Implications of Long-Run Risk for Asset Allocation Decisions*

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Abstract
This paper proposes a structural approach to long-horizon asset allocation. In particular, the investor draws inferences about asset returns from a vector autoregression (VAR) with economic restrictions on the intercept, slope, and covariance matrix implied by the long-run risk model of Bansal and Yaron (2004). Comparing the optimal allocations of investors using the long-run risk VAR versus an unrestricted reduced-form VAR reveals stark differences in portfolio strategies. Long-run risk investors are quite conservative relative to reduced-form investors due to intertemporal hedging concerns. Despite the differing strategies, both investors achieve success in timing the market. The gains of the long-run risk investor appear to arise from his ability to avoid exposure to large negative events, while the reduced-form investor better capitalizes on periods of high average returns.

JEL Classification: E21, E32, G11

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1 Introduction

Long-horizon investors must consider their exposure to risks which affect short-run and long-run performance when choosing optimal portfolios. When expected returns vary with the economic state, investors may optimally allocate more to assets which pay off when the economy is transitioning into less favorable states. Thus, intertemporal considerations become important when current portfolio returns are correlated with expected future performance (e.g., Merton (1971, 1973)). The long-run risk model of Bansal and Yaron (2004) captures the effect of future economic expectations on current returns, inducing intertemporal risk. The long-run risk framework features a small persistent component in the consumption and dividend growth rates, time-varying economic uncertainty, and a representative investor with the recursive preferences of Epstein and Zin (1989, 1991). This formulation is successful in explaining several aggregate and cross-sectional asset pricing puzzles.

In this paper, we investigate the impact of long-run risk on the portfolio decisions of long-horizon investors. We build upon the asset allocation framework of Campbell and Viceira (1999), who propose analytical and numerical solutions to an approximation of the long-horizon portfolio choice problem. In particular, they consider the optimal allocations of an Epstein–Zin investor who draws inferences about current and future returns from a vector autoregression (VAR) of asset returns and state variables. We take a structured approach to the long-horizon portfolio choice problem by appealing to economic theory to determine the form of the VAR as well as its parameters. In particular, we consider the implications of Bansal and Yaron’s (2004) long-run risk economy for the relation between asset returns and macroeconomic variables. In addition to its success in explaining asset pricing regularities, the long-run risk model provides insight into how macroeconomic risk influences asset prices. Thus, we are able to integrate a framework capable of explaining the joint behavior of economic and market variables into the asset allocation problem.

Expected returns in the long-run risk framework are driven by expected economic growth and economic uncertainty. Risk premiums are increasing in uncertainty, while the risk-free rate is
increasing in expected growth and decreasing in uncertainty. Thus, the long-run risk VAR captures the effects of these macroeconomic forces on asset returns. We operationalize the long-run risk VAR by utilizing the fact that the price-dividend ratio and risk-free rate jointly capture the dynamics of the unobservable economic state variables in the long-run risk model (Constantinides and Ghosh (2011)). We therefore estimate a VAR in asset returns and these observable state variables under the parameter restrictions on the intercept, slope, and covariance matrix implied by the long-run risk model. The long-horizon investor then draws inferences about current and future returns from this long-run risk VAR.

We analyze portfolio allocations to the stock market portfolio and a risk-free asset using both the long-run risk VAR and a reduced-form VAR. The long-run risk VAR is estimated subject to the economic restrictions using data on consumption and dividend growth and the two state variables, the price-dividend ratio and risk-free rate. In contrast, the reduced-form VAR leaves the parameters unrestricted and is estimated with stock return data in addition to the two state variables. Thus, differing information about returns and the economy can potentially be reflected by the two VARs.

The optimal portfolio choices of the long-run risk and reduced-form investors display large differences. Reduced-form investors take aggressive positions in stocks due to their belief in strong mean reversion in returns. These investors increase their allocation to stocks because they believe stocks provide a good hedge against intertemporal risk. In contrast, long-run risk investors do not view stocks as an effective means to hedge intertemporal risk, since poor current performance is often not offset by good future performance. Long-run risk investors are thus quite conservative relative to their reduced-form counterparts. For example, a long-run risk investor with a coefficient of relative risk aversion, $\gamma$, of 10 invests 11% of his wealth in stocks versus 137% for a similar reduced-form investor. Additionally, long-run risk investors adjust their allocations less with changes in the economic state variables and the time-series behavior of the long-run risk and reduced-form investors are not strongly related.

Although the long-run risk and reduced-form investors adopt differing portfolio strategies, two investors each perform well. Both investors achieve a Sharpe ratio of about 0.45 relative to
the market Sharpe ratio of 0.34. This performance indicates positive market timing ability, which is confirmed by the Henriksson–Merton (1981) model. Despite their different approaches, both investors achieve success.

We further investigate the market timing abilities of the long-run risk and reduced-form investors. The positive performance of the long-run risk investor appears to be largely driven by his ability to avoid exposure to large negative events. In particular, he substantially reduces his exposure to stocks before the two largest negative market return periods, 1973-74 and 2008. Meanwhile, the reduced-form investor remains highly exposed to negative events. Her positive performance is primarily attributable to her aggressive investments during the relatively high return periods in the 1950s as well as the late-1970s to early-1990s. The investors appear to capitalize on different information in the data.

