy a 1000-strike put (74.201).
• The cost of the position is $49.027.
• The payoff of the position after 6 months is $50.
• So…the implicit interest rate of the CF is ~2%.

<table>
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<th>Instrument</th>
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<th>( S_T &lt; 950 )</th>
<th>( S_T &lt; 950 )</th>
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<td>( S_T - 950 )</td>
<td>( S_T - 950 )</td>
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<tr>
<td>Short 1,000 Call</td>
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<td>0</td>
<td>( 1000 - S_T )</td>
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<tr>
<td>Short 950 Put</td>
<td>( S_T - 950 )</td>
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<td>0</td>
</tr>
<tr>
<td>Long 950 Put</td>
<td>( 1000 - S_T )</td>
<td>( 1000 - S_T )</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
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</tr>
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</table>
Collar

- A collar represents a bet that the price of the underlying asset will decrease and resembles a short position in a stock.
- A zero-cost collar can be created when the premiums of the call and put exactly offset one another.

![Diagram showing profit ($) vs. XYZ Stock Price ($) for a purchased collar.]
Collar

• Consider buying a 40-strike put and selling a 45-strike call – both are with 3 months to expiration. The premiums are $1.99 and $0.97. So the initial net cost is $1.02.
Speculating on Volatility

• Options can be used to create positions that are nondirectional with respect to the underlying asset

• Examples
  – Straddles
  – Strangles
  – Butterfly spreads

• Who would use nondirectional positions?
  – Investors who do not care whether the stock goes up or down, but only how much it moves, i.e., who speculate on volatility
Straddles

- Buying a call and a put with the same strike price and time to expiration
- A straddle is a bet that volatility will be high relative to the market’s assessment
Characteristics of Straddle

- Use when you are bullish on volatility but are unsure of market direction.

- A long straddle is like placing an each-way bet on price action: you make money if the market goes up or down.

- But, the market must move enough in either direction to cover the cost of buying both options.

- Buying straddles is best when you expect the market to make a substantial move before the expiration date - for example, before an earnings announcement.
Strangles

- Buying an out-of-the-money call and put with the same time to expiration
- A strangle can be used to reduce the high premium cost, associated with a straddle
Characteristics of Strangle

- Use when you are bullish on volatility but are unsure of market direction.

- A long strangle is similar to a straddle except the strike prices are further apart, which lowers the cost of putting on the spread but also widens the gap needed for the market to rise/fall beyond in order to be profitable.

- Like long straddles, buying strangles is best when you expect a large movement of market price in either direction.
Written Straddles

- Selling a call and put with the same strike price and time to maturity
- Unlike a purchased straddle, a written straddle is a bet that volatility will be low relative to the market’s assessment
Butterfly Spreads

- Write a straddle + add a strangle = insured written straddle

- A butterfly spread insures against large losses on a straddle
Butterfly Spread: An Example

• Written 40-strike straddle, purchased 45-strike call, and purchased 35-strike put. These positions combined generate the butterfly spread.
Asymmetric Butterfly Spreads

• The asymmetric spread looks like the regular butterfly spread except that it is asymmetric. The peak is closer to the high strike than the low strike.

• The strategy on the next page is created by buying two 35-strike calls, selling ten 43-strike calls, and buying eight 45-strike calls.
Asymmetric Butterfly Spreads

![Graph of Asymmetric Butterfly Spread](image-url)
Summary of Various Strategies

• Different positions, same outcome

<table>
<thead>
<tr>
<th>Position</th>
<th>Is Equivalent To</th>
<th>And Is Called</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index + Put</td>
<td>Zero-Coupon Bond + Call</td>
<td>Insured Asset (floor)</td>
</tr>
<tr>
<td>Index – Call</td>
<td>Zero-Coupon Bond – Put</td>
<td>Covered Written call</td>
</tr>
<tr>
<td>–Index + Call</td>
<td>–Zero-Coupon Bond + Put</td>
<td>Insured Short (cap)</td>
</tr>
<tr>
<td>–Index – Put</td>
<td>–Zero-Coupon Bond – Call</td>
<td>Covered Written Put</td>
</tr>
</tbody>
</table>

• Strategies driven by the view of the market’s direction

<table>
<thead>
<tr>
<th>Volatility Will Increase</th>
<th>No Volatility View</th>
<th>Volatility Will Fall</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price Will Fall</td>
<td>Buy Puts</td>
<td>Sell Underlying</td>
</tr>
<tr>
<td>No Price View</td>
<td>Buy Straddle</td>
<td>Do Nothing</td>
</tr>
<tr>
<td>Price Will Increase</td>
<td>Buy Calls</td>
<td>Buy Underlying</td>
</tr>
</tbody>
</table>
1. A strategy consists of buying a market index product at $830 and longing a Put on the index with a strike of $830. If the Put premium is $18 and the interest rates are 0.5% what is the profit (or loss) at expiration (in 6 months) if the market index is $810?
   A. $20.00 gain  
   B. $18.65 gain  
   C. $36.29 loss  
   D. $43.76 loss
   Answer D

2. A strategy consists of buying a market index product at $830 and longing a Put on the index with a strike of $830. If the Put premium is $18 and the interest rates are 0.5% what is the profit (or loss) from the long index position by itself at expiration (in 6 months) if the market index is $810?
   A. $45.21 loss  
   B. $21.22 loss  
   C. $18.00 gain  
   D. $24.25 gain
   Answer A
3. A strategy consists of buying a market index product at $830 and longing a Put on the index with a strike of $830. If the Put premium is $18 and the interest rates are 0.5% what is the profit (or loss) from the long put position by itself at expiration (in 6 months) if the market index is $810?
   A. $3.45 gain
   B. $1.45 gain
   C. $2.80 loss
   D. $1.36 loss
   Answer B

4. A strategy consists of buying a market index product at $830 and longing a Put on the index with a strike of $830. If the Put premium is $18 and the interest rates are 0.5% what is the estimated price of a call option with an exercise price of $830?
   A. $42.47
   B. $45.26
   C. $47.67
   D. $49.55
   Answer A
Derivative Securities: Applications to Corporate Finance
Corporate Applications:

• We aim to use option pricing based tools, such as the B&S formula, to price a straight zero coupon default-able bond as well as a zero coupon default-able bond which is convertible to stocks.

• We also aim to explain concepts such as default premium, recovery rate, risk neutral probability of default, and hazard rate.
Debt and Equity As Options

• Consider a firm that has no dividend-paying equity outstanding, and a single zero-coupon debt issue.
  – The time $t$ values of the assets of the firm, the debt, and the equity are $A_t$, $B_t$, and $E_t$, respectively
  – The debt matures at time $T$. 
Debt and Equity As Options

- The value of the equity at time $T$ is

$$E_T = \max (0, A_T - \bar{B})$$

- This is the payoff to a call option.
- What is the underlying asset?
- What is the strike price?
Debt and Equity As Options (cont’d)

• The value of the debt is

\[ B_T = \min (A_T, \bar{B}) \]

or

\[ B_T = A_T + \min(0, \bar{B} - A_T) = A_T - \max(0, A_T - \bar{B}) \]

– This implies that the bondholders own the firm assets, but have written a call option on the firm assets to the equityholders

• Summing these equations - equity plus debt - indeed gives the total value of the firm as \( A_T \).
Debt and Equity As Options (cont’d)

• Thus, we can compute the value of debt and equity prior to time $T$ using option pricing methods, with the value of assets taking the place of the stock price and the face value of the debt taking the place of the strike price.

• The equity value at time $t$ is the value of a call option on the firm assets. The value of the debt is then $B_t = A_t - E_t$. 
Pricing Zero Coupon Bonds with Default Risk

• Suppose that a non dividend paying firm issues a zero coupon bond maturing in five years.

• The bond’s face value is $100, the current value of the assets is $90, the risk-free rate (cc) is 6%, and the volatility of the underlying assets is 25%.

• What is the equity and debt value? What is the bond yield to maturity (ytm)?
Default Risk Premium

- The equity value solves the BSCall:
  \[ \text{Equity} = \text{BSCall}(90, 100, 0.25, 0.06, 5, 0) = 27.07 \]
  - The debt value is \( 90 - 27.07 = 62.93 \).
  - The debt cc ytm = \( \frac{1}{5} \times \ln(100/62.93) = 9.26\% \).
  - The ytm is greater than the riskfree rate due to default risk premium.
  - The default risk premium is \( \exp(0.0926) - \exp(0.06) = 0.03518 \).
Default Risk Premium

• What if the asset value is equal to 300?
  Equity = BSCall(300, 100, 0.25, 0.06, 5, 0) = 226.08
  ▪ The debt value is 300 - 226.08 = 73.92
  ▪ The debt credit ytm = $\frac{1}{5}\times \ln\left(\frac{100}{73.92}\right) = 6.04\%$
  ▪ So the risk premium is close to zero.
  ▪ Not a big surprise! The debt is virtually risk free.
Default Risk Premium

• What if the asset volatility is 50%?
  Equity = BSCall(90, 100, 0.5, 0.06, 5, 0) = 43.20
  ▪ The debt value is 90 - 43.20 = 46.80.
  ▪ The debt cc ytm = \(\frac{1}{5} \times \ln(100/46.80)\) = 15.18%.
  ▪ So the risk premium is much higher.
  ▪ Not a big surprise! The debt is much more risky.
Default Risk Premium

• What if the time to maturity is 10 years?
  
  Equity = BSCall(90, 100, 0.25, 0.06, 10, 0) = 43.73

  ▪ The debt value is 90 - 43.20 = 46.26.
  ▪ The debt cc ytm = \frac{1}{10} \times \ln(100/46.26) = 7.7%.
  ▪ So the risk premium is much lower.
  ▪ Now the debt is less risky. Why?
Credit Risk

• The credit risk premium compensates investors for undertaking credit risk.
• We now study several concepts of corporate credit risk or default risk and credit ratings.
• Then, back to derivatives securities...
Risk Neutral Default Probability

• Suppose that the bond has an assumed recovery rate of \( f \) (e.g., \( f=0.50 \)).

• What is the risk neutral default probability \( (q) \)?

• Assuming annual compounding, \( q \) solves

\[
P = q \times \frac{f \times 100}{(1 + y_{riskfree})^T} + (1 - q) \times \frac{100}{(1 + y_{riskfree})^T}
\]
Risk Neutral Default Probability

- Of course if the compounding is semi-annual the formula becomes

\[
P = q \times \frac{f \times 100}{(1 + \frac{\text{Yield}}{2})^{2T}} + (1 - q) \times \frac{100}{(1 + \frac{\text{Yield}}{2})^{2T}}
\]

Now since \( P = \frac{100}{(1 + \frac{\text{Yield}}{2})^{2T}} \), we get a general formula for risk neutral default probability:

\[
q = \frac{1}{1 - f} \times \left[ 1 - \frac{(1 + \frac{\text{Yield}}{2})^{2T}}{(1 + \frac{\text{Yield}}{2})^{2T}} \right]
\]
Recovery Rate

The recovery rate for a bond is usually defined as the price of the bond immediately after default as a percent of its face value.
## Recovery Rates
(Moody’s: 1982 to 2006)

<table>
<thead>
<tr>
<th>Class</th>
<th>Mean (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Senior Secured</td>
<td>54.44</td>
</tr>
<tr>
<td>Senior Unsecured</td>
<td>38.39</td>
</tr>
<tr>
<td>Senior Subordinated</td>
<td>32.85</td>
</tr>
<tr>
<td>Subordinated</td>
<td>31.61</td>
</tr>
<tr>
<td>Junior Subordinated</td>
<td>24.47</td>
</tr>
</tbody>
</table>
The impact of time on default probability

- Assume $f=0.5$, default-able and risk free rates are 8% and 4% (APR), respectively. What is $q$?
- For $T=1$, $q=0.08$.
- For $T=5$, $q=0.35$.
- For $T=10$, $q=0.64$.
- Clearly risk-neutral (and actual) default probability increases with time to maturity.
- Why?
Credit Ratings (Cont’)

Cumulative vs. marginal default probabilities

\[
\begin{align*}
CP1 &= MP1 \\
CP2 &= CP1 + (1 - CP1) \times MP2 \\
CP3 &= CP2 + (1 - CP2) \times MP3
\end{align*}
\]

Given cumulative default probabilities we can compute future marginal default probabilities (analogous to spot vs. forward rates)
### Cumulative Ave Default Rates (%)
(1970-2006, Moody’s)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.02</td>
<td>0.09</td>
<td>0.25</td>
<td>0.52</td>
</tr>
<tr>
<td>Aa</td>
<td>0.01</td>
<td>0.02</td>
<td>0.04</td>
<td>0.11</td>
<td>0.18</td>
<td>0.34</td>
<td>0.52</td>
</tr>
<tr>
<td>A</td>
<td>0.02</td>
<td>0.09</td>
<td>0.22</td>
<td>0.34</td>
<td>0.47</td>
<td>0.76</td>
<td>1.29</td>
</tr>
<tr>
<td>Baa</td>
<td>0.18</td>
<td>0.51</td>
<td>0.93</td>
<td>1.43</td>
<td>1.94</td>
<td>2.96</td>
<td>4.64</td>
</tr>
<tr>
<td>Ba</td>
<td>1.21</td>
<td>3.22</td>
<td>5.57</td>
<td>7.96</td>
<td>10.22</td>
<td>14.01</td>
<td>19.12</td>
</tr>
<tr>
<td>B</td>
<td>5.24</td>
<td>11.30</td>
<td>17.04</td>
<td>22.06</td>
<td>26.80</td>
<td>34.77</td>
<td>43.34</td>
</tr>
<tr>
<td>Caa-C</td>
<td>19.48</td>
<td>30.50</td>
<td>39.72</td>
<td>46.91</td>
<td>52.62</td>
<td>59.94</td>
<td>69.18</td>
</tr>
</tbody>
</table>

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Interpretation

• The table shows the probability of default for companies starting with a particular credit rating
• A company with an initial credit rating of Baa has a probability of 0.181% of defaulting by the end of the first year, 0.506% by the end of the second year, and so on
Default Intensities versus Unconditional Default Probabilities

• The default intensity (also called hazard rate) is the probability of default for a certain time period conditional on no earlier default.

• The unconditional default probability is the probability of default for a certain time period as seen at time zero.

• What are the default intensities and unconditional default probabilities for a Caa rate company in the third year?
Default Intensity (Hazard Rate)

Let \( V(t) \) be the probability of a company surviving to time \( t \).
Default intensity could nicely be modeled as

\[
V(t + \Delta t) - V(t) = -\lambda(t) V(t)
\]
This leads to

\[
V(t) = e^{-\int_0^t \lambda(s) ds}
\]

The cumulative probability of default by time \( t \) is

\[
Q(t) = 1 - e^{-\bar{\lambda}(t) t}
\]
Convertible Bonds

• A convertible bond is a bond that, at the option of the bondholder, can be exchanged for shares in the issuing company.

• The valuation of a convertible bond entails valuing both debt subject to default and a warrant.
Warrant

• If a firm issues a call option on its own stock, it is known as a warrant

• When a warrant is exercised, the warrant-holder receives a share worth more than the strike price. This is dilutive to other shareholders

• The question is how to value a warrant, and how to value the equity given the existence of warrants
Warrant (cont’d)

• Suppose
  – A firm has \( n \) shares outstanding
  – The outstanding warrants are European, on \( m \) shares, with strike price \( K \)
  – The asset value is \( A \)

• At expiration, if the warrant-holders exercise the warrants, they pay \( K \) per share and receive \( m \) shares
Warrant (continued)

• After the warrants are exercised, the firm has assets worth $A + mK$. Thus, exercised warrants are worth

$$\frac{A + mK}{n + m} - K = \frac{n}{n + m} \left( \frac{A}{n} - K \right)$$

dilution correction factor

• Of course, it makes no sense to exercise if the value is negative. Hence, the warrant can be valued by using

$$\frac{n}{n + m} BSCall\left( \frac{A}{n}, K, \sigma, r, t, \delta \right)$$
Terms in CB

Consider a three-period, 9% bond convertible into four shares of the underlying company’s stock.

1. The *conversion ratio* (CR) is the number of shares of stock that can be converted when the bond is tendered for conversion. The conversion ratio for this bond is four.

2. The *conversion value* (CV) is equal to the conversion ratio times the market price of the stock. If the current price of the stock were 92, then the bond’s conversion value would be $CV = (4)(92) = 368$.

3. The *conversion premium* (CP) is equal to the strike price of the bond divided by the share price. See example below.
CV: Valuation

• Convertible bond = straight bond (of the same company) + certain amount of warrants.
Convertible Bonds (cont’d)

• Suppose
  – There are $m$ bonds with maturity payment $M$, each of which is convertible into $q$ shares
  – There are $n$ original shares outstanding
  – The asset value is $A$

• If the bonds are converted, there will be $n + mq$ shares.
Convertible Bonds (cont’d)

• At expiration, the bondholders will convert if the value of the assets after conversion exceeds the value of the maturity payment per share:

\[
\frac{A}{n + mq} - \frac{M}{q} > 0
\]

or

\[
\frac{n}{n + mq} \left( \frac{A}{n} - \frac{M}{q} \frac{n + mq}{n} \right) > 0
\]
Convertible Bonds (cont’d)

• Assuming the convertible is the only debt issue, bankruptcy occurs if $A/m < M$. Thus, for each bond, the payoff of the convertible at maturity, $T$, is

$$M - \max\left(0, M - \frac{A_T}{m}\right) + q \times \frac{n}{n + mq} \times \max\left(0, \frac{A_T}{n} - \frac{M}{q} \frac{n + mq}{n}\right)$$

Bond Written Put $q$ Purchased Warrants

• Therefore, a convertible bond can be valued as

  - Owning a risk-free bond with maturity payment $M$
  - Selling a put on $1/m$ of the firm’s assets, and
  - Buying $q$ warrants with strike $\frac{M}{q} \frac{n + mq}{n}$
Example

• Suppose a firm has issued $m=6$ convertible bonds, each with maturity value $M=1000$ and convertible into $q=50$ shares. The firm has $n=400$ common shares outstanding. Note that the six bonds have a total promised maturity value of $6000$. Conversion occurs when assets exceed $1000 \times 700/50 = 14000$, which is $A > M(n+mq)/q$. The slope of the convertible payoff above 14000 is $mq/(n+mq) = 3/7$.

• Whenever the assets value is below $6000$ the firm is in default. If the value is between $6000$ and $14000$ the warrant component is zero. The bond will be converted into shares only when the assets value exceeds $14000$. 

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Example: convertible bond valuation using the B&S formula

• Suppose a firm has assets worth $10,000 (A=10000) with a single debt issue consisting of six zero-coupon bonds (m=6), each with a maturity value of $1,000 (M=1000) and with five years to maturity. The firm asset volatility is 30% and the risk-free rate is 6%. No dividend payouts.
Example (cont’d)

• Let us first assume that the single debt issue is non convertible. Then the price is the price of a riskless bond minus a put option on 1/6 of the firm assets

$$1000 \times \exp(-0.06 \times 5) - \text{BSPut}(10,000/6,1,000,0.30,0.06,5,0)$$

$$= 701.27$$

Then \( ytm = \frac{1}{5} \times \ln(1000/701.27) = 0.07 > 0.06 \)
Example (cont’d)

• Let us now assume that the debt is convertible. The price of all six bonds is

\[
6 \times 1000 \times \exp(-0.06 \times 5) - \\
\text{BSPut}(10000, 6000, 0.30, 0.06, 5, 0) - \\
6 \times 50 \times 400 / (400 + 6 \times 50) - \\
\times \text{BSCall}(10000/400, 1000/50 \times \\
(400 + 6 \times 50)/400, 0.30, 0.06, 5, 0) = 5276.5
\]
Example (cont’d)

• The price of each convertible bond is 879.39
• Ytm = \( \frac{1}{5} \times \ln(1000/879.39) = 0.0257 \)
• The share value
  \[
  \frac{(10,000 - 5276.35)}{400} = 11.809.
  \]
• Bondholder will convert if the assets are worth more than
  \[
  \frac{M(n + mq)}{q} = 14000.
  \]
• Each bond gives the holder the right to convert into 50
  shares to the strike price is \( \frac{1000}{50} = 20 \). The conversion
  premium is \( \frac{20}{11.809 - 1} = 69.4\% \).
Convertible Bonds (cont’d)

• Valuation of convertible bonds is usually complicated by
  – Early exercise (American options)
  – Callability
  – Changes in interest rates
  – Payment of dividends
  – Payment of coupons (coming up using the Binomial Tree)

• Why do firms issue convertible bonds?
  – One possible explanation is that it helps to resolve a conflict between equityholders and debt-holders
  – Plus, firms would typically like to reduce interest payments
Put Warrant

- Companies can sell **put warrants**: put options on their own stock

- A commonly stated rationale for issuing put warrants is that the put sales are a hedge against the cost of repurchasing shares

- The company intends to buy its own shares in the future – the cost is set up upfront.