In sum, we incorporate the economic restrictions of the long-run risk model into the long-horizon portfolio choice problem. Relative to an investor relying on a reduced-form approach, the long-run risk investor is quite conservative. The two investors adopt very different strategies but both display positive market timing ability, suggesting they use alternative aspects of information about stock returns when they form portfolios.

The remainder of the paper is organized as follows. Section 2 summarizes Campbell and Viceira’s (1999) approach to solving the long-horizon portfolio choice problem and introduces the long-run risk VAR used to determine optimal asset allocations. Section 3 discusses VAR estimation and the resulting long-run risk parameter estimates. Section 4 discusses the estimated reduced-form and long-run risk VARs. Section 5 examines the optimal allocations of long-run risk and reduced-form investors and compares their investment performance. Section 6 concludes.

2 The Model

Our economic setup builds upon the asset allocation framework of Campbell and Viceira (1999) while incorporating the components of the long-run risk economy originated by Bansal and Yaron
(2004). Section 2.1 introduces the long-run risk setup. Section 2.2 defines the asset allocation problem in a long-run risk economy and presents economic restrictions for the VAR in Campbell and Viceira’s (1999) portfolio choice framework.

2.1 Long-Run Risk

The long-run risk economy displays three important features. First, the representative investor has Epstein–Zin (1989, 1991) recursive preferences. Second, both the consumption and dividend growth processes exhibit a small persistent component, which leads to time variation in expected growth. Third, the volatility of consumption and dividend shocks are allowed to vary over time, leading to time-varying economic uncertainty. Recent work has employed the long-run risk framework to offer potential rational explanations for asset-pricing puzzles including the equity premium, risk-free rate, and excess volatility puzzles (Bansal and Yaron (2004)), as well as the value premium (e.g., Hansen, Heaton, and Li (2008), and Ai and Kiku (2010)), and the effects based on analyst forecast dispersion, idiosyncratic volatility, and distress risk (Avramov, Cederburg, and Hore (2011)).

This paper employs the long-run risk framework to impose an equilibrium structure on the VAR of asset returns and state variables and ultimately study optimal portfolio rules. The representative agent is endowed with Epstein–Zin (1989, 1991) recursive preferences,

\[ U_t = \left[ (1 - \delta)C_t^{(1-\gamma)/\theta} + \delta(E_tU_{t+1}^{1-\gamma})^{1/\theta} \right]^{\theta/(1-\gamma)}, \]

where \( C_t \) is consumption at time \( t \), \( E_t \) is the expected value operator, \( \delta \) is a time-preference parameter, \( \gamma \) is the coefficient of relative risk aversion, \( \theta \equiv (1-\gamma)/(1-1/\psi) \), and \( \psi \) is the elasticity of intertemporal substitution. The agent maximizes utility subject to the budget constraint,

\[ W_{t+1} = (1 + R_{a,t+1})(W_t - C_t), \]
where $R_{a,t+1}$ is the return on the aggregate wealth portfolio.

The long-run risk model features time variation in expected economic growth and economic uncertainty. Specifically, consumption and dividend dynamics are formulated as

$$
g_{t+1} = \mu + x_t + \sigma_t \eta_{t+1}, \tag{3}
g_{d,t+1} = \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1} + \pi_d \sigma_t \eta_{t+1}, \tag{4}
x_{t+1} = \rho x_t + \varphi_e \sigma_t e_{t+1}, \tag{5}
\sigma_t^2 = \sigma^2 + \nu_1 (\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1} \tag{6}
e_{t+1}, u_{t+1}, \eta_{t+1}, w_{t+1} \sim N(0, 1). \tag{7}
$$

where $x_t$ is a common, mean-reverting component in consumption growth ($g$) and dividend growth ($g_d$), $\phi$ is a leverage parameter in the spirit of Abel (1999), and $\sigma_t^2$ captures potentially time-varying economic uncertainty.

### 2.2 Long-Horizon Portfolio Choice

Campbell and Viceira (1999) consider the portfolio and consumption choices of a multiperiod investor who can invest in the stock market and a risk-free asset. The investor is endowed with Epstein–Zin (1989, 1991) recursive preferences as in equation (1) and the investor’s intertemporal budget constraint is formulated as

$$W_{t+1} = (1 + R_{p,t+1})(W_t - C_t), \tag{8}$$

where $R_{p,t+1} = \alpha_t R_{m,t+1} + (1 - \alpha_t) R_{f,t+1}$ is the return on a portfolio with a weight of $\alpha_t$ invested in stocks and the remainder invested in a risk-free asset.

Campbell and Viceira (1999) propose a method for finding the optimal portfolio and consumption rules under the assumption that state variables and asset returns jointly follow a vector
autoregression (VAR) of order one,

\[ z_{t+1} = \Phi_0 + \Phi_1 z_t + v_{t+1}, \quad v_{t+1} \sim N(0, \Sigma), \tag{9} \]

where

\[ z_{t+1} \equiv \begin{bmatrix} r_{f,t+1} \\ r_{m,t+1} - r_{f,t+1} \\ s_{t+1} \end{bmatrix}. \tag{10} \]

\( r_{f,t+1} \) and \( r_{m,t+1} \) are the continuously compounded risk-free rate and stock return, respectively, and \( s_{t+1} \) is a vector of economic state variables. In particular, portfolio and consumption rules take the form

\[ \alpha_t = a_0 + A_1 z_t, \tag{11} \]

\[ c_t - w_t = b_0 + b_1' z_t + z'_t B_2 z_t, \tag{12} \]

so the portfolio weight in stocks (consumption-wealth ratio) is a linear (quadratic) function of the state variables. Campbell and Viceira (1999, 2001) and Campbell, Chan, and Viceira (2003) examine asset allocation in this context. In particular, they empirically estimate the VAR in equation (9) for various combinations of assets and state variables to study optimal portfolio choice.