- If the stock price is up the put warrant expires. Then the company buys expensive stocks but the purchase price is diminished by the put premium.
Put Warrant

• Indeed, many companies issue stocks through compensation options, as noted earlier.

• To avoid an increase in the number of shares outstanding the companies buy shares from other shareholders.

• Microsoft, for instance, enhances its repurchase program by selling put warrants. On June 30, 1999 163 million warrants were outstanding with strike prices ranging from $59 to $65.
Fixed Income Securities:
Financial and Interest Rate Forwards and Futures
Introduction

- The underlying assets of financial futures and forwards are
  - Stocks and indexes
  - Bonds
  - Currencies

- Open questions:
  - How are they used?
  - How are they priced?
  - How could they used for hedging?
Alternative Ways to Buy a Stock

- Four different payment and receipt timing combinations
  - Outright purchase: ordinary transaction
  - Fully leveraged purchase: investor borrows the full amount
  - Prepaid forward contract: pay today, receive the share later (you don’t get dividends since you don’t own the stock)
  - Forward contract: agree on price now, pay as well as receive the share later (again, no dividends)
- Let us analyze inflows, outflows, and their timing

<table>
<thead>
<tr>
<th>Description</th>
<th>Pay at Time:</th>
<th>Receive Security at Time:</th>
<th>Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outright Purchase</td>
<td>0</td>
<td>0</td>
<td>( S_0 ) at time 0</td>
</tr>
<tr>
<td>Fully Leveraged Purchase</td>
<td>( T )</td>
<td>0</td>
<td>( S_0 e^{rT} ) at time ( T )</td>
</tr>
<tr>
<td>Prepaid Forward Contract</td>
<td>0</td>
<td>( T )</td>
<td>?</td>
</tr>
<tr>
<td>Forward Contract</td>
<td>( T )</td>
<td>( T )</td>
<td>( ? \times e^{rT} )</td>
</tr>
</tbody>
</table>
Pricing Prepaid Forwards

- If we can price the *prepaid* forward \((F^P)\), then we can calculate the price for a forward contract

\[
F = \text{Future value of } F^P
\]

- Three possible methods to price prepaid forwards
  - Pricing by analogy
  - Pricing by discounted present value
  - Pricing by arbitrage

  Of course, each method should deliver the same price.

- To simplify the analysis, let us assume that there are no dividends. Later we will account for dividends.
Pricing Prepaid Forwards: No Dividends

• Pricing by analogy
  – In the absence of dividends, the timing of delivery is irrelevant
  – Price of the prepaid forward contract same as current stock price
  – \( F^P_{0,T} = S_0 \) (where the asset is bought at time 0, delivered at \( T \))

• Pricing by discounted preset value
  \((\alpha: \text{risk-adjusted discount rate, which is equal to the riskfree rate plus risk premium})\)
  – If expected \( t=T \) stock price at \( t=0 \) is \( E_0(S_T) \), then
  \( F^P_{0,T} = E_0(S_T)e^{-\alpha T} \)
  – Since \( t=0 \) expected value of price at \( t=T \) is \( E_0(S_T) = S_0e^{\alpha T} \)
  – Combining the two, \( F^P_{0,T} = S_0e^{\alpha T} = S_0 \)
Pricing Prepaid Forwards: No Dividends

- Pricing by arbitrage

  - Arbitrage: a situation in which one can generate positive cash flow by simultaneously buying and selling related assets, with no net investment and with no risk → free money!!!
  - If at time \( t=0 \), the prepaid forward price somehow exceeded the stock price, i.e., an arbitrageur could do the following

\[
F_{0,T}^P = S_0
\]

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Cash Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy Stock @ ( S_0 )</td>
<td>(-S_0)</td>
</tr>
<tr>
<td>Sell Prepaid Forward @ ( F_{0,T}^P )</td>
<td>(+F_{0,T}^P)</td>
</tr>
<tr>
<td>Total</td>
<td>( F_{0,T}^P - S_0 )</td>
</tr>
</tbody>
</table>

Thus, an equilibrium relation is

\[
F_{0,T}^P = S_0
\]
Pricing Prepaid Forwards: With Dividends

• What if there are dividends? Is $F_{0,T}^P = S_0$ still valid?

  – No, because the holder of the forward will not receive dividends that will be paid to the holder of the stock

  \[
  F_{0,T}^P = S_0 - PV \quad (\text{all dividends paid from } t=0 \text{ to } t=T)
  \]

  – For discrete dividends $D_{t_i}$ at times $t_i$, $i = 1, \ldots, n$

    • The prepaid forward price: $F_{0,T}^P = S_0 - \sum_{i=1}^{n} PV_{0,t_i}(D_{t_i})$
    • For continuous dividends with an annualized yield $\delta$
    • The prepaid forward price: $F_{0,T}^P = S_0 e^{-\delta T}$
Pricing Prepaid Forwards

• Example
  – XYZ stock costs $100 today and is expected to pay a quarterly dividend of $1.25. If the annual risk-free rate is 10% compounded continuously, how much does a 1-year prepaid forward cost?

\[ F^p_{0,1} = 100 - \sum_{i=1}^{4} 1.25e^{-0.025i} = 95.30 \]
Pricing Prepaid Forwards (cont’d)

• Example
  – The index is $125 and the dividend yield is 3% continuously compounded. How much does a 1-year prepaid forward cost?

\[
F^P_{0,1} = 125e^{-0.03} = 121.31
\]
Prepaid Forwards and LEPOs

In some countries it is possible to buy stock options with very low strike prices – such that it is virtually certain that the option will expire in the money. Such an option is called a low exercise price option.

A LEPO is essentially a prepaid forward contract
Pricing Forwards on Stock

- Forward price is the future value of the prepaid forward
  - No dividends
    \[ F_{0,T} = FV(F^P_{0,T}) = FV(S_0) = S_0e^{rT} \]
  - Discrete dividends
    \[ F_{0,T} = S_0e^{rT} - \sum_{i=1}^{n} e^{r(T-t_i)}D_{t_i} \]
  - Continuous dividends (interest benefit dividend cost):
    \[ F_{0,T} = S_0e^{(r-\delta)T} \]
Does the future price predict the price of the stock in the future?

- According the formula
  \[ F_{0,T} = S_0 e^{(r - \delta)T} \]
  the forward price conveys no additional information beyond what \( S_0, r, \) and \( \delta \) already provide.

- Moreover, the forward price underestimates the future stock price. For example, suppose that a stock index has expected return of 15% while the riskfree is 5%. If the current index price is 100, then on average we expect that the index will be 115 in 1 year. The forward price for a delivery of 1 year will be 105. This does not imply that the forward is a good or bad investment.
Futures Contracts

• Exchange-traded “forward contracts”

• Typical features of futures contracts
  – Standardized, with specified delivery dates, locations, and procedures
  – Each exchange has an associated clearinghouse
    • Matches buy and sell orders
    • Keeps track of members’ obligations and payments
    • After matching the trades, becomes counterparty

• Differences from forward contracts
  – Settled daily through the mark-to-market process \(\Rightarrow\) low credit risk
  – Highly liquid \(\Rightarrow\) easier to offset an existing position
  – Highly standardized structure \(\Rightarrow\) harder to customize (see first set of notes the Starbucks example).
Futures prices versus forward prices

– The difference negligible especially for short-lived contracts

– Can be significant for long-lived contracts and/or when interest rates are correlated with the price of the underlying asset
Example: S&P 500 Futures

- The contract size (notional value) is 250
- Suppose that the future price is 1100 – the notional value of one contract is $250 \times 1100 = 275000$ - the amount you pay at expiration.
- You wish to acquire 2,200,000 of the index – you are long 8 contracts.
- Traded in Chicago Mercantile Exchange.
- Cash-settled contract – no physical delivery
- Daily mark-to-market
- Open interest: total number of buy/sell pairs
- Margin
  - Initial margin
  - Maintenance margin (70 – 80% of initial margin)
  - Margin call
Margins and Mark to Market

- Suppose that the margin is 10%, which amounts to $220,000.
- Suppose also that the future price drops over a week to 1027.99 and that the annual effective riskfree rate is 6%.
- The holder loses \((1100-1027.99) \times 250 \times 8 = 144020\).
- The margin balance \(220,000 \times \exp(0.06/52) - 144020 = 76,233.99\).
- Because we have a 10% margin – the decline in margin is 65%. Margin call arrives.
Example: S&P 500 Futures
(cont’d)

• Mark-to-market proceeds and margin balance for 8 long futures

<table>
<thead>
<tr>
<th>Week</th>
<th>Multiplier ($)</th>
<th>Futures Price</th>
<th>Price Change</th>
<th>Margin Balance($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2000.00</td>
<td>1100.00</td>
<td>—</td>
<td>220,000.00</td>
</tr>
<tr>
<td>1</td>
<td>2000.00</td>
<td>1027.99</td>
<td>-72.01</td>
<td>76,233.99</td>
</tr>
<tr>
<td>2</td>
<td>2000.00</td>
<td>1037.88</td>
<td>9.89</td>
<td>96,102.01</td>
</tr>
<tr>
<td>3</td>
<td>2000.00</td>
<td>1073.23</td>
<td>35.35</td>
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<td>4</td>
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<td>2000.00</td>
<td>1024.74</td>
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<td>2000.00</td>
<td>1007.30</td>
<td>-17.44</td>
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<td>10</td>
<td>2000.00</td>
<td>1011.65</td>
<td>4.35</td>
<td>44,990.57</td>
</tr>
</tbody>
</table>
Uses of Index Futures

• Why buy an index futures contract instead of synthesizing it using the stocks in the index? Lower transaction costs
• Asset allocation: switching investments among asset classes
• Example: invested in the S&P 500 index and temporarily wish to invest in bonds instead of index. What to do?
  – Alternative #1: sell all 500 stocks and invest in bonds
  – Alternative #2: take a short forward position in S&P 500 index

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Cash Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Today</td>
</tr>
<tr>
<td>Own Stock @ $100</td>
<td>−$100</td>
</tr>
<tr>
<td>Short Forward @ $110</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>−$100</td>
</tr>
</tbody>
</table>

Effect of owning the stock and selling forward, assuming that $S_0 = $100 and $F_{0,1} = $110.
Currency Contracts

- Widely used to hedge against changes in exchange rates
- WSJ listing

Prof. Doron Avramov
Derivatives Securities
Currency prepaid forward

- Suppose you want to purchase ¥1 one year from today using $s

\[ F_{0,T}^P = x_0 e^{-r_y T} \]

- Where \( x_0 \) is current (\$/¥) exchange rate, and \( r_y \) is the yen-denominated interest rate
- Let’s understand the economics:
  - You want to buy 1 yen in T years – thus you must have \( x_0 \exp(- r_y \times T) \) in yen today.
  - To obtain this amount, you must exchange \( x_0 \exp(- r_y \times T) \) dollar today.
  - The prepaid forward price is the dollar cost of obtaining 1 yen in the future.
Currency Forward

\[ F_{0,T} = x_0 e^{(r - r_y)t} \]

- \( r \) is the $-denominated domestic interest rate
- \( F_{0,T} > x_0 \) if \( r > r_y \) (domestic risk-free rate exceeds foreign risk-free rate)
Currency Contracts: Pricing (cont’d)

• Example
  – ¥-denominated interest rate is 2% and current ($/¥) exchange rate is 0.009. To have ¥1 in one year one needs to invest today
    • $0.009/¥ x ¥1 x e^{-0.02} = $0.008822

• Example
  – ¥-denominated interest rate is 2% and $-denominated rate is 6%. The current ($/¥) exchange rate is 0.009. The 1-year forward rate
    • $0.009e^{0.06-0.02} = 0.009367
Currency Contracts: Pricing

- Synthetic currency forward: borrowing in one currency and lending in another creates the same cash flow as a forward contract.
- Covered interest arbitrage: offset the synthetic forward position with an actual forward contract.
- The idea here is that a position in a foreign risk free bond with the currency risk hedged pays the same return as a domestic risk free bond.

**Table 5.12**

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Cash Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Year 0</td>
</tr>
<tr>
<td></td>
<td>$</td>
</tr>
<tr>
<td>Borrow $96e^{-r}$ Dollar at 6% ($)</td>
<td>+0.008822</td>
</tr>
<tr>
<td>Convert to Yen @ 0.009 $/¥</td>
<td>—0.008822</td>
</tr>
<tr>
<td>Invest in Yen-Denominated Bill (¥)</td>
<td>—</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>0</strong></td>
</tr>
</tbody>
</table>
Interest Rate Parity

• It is a non-arbitrage condition that relates interest rates and exchange rates:
  \[
  \text{cc. domestic rate} = \text{cc. foreign rate} + \ln(F/S)
  \]

• From the previous example:
  \[
  0.06 + 0.02 + \ln(0.009367/0.009)
  \]

• In words: returns from borrowing in currency I, exchanging currency I, and investing the proceeds in interest bearing instruments of currency II, while simultaneously purchasing futures contracts to convert back to currency I at the end of the holding period should be equal to the returns from holding similar interest bearing instruments of country I.
Interest Rate Futures and Forwards

- **U.S. Treasury**
  - Bills (<1 year), no coupons, sell at discount
  - Notes (1–10 years), Bonds (10–30 years), coupons, sell at par
  - STRIPS: claim to a single coupon or principal, zero-coupon
Bond Basics (cont’d)

• Notation
  – $r_{t1,t2}$: interest rate from time $t_1$ to $t_2$ prevailing at time $t$
  – $P_{t0}(t_1,t_2)$: price of a bond quoted at $t = t_0$ to be purchased at $t = t_1$ maturing at $t = t_2$
Bond Basics (cont’d)

- Zero-coupon bonds make a single payment at maturity

- One year zero-coupon bond: $P(0,1) = 0.943396$
  - Pay $0.943396$ today to receive $1$ at $t=1$
  - Yield to maturity (YTM) = $1/0.943396 - 1 = 0.06 = 6\% = r(0,1)$

- Two year zero-coupon bond: $P(0,2) = 0.881659$
  - YTM = $1/0.881659 - 1 = 0.134225 = (1+r(0,2))^2 \Rightarrow r(0,2) = 0.065 = 6.5\%$
Bond Basics (cont’d)

• Zero-coupon bond price that pays \( C_t \) at \( t \):

\[
P(0,t) = \frac{C_t}{[1 + r(0,t)]^t}
\]

• Yield curve: graph of annualized bond yields against time

Bond Basics: Implied Forward Rates

\[ [1 + r(0, 1)] \times [1 + r(1, 2)] \]

Earn \( r(0, 1) \)

Earn implied forward rate, \( r(1, 2) \)

Earn \( r(0, 2) \) per year

\[ [1 + r(0, 2)]^2 \]
Bond Basics: Implied Forward Rates

- Suppose current one-year rate $r(0,1)$ and two-year rate $r(0,2)$

- Current forward rate from year 1 to year 2, $r_0(1,2)$, must satisfy

$$[1 + r_0(0,1)][1 + r_0(1,2)] = [1 + r_0(0,2)]^2$$
Bond Basics: Implied Forward Rates

• In general

\[ [1 + r_0(t_1, t_2)]^{t_2 - t_1} = \frac{[1 + r_0(0, t_2)]^{t_2}}{[1 + r_0(0, t_1)]^{t_1}} = \frac{P(0, t_1)}{P(0, t_2)} \]

• Example: What are the implied forward rate \( r_0(2,3) \) and forward zero-coupon bond price \( P_0(2,3) \) from year 2 to year 3?

\[ r_0(2,3) = \frac{P(0,2)}{P(0,3)} - 1 = \frac{0.881659}{0.816298} - 1 = 0.0800705 \]

\[ P_0(2,3) = \frac{P(0,3)}{P(0,2)} = \frac{0.816298}{0.881659} = 0.925865 \]
Bond Basics: Coupon Bonds

• Coupon bonds
  – The price at time of issue $t$ of a bond maturing at time $T$ that pays $n$ coupons of size $c$ and maturity payment of $1$

$$B_i(t,T,c,n) = \sum_{i=1}^{n} cP_i(t,t_i) + P_i(t,T)$$

  where $t_i = t + i(T - t)/n$

  – For the bond to sell at par (coupon=yield to maturity) the coupon size must be

$$c = \frac{1 - P_i(t,T)}{\sum_{i=1}^{n} P_i(t,t_i)}$$
Fixed Income Securities: Interest Rates Futures and Swaps
Interest Rate Futures: Forward Rate Agreements (FRAs)

• Let us consider a firm intending to borrow $100m for 91 days beginning 120 days from today, in June. Today is February.

• Suppose that the effective quarterly interest rate in June could be either 1.5% or 2% and the today’s implied June-91 day forward rate (the rate from June to September) is 1.8%.

• Note that there is a variation in the interest rate of $100m \times (0.020 - 0.015) = $0.5m.

• The borrowing firm would like to guarantee the borrowing rate to hedge against increasing interest rates.

• Forward rate agreements (FRA) will do the job!
FRA

• **Forward Rate Agreements** (FRA) are financial contracts that allow counterparties to lock in a forward interest rate.
• Don’t get it wrong: FRA is a forward contract on an interest rate - not on a bond, or a loan.
• The **buyer** of an FRA contract locks in a fixed borrowing rate, the **seller** locks in a fixed lending rate.
• The buyer profits from increasing interest rates. The seller benefits from diminishing rates.
• **Cash settlement** is made either through the maturity of the contract (arrears) or at the time of borrowing.
FRA “Jargon”

- A t1 X t2 FRA: The start date, or delivery date, is in t1 months. The end of the forward period is in t2 months.
- Thus, the loan period is t1-t2 months long.
- E.g., a 9 X 12 FRA has a contract period beginning in nine months and ending in 12 months.
Example 1: settlement in arrears (when the contract matures):

– If the borrowing rate in June is 1.5% - the borrower (the FRA buyer) will pay the FRA seller in September \((0.018-0.015) \times 100m = 300,000\).

– Un-hedged interest expense is \(1,500,000 \times 0.015\). Total cost of borrowing is \(1,800,000\).
Example 1: settlement in arrears (when the contract matures):

– If the borrowing rate in June is 2% the borrower will receive in September

\[(0.02 - 0.018) \times 100\text{m} = 200,000\]. Un-hedged interest expense is \$2,000,000 \[100\text{m} \times 0.02\]. Total cost of borrowing is \$1,800,000.

– The locked in borrowing rate is 1.8% in both cases.
Example 2: settlement at the borrowing time

• Assume that \( r_{\text{qrtly}} = 1.5\% \) then the payment made by the buyer to the seller in June (the time of borrowing) is \( 300,000/1.015=295,566 \).

• If \( r_{\text{qrtly}} = 2\% \) then the payment made by the seller to the buyer in June is \( 200,000/1.02=196,078 \).
Example 3: Locking in Investment Rate

• Your company will receive $100M in 6 M to be invested for 6M.
• The CFO is concerned that the investment rates will fall
• The company could sell a 6×12 FRA on $100M at a rate 5% (assume).
• This effectively locks in an investment rate of 5% starting in 6 months for the following 6 months.
Example 3 (continued)

• Assume that after six months the interest rate is 4%, then your company (the FRA seller) will receive in the end of the next six months:

\[(5\% - 4\%) \times \$100M \times 0.5 = \$500,000\]

which will fully compensate for the lower investment rate.

• Instead, if the settlement is at the time of investment your company will receive \(\text{pv}($500,000,6 \text{ months}, 4\%)\).
Example 4

• A firm sells a 5X8 FRA, with a NP of $300MM, and a contract rate of 5.8% (3-mo. forward LIBOR).

• On the settlement date (five months hence), 3 mo. spot LIBOR is 5.1%.

• There are 91 days in the contract period (8-5=3 months), and a year is defined to be 360 days.

• The settlement is at the time of borrowing.