In contrast to the reduced-form approach of prior studies, we utilize the economic restrictions provided by the long-run risk economy when specifying the VAR in equation (9). The long-run risk framework implies a VAR with four economic state variables: \( x_t \) captures the current state of time-varying expected economic growth, \( \sigma_t^2 \) represents time-varying economic uncertainty, and \( g_t \) and \( g_{d,t} \) are current consumption and dividend growth rates, respectively. The long-run risk VAR
therefore takes the form

\[
\begin{bmatrix}
  r_{f,t+1} \\
r_{m,t+1} - r_{f,t+1} \\
x_{t+1} \\
\sigma_{t+1}^2 \\
g_{t+1} \\
g_{d,t+1}
\end{bmatrix}
= \Phi_{LRR,0} + \Phi_{LRR,1}
\begin{bmatrix}
r_{f,t} \\
r_{m,t} - r_{f,t} \\
x_t \\
\sigma_t^2 \\
g_t \\
g_{d,t}
\end{bmatrix}
+ \nu_{t+1}, \quad \nu_{t+1} \sim N(0, \tilde{\Sigma}_{LRR,t}),
\]

with the full specifications of $\Phi_{LRR,0}$, $\Phi_{LRR,1}$, and $\tilde{\Sigma}_{LRR,t}$ appearing in Appendix A.

The long-run risk model captures interesting dynamics in asset returns. As in the data, the risk-free rate is generally pro-cyclical since expected consumption growth is an important determinant of the risk-free rate. As $x_t$ increases, investors must receive extra compensation in order to delay consumption. Economic uncertainty also impacts the risk-free rate, as higher $\sigma_t^2$ causes investors to increase precautionary savings. Risk premiums in the long-run risk economy are counter-cyclical as a result of time-varying economic uncertainty. When $\sigma_t^2$ is relatively high, investors require larger risk premiums for taking on exposure to the stock market because the risk levels of both consumption and stock investments are high. In sum, the two most important state variables in the long-run risk economy are $x_t$ and $\sigma_t^2$. Indeed, $x_t$ governs time variation in the risk-free rate, while $\sigma_t^2$ causes fluctuations in the risk-free rate as well as the risk premium on stocks. These state variables, however, are unobservable and thus must either be estimated or otherwise substituted out of the VAR.

To circumvent the unobservable state variable issue, we draw upon the analysis of Constantinides and Ghosh (2011). They point out that since the log price-dividend ratio and log risk-free rate are
linear in \( x_t \) and \( \sigma_t^2 \) in a long-run risk economy,

\[
p_t - d_t - E(p - d) = A_{1,m}x_t + A_{2,m}(\sigma_t^2 - \sigma^2),
\]

\[
r_{f,t+1} - E(r_f) = A_{1,f}x_t + A_{2,f}(\sigma_t^2 - \sigma^2),
\]

then these relations can be inverted to express the unobserved state variables in terms of observed variables,

\[
x_t = \alpha_1(p_t - d_t - E(p - d)) + \alpha_2(r_{f,t+1} - E(r_f)),
\]

\[
\sigma_t^2 - \sigma^2 = \beta_1(p_t - d_t - E(p - d)) + \beta_2(r_{f,t+1} - E(r_f)).
\]

We exploit these relations in conjunction with the long-run risk VAR in equation (13) to develop an alternative long-run risk VAR which uses \( p - d \) and \( r_f \) as observable state variables which forecast returns,

\[
\begin{bmatrix}
  r_{f,t+1} \\
  r_{m,t+1} - r_{f,t+1} \\
  p_{t+1} - d_{t+1} - E(p - d) \\
  r_{f,t+2} - E(r_f) \\
  g_{t+1} \\
  g_{d,t+1}
\end{bmatrix} = \Phi_{LRR,0} + \Phi_{LRR,1} \begin{bmatrix}
  r_{f,0} \\
  r_{m,0} - r_{f,0} \\
  p_0 - d_0 - E(p - d) \\
  r_{f,1} - E(r_f) \\
  g_0 \\
  g_{d,0}
\end{bmatrix} + \nu_{t+1}, \quad \nu_{t+1} \sim N(0, \Sigma_{LRR,t}),
\]

(18)
where

\[
\Phi_{LRR,0} = \begin{bmatrix}
A_0 + A_1 f \\
\beta_1 \left( \beta_{m,n} \lambda_0 + \beta_{m,e} \lambda_e - \frac{1}{2} \left( \beta_{m,n}^2 + \beta_{m,e}^2 + \beta_{m,u}^2 \right) \right) \\
0 \\
\mu \\
\sigma^2
\end{bmatrix},
\]

\[
\Phi_{LRR,1} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
\beta_1 \left( \beta_{m,n} \lambda_0 + \beta_{m,e} \lambda_e - \frac{1}{2} \left( \beta_{m,n}^2 + \beta_{m,e}^2 + \beta_{m,u}^2 \right) \right) & 1 & 0 & 0 & 0 \\
0 & \mu_1 A_{1,1,m} + \mu_2 A_{1,2,m} & \rho_{A_{1,1,1}} + \rho_{A_{1,2,1}} & 0 & 0 \\
0 & \mu_1 A_{1,1,f} + \mu_2 A_{1,2,f} & \rho_{A_{1,1,f}} + \rho_{A_{1,2,f}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
\Sigma_{LRR,1} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
\beta_{m,n} \sigma_1^2 & A_{1,1,m} \sigma_{1,1}^2 + A_{1,2,m} \sigma_{1,2}^2 + A_{1,1,f} \sigma_{1,1}^2 + A_{1,2,f} \sigma_{1,2}^2 & A_{1,1,m} \sigma_{1,1}^2 + A_{1,2,m} \sigma_{1,2}^2 & A_{1,1,m} \sigma_{1,1}^2 + A_{1,2,m} \sigma_{1,2}^2 & \beta_{m,n} \sigma_1^2 \\
0 & \beta_{m,n} \sigma_2^2 & A_{1,1,m} \sigma_{2,1}^2 + A_{1,2,m} \sigma_{2,2}^2 + A_{1,1,f} \sigma_{2,1}^2 + A_{1,2,f} \sigma_{2,2}^2 & A_{1,1,m} \sigma_{2,1}^2 + A_{1,2,m} \sigma_{2,2}^2 & \beta_{m,n} \sigma_2^2 \\
0 & \beta_{m,n} \sigma_3^2 & A_{1,1,m} \sigma_{3,1}^2 + A_{1,2,m} \sigma_{3,2}^2 + A_{1,1,f} \sigma_{3,1}^2 + A_{1,2,f} \sigma_{3,2}^2 & A_{1,1,m} \sigma_{3,1}^2 + A_{1,2,m} \sigma_{3,2}^2 & \beta_{m,n} \sigma_3^2 \\
0 & \beta_{m,n} \sigma_4^2 & A_{1,1,m} \sigma_{4,1}^2 + A_{1,2,m} \sigma_{4,2}^2 + A_{1,1,f} \sigma_{4,1}^2 + A_{1,2,f} \sigma_{4,2}^2 & A_{1,1,m} \sigma_{4,1}^2 + A_{1,2,m} \sigma_{4,2}^2 & \beta_{m,n} \sigma_4^2 \\
(\phi_d \beta_{m,u} + \phi_d \beta_{m,n}) \sigma_1^2 & 0 & 0 & 0 & \phi_d \sigma_1^2 \\
(\phi_d \beta_{m,u} + \phi_d \beta_{m,n}) \sigma_2^2 & 0 & 0 & 0 & \phi_d \sigma_2^2 \\
(\phi_d \beta_{m,u} + \phi_d \beta_{m,n}) \sigma_3^2 & 0 & 0 & 0 & \phi_d \sigma_3^2 \\
(\phi_d \beta_{m,u} + \phi_d \beta_{m,n}) \sigma_4^2 & 0 & 0 & 0 & \phi_d \sigma_4^2
\end{bmatrix},
\]

and \( \sigma_i^2 \equiv \left( \sigma^2 + \frac{A_{1,m}(r_{f,t+1} - E(r_f)) - A_{1,f}(p_t - d_t - E(p - d))}{A_{1,m}A_{2,f} - A_{2,m}A_{1,f}} \right)^2 \). For portfolio choice purposes, we use the VAR in equation (18) with the following difference. Current consumption and dividend growth do not affect future expected returns. Further, the relevant current information from these variables is also reflected by current return realizations. Thus, when determining optimal portfolio weights we simply exclude consumption and dividend growth from the VAR without losing any relevant information.

### 3 Data and Estimation

We use US data over the post-war period of 1947 to 2009 to estimate the long-run risk VAR. Consumption data is from the Bureau of Economic Analysis, dividend growth and the price-dividend ratio are calculated from the CRSP value-weighted portfolio following Cochrane (2008), and the

\[^1\text{We express the VAR in terms of } p_t - d_t - E(p - d) \text{ and } r_{f,t+1} - E(r_f) \text{ for expositional clarity. In practice, } E(p - d) \text{ and } E(r_f) \text{ are unobserved functions of the underlying parameters. The VAR used for estimation purposes incorporates this fact and can be viewed in the appendix.}\]
risk-free rate is the 90-day Treasury bill rate from the St. Louis Federal Reserve. All growth rates and returns are in logs and have been adjusted for inflation. Consumption and dividend growth rates are per capita figures.

We estimate the long-run risk VAR in equation (18) using maximum likelihood. As shown by equations (19) to (21), the VAR parameters are functions of the underlying investor preference parameters as well as the parameters which govern economic dynamics in a long-run risk economy. Thus, we estimate a total of 13 underlying parameters which in turn imply values of the parameters in the long-run risk VAR. While the VAR contains six variables, there are four unique shocks in the economic system so the covariance matrix in equation (21) is singular. We therefore estimate the portion of the VAR containing \( p_{t+1} - d_{t+1}, r_{f,t+2}, g_{t+1}, \) and \( g_{d,t+1} \). The parameter estimates reflect information about the economic dynamics of consumption and dividend growth as well as their impact on the price-dividend ratio and risk-free rate. The VAR parameters regulating asset returns are functions of these parameter estimates. Thus, the long-run risk VAR that we utilize is estimated without using market return data. Rather, the dynamics of market returns are inferred from the interrelations of consumption and dividend growth with the price-dividend ratio and risk-free rate, as implied by the long-run risk model. Further estimation details are available in Appendix B.