• How much does the firm receive in that time?
Example 4 (continued)

• $524,077.11, which is calculated as:

\[
\frac{(300,000,000)(0.058-0.051)(91/360)}{1+[(0.051)(91/360)]}
\]
FRA: A Challenging One.

- A firm sells a 5X8 FRA, with a NP of $300MM, and a contract rate of 5.8% (3-mo. forward LIBOR). Settlement is at maturity.

- Suppose that one month after the FRA origination 4X7 FRAs are priced at 5.5% and 5X8 FRAs are priced at 5.6%.

- Also assume that after that month, the annual spot LIBOR rate corresponding to seven month discounting is 4.9%.

- What is the original FRA worth? Recall upon inception the contract value is zero.
Eurodollar Futures

• When foreign banks receive dollar deposits, those dollars are called Eurodollars.
• Like FRA, the Eurodollar contract is used to hedge interest rate risk.
• Similar to FRA, Eurodollar contracts involve 3-month forward rates.
• The Eurodollar is linked to a LIBOR rate.
• Libor (the London Interbank Offer Rate) is the average borrowing rate faced by large international London banks.
• The contract exists on many currencies (dollar, yen, euro, sterling, Swiss franc).
• Traded at the Chicago Mercantile Exchange (CME).
Eurodollar Futures

• Every contract has a face value of $1 million.
• A borrower, who would like to hedge against increasing interest rates, would take a short position.
• This is different from FRA wherein the borrower is the contract buyer (long) - why?
• Because FRA is a contract on interest rate, while Eurodollar is a contract on the loan itself.
• Four months prior to the delivery date, a Eurodollar futures contract is equivalent to a 4X7 FRA.
• 12 months prior to delivery a Eurodollar futures contract is equivalent to 12X15 FRA.
Eurodollar Futures

• Suppose the current LIBOR is 1.5% per 3 months. By convention this is annualized by multiplying by 4, so the quoted LIBOR rate is 6%. (APR - not effective!)
• The Eurodollar future price at any time is
• 100-Annualized 3-month LIBOR.
• Thus, if LIBOR is 6% at maturity of the Eurodollar futures contract, the final futures price will be 100-6=94.
• Three-month Eurodollar contracts have maturities out to 10 years, which means that it is possible to use the contract to lock in a 3-month rate for 10 years.
Eurodollar Futures: The Contract Price

• The contract price is calculated by
  \[ P_t = 10,000 \times [100 - 0.25(100 - Fut_t)] \]
  where \( Fut_t \) is the quoted Eurodollar futures rate, or \( Fut_t = 100 - \text{Annualized 3-month LIBOR} \),
  • and 0.25 represents 3M maturity.
• You can also express the contract price as
  \[ P_t = 1,000,000 - 2500 \times \text{Annualized 3-month LIBOR} \]
Eurodollar Futures: The Contract Price

• For example, if the market quotes Fut=94.47, the contract price is then
  
  \[ P = 10,000 \times [100 - 0.25 \times 5.53] = 986,175 \]

• What if the annual Libor increases by 1 bp? The contract price diminishes by
  
  \[ 1,000,000 \times 0.01\% \times 0.25 = 25 \]
Eurodollar Futures: Hedging

• Suppose that in 7 months you plan to borrow $1 million for 90 days when the borrowing rate is LIBOR.
• The Eurodollar price for 7 months from today is 94, implying a 90-day rate of \((100-94) \times 0.25 \times 0.01 = 1.5\%\).
• As noted earlier, the borrower is shorting the contract.
• Suppose now that after 7 months the three-month LIBOR is 8%, which implies a Eurodollar futures price of 92.
• Your extra borrowing expense will be \((0.02 - 0.015) \times 1,000,000 = $5,000\).
• But you gain on the Eurodollar contract $5,000, which is the difference
  • between \(P = 10,000 \times [100 - 0.25 \times 6] = $985,000\) and
  • \(P = 10,000 \times [100 - 0.25 \times 8] = $980,000\)
• The short position fully compensates for the increasing borrowing cost.
Eurodollar Futures: An Impressive Success

• Recently Eurodollar futures took over T-bill futures as the preferred contract to manage interest rate risk

• LIBOR tracks the corporate borrowing rates better than the T-bill rate

![FIGURE 7.3](source: Datastream)

Three-month LIBOR rate, 1982-2004, and the difference between 3-month LIBOR and the yield on the 3-month Treasury bill.

Source: Datastream.
Eurodollar Futures: Listing and Specifications

• WSJ listing

<table>
<thead>
<tr>
<th>Month</th>
<th>Open</th>
<th>High</th>
<th>Low</th>
<th>Settle</th>
<th>Change</th>
<th>YIELD</th>
<th>CHG</th>
<th>INT.</th>
</tr>
</thead>
<tbody>
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<td>98.02</td>
<td>98.03</td>
<td>98.01</td>
<td>98.04</td>
<td>+0.01</td>
<td>1.91</td>
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<td>97.96</td>
<td>97.94</td>
<td>97.96</td>
<td>+0.04</td>
<td>2.04</td>
<td>-0.04</td>
<td>4,565</td>
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<tr>
<td>June</td>
<td>97.67</td>
<td>97.74</td>
<td>97.65</td>
<td>97.73</td>
<td>+0.05</td>
<td>2.27</td>
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<td>Sept</td>
<td>97.17</td>
<td>97.30</td>
<td>97.17</td>
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<td>+0.03</td>
<td>2.72</td>
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<td>96.74</td>
<td>96.58</td>
<td>96.72</td>
<td>+0.12</td>
<td>3.28</td>
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<td>Mr03</td>
<td>96.05</td>
<td>96.15</td>
<td>96.05</td>
<td>96.14</td>
<td>+0.12</td>
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<td>95.57</td>
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<td>+0.11</td>
<td>4.35</td>
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<td>95.26</td>
<td>95.19</td>
<td>95.25</td>
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<td>94.46</td>
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<td>5.54</td>
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<td>94.28</td>
<td>94.22</td>
<td>94.27</td>
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<td>5.73</td>
<td>-0.09</td>
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<td>94.08</td>
<td>94.02</td>
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<td>Mr05</td>
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<td>93.97</td>
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<td>6.10</td>
<td>-0.08</td>
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<td>93.76</td>
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<td>-0.08</td>
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<tr>
<td>Dec</td>
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<td>93.66</td>
<td>93.63</td>
<td>93.67</td>
<td>+0.03</td>
<td>6.33</td>
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<td>Mr06</td>
<td>93.33</td>
<td>93.68</td>
<td>93.62</td>
<td>93.67</td>
<td>+0.07</td>
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<tr>
<td>June</td>
<td>93.59</td>
<td>93.61</td>
<td>93.56</td>
<td>93.61</td>
<td>+0.07</td>
<td>6.39</td>
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<tr>
<td>Sept</td>
<td>93.54</td>
<td>93.56</td>
<td>93.54</td>
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<td>+0.05</td>
<td>6.45</td>
<td>-0.05</td>
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</tr>
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<td>Dec</td>
<td>93.43</td>
<td>93.45</td>
<td>93.42</td>
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<td>+0.06</td>
<td>6.56</td>
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<tr>
<td>Jo07</td>
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<td>93.38</td>
<td>93.42</td>
<td>+0.05</td>
<td>6.58</td>
<td>-0.05</td>
<td>18,263</td>
</tr>
<tr>
<td>Sept</td>
<td>93.35</td>
<td>93.39</td>
<td>93.35</td>
<td>93.38</td>
<td>+0.05</td>
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<td>-0.05</td>
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<tr>
<td>Dec</td>
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<td>93.24</td>
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<td>+0.04</td>
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<td>-0.04</td>
<td>13,227</td>
</tr>
<tr>
<td>Jo08</td>
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<td>93.28</td>
<td>93.23</td>
<td>93.26</td>
<td>+0.04</td>
<td>6.75</td>
<td>-0.04</td>
<td>11,236</td>
</tr>
<tr>
<td>Jo09</td>
<td>93.04</td>
<td>93.09</td>
<td>93.04</td>
<td>93.08</td>
<td>+0.02</td>
<td>6.92</td>
<td>-0.02</td>
<td>2,406</td>
</tr>
</tbody>
</table>

Est vol: 568,992; vol Fri 1,183,059; open int 4,864,582. +85,457.

• Contract Specifications

Specifications for the Eurodollar futures contract.

Where traded: Chicaco Mercantile Exchange
Size: 3-month Eurodollar time deposit, $1 million principal
Months: Mar, Jun, Sep, Dec, out 10 years, plus 2 serial months and spot month
Trading ends: 5 A.M. (11 A.M. London) on the second London bank business day immediately preceding the third Wednesday of the contract month.
Delivery: Cash settlement
Settlement: 100 — British Banker’s Association Futures Interest Settlement Rate for 3-Month Eurodollar Interbank Time Deposits. (This is a 3-month rate annualized by multiplying by 360/90.)
Eurodollar – A Question

• Suppose you buy (take a long position in) a contract on November 1
• The contract expires on December 21
• The prices are as shown
• How much do you gain or lose a) on the first day, b) on the second day, c) over the whole time until expiration?
Example

<table>
<thead>
<tr>
<th>Date</th>
<th>Quote</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nov 1</td>
<td>97.12</td>
</tr>
<tr>
<td>Nov 2</td>
<td>97.23</td>
</tr>
<tr>
<td>Nov 3</td>
<td>96.98</td>
</tr>
<tr>
<td>......</td>
<td>......</td>
</tr>
<tr>
<td>Dec 21</td>
<td>97.42</td>
</tr>
</tbody>
</table>
Example continued

• If on Nov. 1 you know that you will have $1 million to invest on for three months on Dec 21, the contract locks in a rate of

\[
100 - 97.12 = 2.88\%
\]

• In the example you earn \(100 - 97.42 = 2.58\%\) on $1 million for three months (= $6,450) and make a gain day by day on the futures contract of \(30 \times 25 = 750\).
Introduction to Swaps

• Thus far we have been dealing with a single transaction that will occur on a specific date in the future.
• A forward contract, for example, fixes a price for a transaction that will occur on a specific date in the future.
• What if many transactions occur repeatedly? For example,
  - Firms that issue bonds make periodical payments.
  - Multinational firms frequently exchange currencies.
• What is the easiest way to hedge risk in the presence of a risky payment stream – as opposed to a single risky payment?
• Perhaps we can enter into multiple future contracts for each payment we wish to hedge.
• Better solution – a swap contract!
Introduction to Swaps

• A swap is a contract calling for an exchange of one or multiple payments, on one or more dates, determined by the difference in two prices.