Table I shows the maximum likelihood estimates of the long-run risk parameters. For comparison purposes, the calibrated parameters from Bansal and Yaron (2004) (henceforth BY) and Bansal, Kiku, and Yaron (2007a, 2012) (henceforth BKY) are also displayed. Investor preferences are marked by high risk aversion and high elasticity of intertemporal substitution. The estimate of \( \gamma \) is nearly 31, compared to calibrated values of 7.5 or 10 typically used in the long-run risk literature. The \( \psi \) estimate, which must be greater than one in a long-run risk model (when \( \gamma > 1 \)), is 4.5 which indicates investors are willing to accept relatively large fluctuations in consumption over time.

The economic parameters display the important characteristics of a long-run risk economy. The parameter estimates indicate that the state variables \( x_t \) and \( \sigma_t^2 \) are highly persistent. The estimate
of $\rho$ is only slightly less than one, suggesting that shocks to $x_t$ persist for long periods. Similarly, the $\nu_1$ estimate of 0.991 indicates that $\sigma^2_t$ is more persistent than was assumed by BY, but less so than with the BKY parameters. Meanwhile, the estimated $\sigma_w$ implies more variation in economic uncertainty than in BY and BKY, so risk premiums and volatility may display more time variation as well. Finally, the high volatility of the dividend growth time series is reflected by large estimates of $\phi$, $\varphi_d$, and $\pi$ relative to the calibrated values.

4 Vector Autoregression Estimates

In this section, we examine the estimated reduced-form and long-run risk VARs. The long-run risk VAR implied by the parameter estimates is shown in Panel A of Table II. The VAR parameters reflect highly persistent price-dividend ratio and risk-free rate time series. Meanwhile, the equity premium displays some forecastability by the price-dividend ratio and risk-free rate. In the forecasting equation, the price-dividend ratio comes in with a positive sign while the sign on the risk-free rate is negative.

The positive effect of the price-dividend ratio on expected returns may be initially surprising. This result is explained by the close relation between the price-dividend ratio and the risk-free rate. Both variables are determined by the long-run risk state variables $x_t$ and $\sigma^2_t$. The estimated functions are given by,

\begin{align*}
 p_t - d_t &= 4.36 + 209.03x_t - 26.37\sigma^2_t, \tag{22} \\
 r_{f,t+1} &= -0.00 + 0.22x_t - 18.63\sigma^2_t, \tag{23}
\end{align*}

so the price-dividend ratio and risk-free rate tend to move together. Both variables tend to be higher in periods of high expected economic growth and lower when economic uncertainty is high, consistent with intuition. Time variation in expected returns is driven by variation in $\sigma^2_t$, so both the price-dividend ratio and risk-free rate tend to be high when expected returns are low. When the
two variables are used jointly to forecast returns, however, the marginal effect of the price-dividend ratio becomes positive.

Panel B of Table II shows reduced-form VAR estimates. In contrast to the long-run risk VAR, the VAR parameters are not restricted by economic theory. The reduced-form parameters also suggest that the price-dividend ratio and risk-free rate are persistent, although the persistence is markedly lower than in the long-run risk VAR. Additionally, the signs of the predictive coefficients on the price-dividend ratio and risk-free rate flip in the reduced-form VAR.

Perhaps the largest discrepancy between the long-run risk and reduced-form VARs occurs in the covariance matrices. The reduced-form VAR suggests that current market return shocks are highly correlated with changes in the price-dividend ratio and risk-free rate. Meanwhile, the long-run risk VAR displays weaker relations between the shocks. This difference affects optimal portfolio choice through the intertemporal hedging channel. We therefore further explore this difference between the two models.

Intertemporal hedging will tend to affect portfolio choice if two conditions hold. First, expected returns must be time varying so the investor is exposed to intertemporal risk. Second, current returns must be correlated with changes in expected future returns so the investor can use positions in the available assets to hedge against intertemporal risk. Both the reduced-form and long-run risk VARs imply that returns are predictable, so intertemporal risk arises in both frameworks. The primary difference between the two investors comes from their views on the extent to which stocks hedge against intertemporal risk.

The reduced-form investor believes that stocks provide a very good intertemporal hedge. In his view, poor current returns are typically offset by higher expected returns moving forward. Thus, the risk of stocks declines relatively quickly as the investment horizon expands. In contrast, the long-run risk investor views stocks as a relatively poor hedge against intertemporal risk. In the long-run risk framework, poor current performance often reflects a lowering of expectations of future economic growth. This downward revision in expectations is not accompanied by an increase in expected returns, so the poor returns are not offset by better future performance. As a result, the
risk of stocks does not decline as quickly as the investment horizon increases in a long-run risk economy.

The discussion above is related to the literature originating with Campbell (1991) which attributes return variance to discount-rate and cash-flow risk. When much of the variation in current returns is driven by discount-rate risk, low current returns are followed by high expected returns. In contrast, low current returns which are attributable to cash-flow shocks are not offset by better future performance. Table III confirms that discount-rate risk is relatively more important in the reduced-form VAR than in the long-run risk VAR. The reduced-form VAR attributes 106% of return variance to discount-rate risk, while the long-run risk VAR interprets 107% of variance as cash-flow risk. Campbell (1991) and Campbell and Vuolteenaho (2004) find that the large majority of stock market return variance is attributable to changing discount rates, while Chen and Zhao (2009) demonstrate that these inferences are quite sensitive to small, reasonable revisions to the model for stock returns. Depending on the specification, Chen and Zhao (2009) find that discount-rate risk accounts for anywhere from less than 10% to more than 80% of return variance. Further, discount-rate or cash-flow shocks may dominate during different periods in the market. For example, Campbell, Giglio, and Polk (2011) attribute the market crash in the early 2000s to changing discount rates, while the financial crisis of 2007 to 2009 was caused by a decline in expected cash flows. In sum, from an empirical perspective there exists substantial uncertainty about the overall attribution of variance to discount-rate and cash-flow risk, as well as which risk drives individual events in the stock market. In the next section we therefore appeal to the implications of economic theory for return dynamics while studying the long-horizon portfolio choice problem.

5 Optimal Allocations in a Long-Run Risk Economy

This section studies asset allocation under the long-run risk and reduced-form VARs. Section 5.1 investigates optimal portfolio choice and its sensitivity to economic dynamics and investor preferences. Section 5.2 compares the investment performance of the long-run risk and reduced-
form investors.

5.1 Optimal Portfolio Choice

Here we analyze a simple asset allocation problem in which the investment universe consists of a risk-free asset and a stock market portfolio. Investors choose a portfolio conditional on either the long-run risk VAR or the reduced-form VAR. Therefore, the long-horizon investor faces a time-varying investment opportunity set and must choose an optimal portfolio while accounting for intertemporal hedging concerns.

We initially consider a range of preference parameters. The coefficient of relative risk aversion, \( \gamma \), is the most crucial parameter for asset allocation. Hence, we display results as a function of \( \gamma \). On the other hand, portfolio weights are less sensitive to the elasticity of intertemporal substitution, \( \psi \), and the time-preference parameter, \( \delta \). In fact, Campbell, Chan, and Viceira (2003) show that portfolio choice is virtually independent of \( \psi \) with the exception that \( \psi \) affects the log-linearization constant due to its influence on the mean consumption-wealth ratio. In practice, however, this effect is economically insignificant. Similarly, \( \delta \) does not strongly affect optimal portfolio weights. Therefore, we concentrate on the effect of \( \gamma \) and use \( \psi = 1.5 \) and \( \delta = 0.9989 \) to match the preferences often assumed in the long-run risk literature (e.g., Bansal, Kiku, and Yaron (2007a, 2012)).

Figure 1 displays the optimal allocation to stocks for the long-run risk and reduced-form VARs. The allocation is further divided into two components: a mean-variance component and an intertemporal hedging component. The mean-variance component captures the optimal allocation of a myopic investor, while the hedging component adjusts the optimal allocation for the effect of intertemporal risk.

As expected, risk aversion plays a large role in optimal portfolio choice. Panel A of Figure 1 shows that a long-run risk investor with \( \gamma = 1 \) invests 132% of his portfolio in stocks versus only 11% when \( \gamma = 10 \) (risk tolerance of \( 1/\gamma = 1/10 \)). Meanwhile, the reduced-form investor’s allocations in Panel B are less sensitive to \( \gamma \), but a substantial effect still exists. His allocation in stocks is 226% when \( \gamma = 1 \) and 137% when \( \gamma = 10 \).
The optimal allocations under the long-run risk and reduced-form VARs are clearly quite different. The reduced-form investors take more aggressive positions in stocks at almost all levels of risk aversion. Further analysis reveals that the differences are primarily driven by intertemporal hedging concerns. While long-run risk investors optimally invest similar to myopic investors, most reduced-form investors substantially increase their allocations to stocks because of intertemporal risk.

An investor’s response to intertemporal risk depends on his risk aversion. Campbell (1996) points out that an investor faces a tradeoff with respect to time-varying investment opportunities. On one hand, the investor would like to be wealthy when the productivity of wealth is high during periods of high expected returns. On the other hand, the investor attempts to hedge against diminishing expected future returns. The optimal response depends on $\gamma$. When $\gamma = 1$, the two effects balance and the investor acts myopically. Investors with $\gamma < 1$ prefer to buy assets which pay off when expected returns increase, taking on a substantial amount of risk in the process. For $\gamma > 1$, investors wish to hedge against adverse changes in investment opportunities. These investors would like to buy assets which pay off when expected returns decrease, which leads to a smoother consumption stream.

As previously discussed, the reduced-form VAR attributes much of the return variance to discount-rate risk. These investors therefore believe that risk in current returns is largely offset by changes in expected future returns. Thus, investors with $\gamma > 1$ ($\gamma < 1$) optimally increase (decrease) their allocation to stocks. This effect is illustrated by the hedging component in Panel B. In contrast, long-run risk investors attribute most volatility in returns to cash-flow risk which is not offset by future returns. Therefore, optimal allocations in Panel A are largely unaffected by the hedging component.

The optimal allocations in Figure 1 assume the price-dividend ratio and risk-free rate are at their long-run averages. The optimal portfolio weight in stocks will, however, vary with the state of the economy. In Figure 2, we demonstrate how optimal allocations vary as the price-dividend ratio and risk-free rate change.
Figure 2 shows that the long-run risk and reduced-form investors also respond quite differently to time variation in economic conditions. Long-run risk investors modestly increase their allocation to stocks when the price-dividend ratio increases and invest less in stocks when the risk-free rate is high. The magnitude of the risk-free rate effect is about four times larger than that of the price-dividend ratio. For instance, an investor with \( \gamma = 1 \) increases the weight on stocks by about 24% when the risk-free rate decreases by one standard deviation, versus a decrease of only 6% when the price-dividend ratio is low.