• A swap provides a means to hedge a stream of risky payments.

• A forward contract is a single-payment swap.
An Example of a Commodity Swap

• An industrial producer, IP Inc., needs to buy 100,000 barrels of oil 1 year from today and 2 years from today.

• The forward prices for delivery in 1 year and 2 years are $20 and $21/barrel.

• The 1- and 2-year zero-coupon bond yields are 6% and 6.5%.
An Example of a Commodity Swap (cont’d)

• IP can guarantee the cost of purchasing oil for the next 2 years by buying forward contracts for 100,000 barrels in each of the next 2 years.
• The PV of this cost per barrel is

$$\frac{20}{1.06} + \frac{21}{1.065^2} = 37.383$$

• Thus, IP could pay an oil supplier $37.383, and the supplier would commit to delivering one barrel in each of the next two years
• This is a prepaid swap contract, which is the multi-period version of a prepaid futures contract.
• A prepaid swap entails a single payment made today for multiple deliveries of oil in the future
An Example of a Commodity Swap (cont’d)

• With a prepaid swap, the buyer might worry about the resulting credit risk – what if the seller defaults? Such credit risk is nonexistent in financial contracts.

• A better solution would be to defer payments until the oil is delivered, while still fixing the total price upfront.

• Any payment stream with a PV of $37.383 is acceptable. Typically, a swap will call for equal payments in each year – that is the swap price.

• For example, the payment per year per barrel, x, will have to be $20.483 to satisfy the following equation

\[
\frac{x}{1.06} + \frac{x}{1.065^2} = $37.383
\]

We then say that the 2-year swap price is $20.483. The swap price is not the simple average of 20 and 21. It is closer to 20 since 20 is a nearby cash flow.
Physical Settlement

- **Physical settlement** of the swap: the buyer pays $20.483 and receives one barrel of oil each year.
Financial Settlement

- **Financial settlement** of the swap
  - The oil buyer, IP, pays the swap counterparty the difference between $20.483 and the spot price, and the oil buyer then buys oil at the spot price.
  
  - If the difference between $20.483 and the spot price is negative, then the swap counterparty pays the buyer.
Financial Settlement

• Whatever the market price of oil, the net cost to the buyer is the swap price, $20.483

\[
\text{Spot price} - \text{Swap price} - \text{Spot price} = -\text{Swap price}
\]

Swap Payment Spot Purchase of Oil

• Note that 100,000 is the **notional amount** of the swap, meaning that 100,000 barrels is used to determine the magnitude of the payments when the swap is settled financially.
Physical Versus Financial Settlement (cont’d)

• The results for the buyer are the same whether the swap is settled physically or financially. In both cases, the net cost to the oil buyer is $20.483
Swaps are forward contracts coupled with a borrowing/lending money agreement

- Consider the swap price of $20.483/barrel. Relative to the forward curve price of $20 in 1 year and $21 in 2 years, we are overpaying by $0.483 in the first year, and we are underpaying by $0.517 in the second year.
- Thus, by entering into the swap, we are lending the counterparty money for 1 year. The interest rate on this loan is:

\[ \frac{0.517}{0.483} - 1 = 7\% \]

- Given 1- and 2-year zero-coupon bond yields of 6% and 6.5%, 7% is the 1-year implied forward yield from year 1 to year 2:

\[ \frac{(1 + 0.065)^2}{1.06} - 1 \]

- That is, if the deal is priced fairly, the interest rate on this loan should be the implied forward interest rate.
Swaps are forward contracts coupled with a borrowing/lending money agreement

• So let us analyze the following two transactions.

• Buy swap where the spot price is $20.483.

• Buy one year ($20) and two year ($21) futures contracts, and enter an agreement where you lend $0.483 in the end of year one till the end of year two (7%).

• The two transactions are indeed equivalent.

• One swap contract is equivalent to the combination of three futures contracts.
The Swap Counterparty

• The swap counterparty is a dealer, who is, in effect, a broker between buyer and seller.

• The fixed price paid by the buyer, usually, exceeds the fixed price received by the seller.

• This price difference is a bid-ask spread, and is the dealer’s fee.

• The dealer bears the credit risk of both parties, but is not exposed to price risk.
The Swap Counterparty (cont’d)

• The situation where the dealer matches the buyer and seller is called a back-to-back transaction or “matched book” transaction.
The Swap Counterparty (cont’d)

- Alternatively, the dealer can serve as counterparty and hedge the transaction by entering into long forward or futures contracts.

  - Note that the net cash flow for the hedged dealer is a loan, where the dealer receives cash in year 1 and repays it in year 2.
  - Thus, the dealer also has interest rate exposure (which can be hedged by using Eurodollar contracts or forward rate agreements).

<table>
<thead>
<tr>
<th>Year</th>
<th>Payment from Oil Buyer</th>
<th>Long Forward</th>
<th>Net</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$20.483 – Year 1 Spot Price</td>
<td>Year 1 Spot Price – $20</td>
<td>$0.483</td>
</tr>
<tr>
<td>2</td>
<td>$20.483 – Year 2 Spot Price</td>
<td>Year 2 Spot Price – $21</td>
<td>−$0.517</td>
</tr>
</tbody>
</table>
The Market Value of a Swap

• The market value of a swap is zero at interception

• Once the swap is struck, its market value will generally no longer be zero because
  – the forward prices for oil and interest rates will change over time
  – even if prices do not change, the market value of swaps will change over time due to the implicit borrowing and lending

• A buyer wishing to exit the swap could enter into an offsetting swap with the original counterparty or whomever offers the best price

• The market value of the swap is the difference in the PV of payments between the original and new swap rates
The Market Value of a Swap: Example

- Suppose that immediately after the buyer enters the swap contract, the one and two year forward prices become 22 and 23.
- The PV of these two future payments is
  \[ \frac{22}{1.06} + \frac{23}{1.065^2} = 41.033 \]
- Recall with 21 and 22 forward prices the PV is $37.383
- With the new forward prices, the PV is $3.65 higher from the previous PV.
- So the Swap market value is $3.65.
More Examples

• Q: Suppose that corn forward prices for 1, 2, and 3 years are $2.1, $2.2, and $2.35. The corresponding effective rates are 5.2%, 5.5%, and 5.8%. What is the prepaid swap price?
  • A: The PV of the forward prices is $5.96. This is the prepaid swap price.

• Q: Assume now that interest rates for the next 3 years are 6.2%, 6.5%, and 6.8%. The prepaid swap price, which is the PV of the forward prices, is given as $6.50. What is the 3-year swap price on corn?
  • A: You are looking for \( x \) such that the present value of \( x \) to be paid over the next three years is equal to the prepaid swap price. \( x = $2.46 \).

• Q: Assume corn spot prices over the next 3 years are $2.20, $2.35, and $2.28, respectively. The original swap price was $2.30 per bushel. If cash settlement occurs, what transaction will the counter-party make in year 2 on a 5,000-bushel swap agreement?
  • A: Compute \((2.3-2.35)*5000\), which means payment of $250.
More Examples

Q: Suppose the 1-year, 2-year, and 3-year oil forward prices are 50.85, 45.94, and 42.97, respectively. The corresponding interest rates are 3%, 3.5%, and 3.75%. What is the 2-year and the 3-year swap prices?

A1: First solve the PV of the cost per two barrels: 
\[
\frac{50.85}{1.03} + \frac{45.94}{1.035^2} = 92.254.
\]
Next solve 
\[
\frac{x}{1.03} + \frac{x}{0.1035^2} = 92.254
\]
which gives 
\[
x = 48.443.
\]

A2: Following the same idea you get 
\[
x = 46.693
\]
for the 3-year swap.
Interest Rate Swaps

- Companies use interest rate swaps to modify their interest rate exposure.

- The notional principle (NP) of the swap is the amount on which the interest payments are based.

- The life of the swap is the swap term or swap tenor

- If swap payments are made at the end of the period (when interest is due), the swap is said to be settled in arrears
Basic terms of IRS

• Payer and receiver - quoted relative to fixed interest (i.e. payer = payer of fixed rate)
• Short party = payer of fixed interest.
• Long party = receiver of fixed interest.
An Example of an IRS

- XYZ Corp. has $200M of floating-rate debt at LIBOR, i.e., every year it pays that year’s current LIBOR.

- XYZ would prefer to have fixed-rate debt with 3 years to maturity.

- One possibility is to retire the floating rate debt and issue fixed rate debt in its place. That could be too costly.

- XYZ could enter a swap, in which they receive a floating rate and pay the fixed rate, which is, say, 6.9548%.
An Example of an IRS (cont’d)

• On net, XYZ pays 6.9548%

\[
\text{XYZ net payment} = -\text{LIBOR} + \text{LIBOR} - 6.9548\% = -6.9548\%
\]

Floating Payment Swap Payment

Prof. Doron Avramov
Derivatives Securities
Computing the Swap Rate

- Suppose there are $n$ swap settlements, occurring on dates $t_i$, $i = 1, \ldots, n$
- The implied forward interest rate from date $t_{i-1}$ to date $t_i$, known at date 0, is $r_0(t_{i-1}, t_i)$
- The price of a zero-coupon bond maturing on date $t_i$ is $P(0, t_i)$
- The fixed swap rate is $R$

- The market-maker is a counterparty to the swap in order to earn fees, not to take on interest rate risk. Therefore, the market-maker will hedge the floating rate payments by using, for example, forward rate agreements.
Computing the Swap Rate

• The requirement that the hedged swap have zero net PV is

\[ \sum_{i=1}^{n} P(0,t_i) \left[ R - r_0(t_{i-1},t_i) \right] = 0 \]

• which can be rewritten as

\[ R = \frac{\sum_{i=1}^{n} P(0,t_i)r(t_{i-1},t_i)}{\sum_{i=1}^{n} P(0,t_i)} \]
Computing the Swap Rate

• We can further rewrite the equation to make it easier to interpret

\[ R = \sum_{i=1}^{n} \left[ \frac{P(0,t_i)}{\sum_{j=1}^{n} P(0,t_j)} \right] r(t_{i-1},t_i) \]

• Thus, the fixed swap rate is as a weighted average of the implied forward rates, where zero-coupon bond prices are used to determine the weights
Computing the Swap Rate

• Alternative way to express the swap rate is

\[ R = \frac{1 - P_0(0,t_n)}{\sum_{i=1}^{n} P_0(0,t_i)} \]

• This equation is equivalent to a formula for the coupon on a par coupon bond.

• Thus, the swap rate is the coupon rate on a par coupon bond.
Example: Computing the Swap Rate

- Define $r(0,t)$ as the spot interest rate for a zero coupon bond maturing at time $t$.

- Define $r(t_1,t_2)$ as the forward interest rate from time $t_1$ until time $t_2$.

- Assume the zero (spot) term structure is:
  - $r(0,1) = 5\%$, $r(0,2) = 6\%$, $r(0,3) = 7.5\%$.

- Therefore the forward rates are:
  - $fr(1,2) = 7.01\%$; and $fr(2,3) = 10.564\%$.

- These forward rates should exist in the FRA and futures markets.
Example: Computing the Swap Rate

• Now consider a swap with a tenor of 3 years.
• The floating rate is the one-year LIBOR.
• Settlement is yearly.
• What is the “fair” fixed rate?
• Let the forward rates be the expected future spot rates.
• Arbitrarily set the loan size to be $100.
• Thus, the “expected” floating rate cash flows are:
  • \( CF_1 = (0.05)(100) = 5 \)
  • \( CF_2 = (0.0701)(100) = 7.01 \)
  • \( CF_3 = (0.1056)(100) + 100 = 110.564 \)
Example: Computing the Swap Rate

• Value these expected cash flows at the appropriate discount rates: the spot zero coupon interest rates:

\[
\frac{5}{1.05} + \frac{7.01}{(1.06)^2} + \frac{10.564}{(1.075)^3} + \frac{100}{(1.075)^3} = 100
\]

• An important lesson: The value of the floating rate side of the swap equals the value of the fixed side of the swap immediately after a floating payment has been made.
Example: Computing the Spot Rate

• Because the value of a swap at origination is set to zero, the fixed rate payments must satisfy:

\[
\frac{100A}{(1.05)} + \frac{100A}{(1.06)^2} + \frac{100A}{(1.075)^3} + \frac{100}{(1.075)^3} = 100
\]

\[
A\left\{\frac{1}{(1.05)} + \frac{1}{(1.06)^2} + \frac{1}{(1.075)^3}\right\} = 1 - 0.80496
\]

\[
A\{2.6473\} = 0.19504
\]

\[
A = 7.367\%
\]
Valuing a Swap after Origination

- Consider the previous interest rate swap example.

- Suppose that 3 months after the origination date, the yield curve flattens at 7%.

- The next floating cash flow is known to be 5.

- Immediately after this payment is paid, \( PV(\text{remaining floating payments}) = NP = 100 \). Recall, at the reset date the contract is valued at par.
Valuing a Swap after Origination.

\[ V_{\text{floating}} = \frac{5}{1.07^{0.75}} + \frac{100}{1.07^{0.75}} = 99.805. \]

\( V_{\text{floating}} = PV(\text{Next Payment}) + PV(\text{NP}) \)

So, after 3 - months, \( V_{\text{floating}} \) has decreased.

\[ V_{\text{fixed}} = \frac{7.367}{1.07^{0.75}} + \frac{7.367}{1.07^{1.75}} + \frac{107.367}{1.07^{2.75}} = 102.685. \]

So, after 3 - months, \( V_{\text{fixed}} \) has increased.

- The swap’s value is $2.88, per $100 of NP. (102.685 – 99.805).
- That is, one would have to pay $2.88 today to eliminate this swap. Who pays whom?
The Swap Curve

- A set of swap rates at different maturities is called the *swap curve*.

- The swap curve should be consistent with the interest rate curve implied by the Eurodollar futures contract, which is used to hedge swaps.

- Recall that the Eurodollar futures contract provides a set of 3-month forward LIBOR rates. In turn, zero-coupon bond prices can be constructed from implied forward rates. Therefore, we can use this information to compute swap rates.
### The Swap Curve (cont’d)

- Three months LIBOR forward rates implied by Eurodollar futures prices with maturity dates given in the first column. Prices are from June 2\textsuperscript{nd}, 2004.

<table>
<thead>
<tr>
<th>Maturity Date</th>
<th>Eurodollar Futures Price</th>
<th>Implied Quarterly Rate $r(t_i, t_{i+1})$</th>
<th>Implied June 2004 Price of $1$ Pain on Maturity Date, $t_i, P(0, t_i)$</th>
<th>Swap Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jun – 04</td>
<td>98.555</td>
<td>0.0037</td>
<td>-</td>
<td>1.4611%</td>
</tr>
<tr>
<td>Sep – 04</td>
<td>98.010</td>
<td>0.0050</td>
<td>0.9964</td>
<td>1.7359%</td>
</tr>
<tr>
<td>Dec – 04</td>
<td>97.495</td>
<td>0.0063</td>
<td>0.9914</td>
<td>2.0000%</td>
</tr>
<tr>
<td>Mar – 05</td>
<td>97.025</td>
<td>0.0075</td>
<td>0.9851</td>
<td>2.2495%</td>
</tr>
<tr>
<td>Jun – 05</td>
<td>96.600</td>
<td>0.0086</td>
<td>0.9778</td>
<td>2.4836%</td>
</tr>
<tr>
<td>Sep – 05</td>
<td>96.235</td>
<td>0.0095</td>
<td>0.9695</td>
<td>2.6997%</td>
</tr>
<tr>
<td>Dec – 05</td>
<td>95.910</td>
<td>0.0103</td>
<td>0.9603</td>
<td>2.8995%</td>
</tr>
<tr>
<td>Mar – 06</td>
<td>95.650</td>
<td>0.0110</td>
<td>0.9505</td>
<td>3.0808%</td>
</tr>
</tbody>
</table>
Computing the Implied 3M Future Rate

- Of course, the futures prices are set by the market.
- The implied quarterly rate is simply computed as the difference between 100 and the Futures price divided by 4 and multiplied by 1%:

\[
ImpliedRate = \frac{100 - \text{Future Price}}{4} \times \frac{1}{100}
\]

- For instance, the Jun-04 rate is computed as:

\[
ImpliedRate = \frac{100 - 98.555}{4} \times \frac{1}{100} = 0.0037
\]
Computing the Price of Zeros

- Next, the Zero Coupon Price is computed by dividing one by the product of future rates (which is the spot rate). For instance

\[
\frac{1}{1.0037} = 0.9964
\]

\[
\frac{1}{1.0037 \times 1.005} = 0.9914
\]

\[
\frac{1}{1.0037 \times 1.005 \times 1.0063} = 0.9851
\]
Computing the Swap Rate

- Finally, the Swap Rate is based upon the formulas exhibited earlier.
- For Jun-04, the Swap Rate is computed as

\[
\frac{100 \times R + 100}{1.0037} = 100
\]

which gives a quarterly rate \( R = 0.0037 \) and an annual rate of 1.4579%.
- For Sep-2004, the Swap Rate computed as

\[
\frac{100R}{1.0037} + \frac{100R + 100}{1.0037 \times 1.005} = 100
\]

- It is important to note that the futures rate reported establish the term structure of futures rates. From this term structure we could generate the term structure of spot rates taught in the term structure section.
Unwinding an existing swap

• Enter into an offsetting swap at the prevailing market rate.
• If we are between two reset dates the offsetting swap will have a short first period to account for accrued interest.
• It is important that floating payment dates match!!
Risks of Swaps

- Price risk - value of fixed side may change due to changing interest rates.
- Credit risk - default or change of rating of counterparty
- Mismatch risk - payment dates of fixed and floating side are not necessarily the same
- Basis risk and Settlement risk
Credit risk of a swap contract

• Default of counterparty (change of rating).
• Exists when the value of swap is positive
• Frequency of payments reduces the credit risk, similar to mark to market.
• Credit exposure changes during the life of a swap.
Why Swap Interest Rates?

- Interest rate swaps permit firms to separate credit risk and interest rate risk
  - By swapping its interest rate exposure, a firm can pay the short-term interest rate it desires, while the long-term bondholders will continue to bear the credit risk
Deferred Swap

- A **deferred swap** is a swap that begins at some date in the future, but its swap rate is agreed upon today.

- The fixed rate on a deferred swap beginning in $k$ periods is computed as

$$R = \sum_{i=k}^{T} \frac{P_0(0,t_i)r_0(t_{i-1},t_i)}{\sum_{i=k}^{T} P(0,t_i)}$$

- Previously we dealt with $k = 1$
Amortizing and Accreting Swaps

- An **amortizing swap** is a swap where the notional value is *declining* over time (e.g., floating rate mortgage)

- An **accreting swap** is a swap where the notional value is *growing* over time

- The fixed swap rate is still a weighted average of implied forward rates, but now the weights also involve changing notional principle, $Q_t$

$$R = \frac{\sum_{i=1}^{n} Q_t P(0,t_i) r(t_{i-1},t_i)}{\sum_{i=1}^{n} Q_t P(0,t_i)}$$ (8.7)
Currency Swaps

- Firms often issue debt denominated in a foreign currency either to hedge against revenues received in that currency or because borrowing costs in that country are cheaper.

- Suppose the firm later wants to change the currency to which they have exposure.

- A currency swap entails an exchange of payments in different currencies.

- A currency swap is equivalent to borrowing in one currency and lending in another.
Currency Swap Formulas

• Consider a swap in which a dollar annuity, $R$, is exchanged for an annuity in another currency, $R^*$

• There are $n$ payments

• The time-0 forward price for a unit of foreign currency delivered at time $t_i$ is $F_{0,t_i}^*$

• The dollar-denominated zero-coupon bond price is $P_{0,t_i}$
Currency Swap Formulas (cont’d)

- Given $R^*$, what is $R$?

$$R = \frac{\sum_{i=1}^{n} P_{0,t_i} \times R \times F_{0,t_i}}{\sum_{i=1}^{n} P_{0,t_i}}$$  

(8.8)

- This equation is equivalent to equation (8.2), with the
Currency Swap Formulas (cont’d)

– When coupon bonds are swapped, one has to account for the difference in maturity value as well as the coupon payment
– If the dollar bond has a par value of $1, the foreign bond will have a par value of $1/x_0$, where $x_0$ is the current exchange rate expressed as dollar per unit of the foreign currency

• The coupon rate on the dollar bond, $R$, in this case is

$$R = \frac{\sum_{i=1}^{n} P_{0,t_i} \times R \times \frac{F_{0,t_i}}{X_0} + P_{0,t_n} \times \left(\frac{F_{0,t_n}}{X_0} - 1\right)}{\sum_{i=1}^{n} P_{0,t_i}}$$

(8.9)
Currency Swaps: Example

• Suppose the effective annual euro denominated interest rate is 3.5% whereas the dollar denominated rate is 6%. The spot exchange rate is $0.9 per euro. A US based firm has a 3-year 3.5% euro denominated bond with a 100 euro per value and price. The firm wishes to guarantee the dollar value of the payments. The firm therefore buys the euro forward to eliminate currency exposure.