Reduced-form investors take the opposite actions when the state variables change, and are also more responsive to changing conditions. When the risk-free rate is low, an investor with \( \gamma = 1 \) decreases his allocation to stocks by 23% with a one-standard-deviation drop. Meanwhile, he reacts strongly to changes in the price-dividend ratio, increasing the portfolio weight by 205%.

In sum, the optimal allocations in Figures 1 and 2 show that the strategies of long-run risk and reduced-form investors differ dramatically. Their perceptions of and reactions to intertemporal risk produce quite different allocations to stocks for a given price-dividend ratio and risk-free rate. The reduced-form investor takes aggressive positions compared to the long-run risk investor, believing that mean reversion in returns offsets much of the risk of stocks. Further, the two investors display differing responses to time variation in the economic state variables. The reduced-form investor makes large adjustments to his stock allocation when the state variables change, while the long-run risk investor adopts more moderate actions. In the next section, we evaluate the performance of these investors’ portfolios. Interestingly, we find that despite the large differences in their strategies, both investors perform quite well and are able to successfully time the market.

5.2 Portfolio Performance

We now investigate the performance of long-run risk and reduced-form investors who adopt their respective optimal strategies over the period 1948 to 2009. That is, at the end of each year the investors choose the optimal allocation to stocks based on the prevailing price-dividend ratio and risk-free rate. They buy and hold this portfolio for the next calendar year, and we evaluate their
performance based on the time series of portfolio returns. We investigate investors with $\gamma = 10$, but results are similar for other levels of risk aversion.

Figure 3 shows the time series of allocations to stocks, along with the mean-variance and hedging components. As expected from the results in the previous section, the long-run risk investor’s allocations in Panel A are driven primarily by the mean-variance component while the hedging component is more important for the reduced-form investor in Panel B. Additionally, we see relatively muted responses of the long-run risk investor to changes in the price-dividend ratio and risk-free rate when compared to the reduced-form investor. Panel A of Table IV confirms these patterns as the long-run risk investor’s allocation to stocks varies from 3% to 31% while the reduced-form investor takes positions of 23% to 257%.

Despite their differences, we find that both investors do quite well. Panel B of Table IV shows summary statistics for the returns of the investors’ portfolios as well as the market portfolio. As expected, the conservative long-run risk investor realizes low average returns (2.4% per year) and volatility (3.9% per year). The portfolio Sharpe ratio is 0.45 versus only 0.34 for the market portfolio. As the investor can only invest in the stock market or a risk-free asset, this performance indicates positive market timing. Meanwhile, the aggressive reduced-form investor earns an average return of 12.3% with a 25.4% standard deviation and also achieves a Sharpe ratio of about 0.45. Panel C shows that both investors earn significantly positive alphas in an unconditional CAPM regression (0.5% for long-run risk and 3.2% for reduced-form), while Panel D confirms that the positive performance arises due to market timing. We estimate the Henriksson–Merton (1981) model for each portfolio, which measures the difference in market exposure during up and down markets. For example, the market timing estimate of 0.17 for the long-run risk investor indicates that his portfolio’s beta was lower by 0.17 when market returns were negative than when returns were positive. Both investors display significant timing ability, which accounts for their high Sharpe ratios and CAPM alphas.

We further investigate the portfolio returns in Figure 4. This figure displays the time series of returns for each strategy. The portfolio returns are scaled by the standard deviation of returns to
adjust for the difference in scale between the strategies.

One striking feature in Figure 4 is the lack of extreme negative events for the long-run risk investor. His worst performance occurs in 1969 when he realizes a return of −5.2%. During the largest negative market shocks in 1973-74 (41.5% drop in the stock market) and 2008 (39.9% drop), the long-run risk investor is well insulated and only loses 5.8% and 4.0%. In these large negative events, the long-run risk investor successfully reduces his allocation to stocks in anticipation of the drop in stock prices.

The success of the reduced-form investor appears to be primarily attributable to the aggressive timing of up markets in the 1950s as well as the mid-1970s to early-1990s. This investor is also relatively unharmed by the aftereffects of the “internet bubble,” although he also misses out on much of the gain throughout the 1990s. While the long-run risk investor largely avoids the bad periods of 1973-74 and 2008, the reduced-form investor loses 72.4% of his wealth in the former period and 45.7% in the latter.

To summarize, the long-run risk and reduced-form investors both display positive market timing despite their stark differences. Each investor achieves a Sharpe ratio of about 0.45 versus the market Sharpe ratio of 0.34. They each achieve this success even as their portfolio allocations are quite different. Based on the return time series, there is some evidence that the positive performance of the long-run risk investor lies with his ability to avoid exposure to large negative events. Meanwhile, the reduced-form investor is better able to capitalize on periods of high average returns.

6 Conclusion

In this paper, we extend the long-horizon asset allocation literature to consider the effects of long-run risk. The long-run risk framework has implications for the determination of risk premiums, interrelations among assets, and relevance of state variables for asset allocation. In particular, the long-run risk framework of Bansal and Yaron (2004) imposes a tight structure on the VAR of asset returns and state variables. This structural VAR can be used within the asset allocation framework
of Campbell and Viceira (1999) in place of the typical reduced-form VAR.