• What are the one two and three year forward exchange rate? We studied earlier that the
  
  forward rate=spot rate*exp[T*(domestic-foreign)].

• Note that the formula requires continuously compounded rates, which are ln(1.06) and ln(1.035), respectively. So we get forward exchange rates of 0.9217, 0.9440, and 0.9668.

• The future debt payments in $ are 3.226, 3.304, and 100.064.

• Using a 6% rate – the PV of those payments is $90.

• Clearly, hedging does not change the value of debt.
Currency Swaps: Example

• As an alternative to hedging each euro-denominated payment with a forward contract, a firm wishing to change its currency exposure can enter into a currency swap.
• The firm makes debt payments in one currency and receives debt payments in a different currency.
• So let us assume that the firm considered in the previous example uses a swap rather than forward contracts to hedge its euro exposure.
Example of a Currency Swap

• That is, we assume the firm enters into a swap where it pays dollars (6% on a $90 bond) and receives euros (3.5% on a €100 bond).
• The position of the market-maker is summarized below:
• Unhedged and hedged cash flows for a dollar-based firm with euro-denominated debt. The effective annual dollar-denominated rate is 6% and the effective annual euro-denominated interest rate is 3.5%.

<table>
<thead>
<tr>
<th>Year</th>
<th>Forward Exchange Rate ($/€)</th>
<th>Receive Dollar Interest</th>
<th>Pay Hedged Euro Interest</th>
<th>Net Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9217</td>
<td>$5.40</td>
<td>$-€3.5 \times 0.9217</td>
<td>$2.174</td>
</tr>
<tr>
<td>2</td>
<td>0.9440</td>
<td>$5.40</td>
<td>$-€3.5 \times 0.9440</td>
<td>$2.096</td>
</tr>
<tr>
<td>3</td>
<td>0.9668</td>
<td>$95.40</td>
<td>$-€103.5 \times 0.9668</td>
<td>$-4.664</td>
</tr>
</tbody>
</table>

• The PV of the market-maker’s net cash flows is $(2.174 / 1.06) + (2.096 / 1.06^2) - (4.664 / 1.06^3) = 0$
One more Example of Hedging Currency Risk

- Suppose the annually cc Euro and Dollar dominated interest rate are 4% and 3.25% respectively.
- The spot exchange rate is $1.27/Euro.
- Europia, a Paris based firm conducting business in Euros has issued a 3-year 3.25% dollar dominated bond at par in the amount of $250M.
- The firm wishes to guarantee the Euro value of the payments:
  a. Construct the Euro/Dollar forward curve for the next 3 years.
  b. What should Europia do to guarantee a Euro value of its payments?
  c. What is the PV of the hedged Euro cash flow?
Hedging Currency Risk – continued

a) The forward rates for year 1, 2 and 3 (here we have to be careful about using the right currency in the numerator)

\[ F_{0,1} = 0.7874 \times \exp\left( (0.04 - 0.0325) \times 1 \right) = 0.7933 \]
\[ F_{0,2} = 0.7874 \times \exp\left( (0.04 - 0.0325) \times 2 \right) = 0.7993 \]
\[ F_{0,3} = 0.7874 \times \exp\left( (0.04 - 0.0325) \times 3 \right) = 0.8053 \]

b) Since the firm will make debt payments in Dollars it should buy three Euro forwards to eliminate currency exposure:

<table>
<thead>
<tr>
<th>Year</th>
<th>Un-hedged Dollar CF</th>
<th>Forward Exchange Rate</th>
<th>Hedged Euro CF</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-8.125M</td>
<td>0.7933</td>
<td>-6.4456M</td>
</tr>
<tr>
<td>2</td>
<td>-8.125M</td>
<td>0.7993</td>
<td>-6.4943M</td>
</tr>
<tr>
<td>3</td>
<td>-258.125M</td>
<td>0.8053</td>
<td>-207.8705M</td>
</tr>
</tbody>
</table>

c) \[ PV (€) = -6.4456/\exp(0.04) - 6.4943/\exp(0.04\times2)-207.8705/\exp(0.04\times3) = -196.8294 \]
What is the $ value of this PV? \[ €196.8294 \times 1.27 = $250K \]
Swaptions

• A swaption is an *option* to enter into a swap with specified terms. This contract will have a premium

• A swaption is analogous to an ordinary option, with the PV of the swap obligations (the price of the prepaid swap) as the underlying asset

• Swaptions can be American or European
Swaptions (cont’d)

• A **payer swaption** gives its holder the right, but not the obligation, to pay the fixed price and receive the floating price
  – The holder of a receiver swaption would exercise when the fixed swap price is above the strike

• A **receiver swaption** gives its holder the right to pay the floating price and receive the fixed strike price.
  – The holder of a receiver swaption would exercise when the fixed swap price is below the strike
Summary

• The swap formulas in different cases all take the same general form

• Let $f_0(t_i)$ denote the forward price for the floating payment in the swap. Then the fixed swap payment is

$$ R = \frac{\sum_{i=1}^{n} P(0,t_i) f_0(t_i)}{\sum_{i=1}^{n} P(0,t_i)} $$
T-bill Futures

- One can use T-bill and Eurodollar futures to speculate on, or hedge against changes in, short-term (3-months to a year) interest rates.
- The longs profit when interest rates fall; the shorts profit when interest rates rise (and fixed income instrument prices fall).
- The T-bill futures market is thinly traded (illiquid).
- LIBOR futures contracts are quite liquid.
T-bill Futures: Example


– Then, someone who goes long a T-bill futures contract has “essentially” agreed to buy $1 million face value of 3-month T-bills on June 12, 1995, at a forward discount yield (APR) of 6.1% [100-93.99].

– The price of the futures contract is

\[ 1,000,000 [1-0.061 \times 91/360] = 984,808 \]

The implied effective YTM is 6.33%. Check!
T-bill Futures: The Underlying Index

Define: index price = 100 \( (1 - Y_d) \), where \( Y_d \) is the yield on a bank discount basis. \( Y_d \) is also the quoted price of the T-bill in the market.

The corresponding dollar discount is

\[
D = Y_d \cdot 1,000,000 \cdot \frac{t}{360}
\]
T-bill futures, example

Assume that the index price is 92.52. The corresponding yield on a bank discount basis is 7.48% 
\[
(100-92.52)/100
\]
Assume there are 91 days to maturity, then the discount is
\[
D = 0.0748 \times $1,000,000 \times 91/360 = $18,907.78
\]
The invoice price is then $981,092.22 (nominal–D). Here, one tick change in the value of futures contract 0.01 
will change the value by $25.28 (since 91/360).
T-bond Futures

• T-bond Futures are futures contracts tied to a pool of Treasury bonds, with a remaining maturity greater than 15 years (and non callable within 15 years).

• Similar contracts exists of 2, 5, and 10 year Notes.

• Physical Delivery.
T-bond Futures

• Futures contracts are quoted like T-bonds, e.g. 97-02, in percent plus 1/32, with a notional of $100,000.

• Thus the price of the contract will be

$100,000 \times (97 + 2/32)/100 = $97,062.50

• Assume that the next day the yield goes up and the price drops to 95-0, the new price is $95,000 and the loss of a long side is $2,062.50.
Treasury Bond/Note Futures

- WSJ listings for T-bond and T-note futures

- Contract specifications

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Treasury Bond/Note Futures (cont’d)

- Long T-note futures position is an obligation to buy a 6% bond with maturity between 6.5 and 10 years to maturity.
- The short party is able to choose from various maturities and coupons: the “cheapest-to-deliver” bond.
- In exchange for the delivery the long pays the short the “invoice price.”
  - Invoice price = (Futures price x conversion factor) + accrued interest.

### Table 7.5

<table>
<thead>
<tr>
<th>Description</th>
<th>8-Year 7% Coupon, 6.4% Yield</th>
<th>7-Year 5%, 6.3% Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market Price</td>
<td>103.71</td>
<td>92.73</td>
</tr>
<tr>
<td>Price at 6% (Conversion Factor)</td>
<td>106.28</td>
<td>94.35</td>
</tr>
<tr>
<td>Invoice Price (Futures × Conversion Factor)</td>
<td>103.71</td>
<td>92.09</td>
</tr>
<tr>
<td>Invoice — Market</td>
<td>0</td>
<td>-0.66</td>
</tr>
</tbody>
</table>

Prices, yields, and the conversion factor for two bonds. The futures price is 97.583. The short would break even delivering the 8-year 7% bond, and lose money delivering the 7-year 5% bond. Both bonds make semiannual coupon payments.
Treasury Bond/Note Futures (cont’d)

- Long T-note futures position is an obligation to buy a 6% bond with maturity between 6.5 and 10 years to maturity
- The short party is able to choose from various maturities and coupons: the “cheapest-to-deliver” bond
- In exchange for the delivery the long pays the short the “invoice price.”

\[
\text{Invoice price} = (\text{Futures price} \times \text{conversion factor}) + \text{accrued interest}
\]

**Table 7.3**

<table>
<thead>
<tr>
<th>Description</th>
<th>8-Year 7% Coupon, 6.4% Yield</th>
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</tr>
</tbody>
</table>
Repurchase Agreements

- A repurchase agreement or a repo entails selling a security with an agreement to buy it back at a fixed price.
- The underlying security is held as collateral by the counterparty => A repo is collateralized borrowing.
- Frequently used by securities dealers to finance inventory.
- Speculators and hedge funds also use repos to finance their speculative positions.
- A “haircut” is charged by the counterparty to account for credit risk.