Investors in a long-run risk economy are faced with a complicated investment problem as the
time-varying risk-free rate and risk premiums lead to intertemporal hedging concerns among in-
vestors. In comparison with the reduced-form results of Campbell and Viceira (1999), however,
temporal hedging concerns are moderate. Long-run risk investors optimally take more conserva-
tive positions. The behavior of long-run risk and reduced-form investors is quite different, but
both investors experience success in investing. They both display positive market timing ability,
and the results suggest the investors capitalize on different aspects of the information about returns
when forming portfolios.

The long-run risk framework of Bansal and Yaron (2004) offers explanations of several aggregate
and cross-sectional asset pricing puzzles. Here, we begin to investigate the effects of long-run risk
on optimal portfolio choice, but there are several remaining issues for future research. Our results
suggest that the long-run risk investor can time the market despite disagreeing with the reduced-
form investor about return dynamics. The implications of long-run risk for return predictability
should be further explored. In addition, it would be interesting to study how Bayesian investors
with various beliefs about the validity of the long-run risk model optimally respond to the empirical
evidence about predictability while forming portfolios.
A Appendix

The forms of the VAR parameters from the long-run risk VAR in equation (13) are given by

\[ \Phi_{LRR,0} = \begin{bmatrix} -\log \delta + \frac{1}{2} \mu + \left( \frac{1}{2} \mu \right) \left( \beta_{a,\lambda} \lambda_w - \frac{1}{2} \beta_{a,\lambda}^2 \right) - \frac{1}{2 \sigma_w^2} \right) & 0 \\ \left( \beta_{m,\lambda} \lambda_w - \frac{1}{2} \beta_{m,\lambda}^2 \right) & 0 \\ \mu & (1 - \nu_1) \sigma^2 \\ \mu_d & 0 \\ \end{bmatrix}, \quad (A1) \]

\[ \Phi_{LRR,1} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \rho \left( \beta_{a,\lambda} \lambda_e + \beta_{a,\varepsilon} \lambda_e - \frac{1}{2} \left( \beta_{a,\lambda}^2 + \beta_{a,\varepsilon}^2 \right) \right) - \frac{1}{2 \sigma_e^2} \left( \lambda_e^2 + \lambda_v^2 \right) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \nu_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \phi & 0 \\ \end{bmatrix}, \quad (A2) \]

\[ \Phi_{LRR,2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\beta_{m,\lambda} \lambda_e + \beta_{m,\varepsilon} \lambda_e - \frac{1}{2} \left( \beta_{m,\lambda}^2 + \beta_{m,\varepsilon}^2 + \beta_{m,\lambda}^2 + \beta_{m,\varepsilon}^2 \right) \right) & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_{m,\lambda} \beta_{m,\varepsilon} \sigma_1^2 & \beta_{m,\lambda} \beta_{m,\varepsilon} \sigma_1^2 & \beta_{m,\lambda} \beta_{m,\varepsilon} \sigma_1^2 & \beta_{m,\lambda} \beta_{m,\varepsilon} \sigma_1^2 & \beta_{m,\lambda} \beta_{m,\varepsilon} \sigma_1^2 & \beta_{m,\lambda} \beta_{m,\varepsilon} \sigma_1^2 & \beta_{m,\lambda} \beta_{m,\varepsilon} \sigma_1^2 & \beta_{m,\lambda} \beta_{m,\varepsilon} \sigma_1^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\phi_d \beta_{m,\lambda} + \pi_d \beta_{m,\eta}) \sigma_1^2 & 0 & 0 & \sigma_d \sigma_1^2 & 0 & 0 & \sigma_d \sigma_1^2 & 0 & 0 \\ \end{bmatrix}, \quad (A3) \]

The prices of risk and risk exposures in the long-run risk VARs are

\[ \lambda_e = (1 - \theta) \kappa_1 A_1 \varphi_e, \quad (A4) \]
\[ \lambda_\eta = \gamma, \quad (A5) \]
\[ \lambda_w = (1 - \theta) \kappa_1 A_2, \quad (A6) \]
\[ \beta_{a,e} = \kappa_1 A_1 \varphi_e, \quad (A7) \]
\[ \beta_{a,\eta} = 1, \quad (A8) \]
\[ \beta_{a,w} = \kappa_1 A_2, \quad (A9) \]
\[ \beta_{m,e} = \kappa_1 A_1 \varphi_e, \quad (A10) \]
\[ \beta_{m,u} = \varphi_d, \quad (A11) \]
\[ \beta_{m,\eta} = \pi_d, \quad (A12) \]
\[ \beta_{m,w} = \kappa_1 A_2 m, \quad (A13) \]

20
where $A_1$ and $A_2$ are constants in the log wealth-consumption ratio,

$$w_t - c_t = A_0 + A_1 x_t + A_2 \sigma_t^2,$$  \hspace{1cm} (A14)

and $A_{1,m}$ and $A_{2,m}$ are the analogous constants for the log price-dividend ratio,

$$p_t - d_t = A_{0,m} + A_{1,m} x_t + A_{2,m} \sigma_t^2,$$  \hspace{1cm} (A15)

the forms of which can be found in Bansal and Yaron (2004). The $\kappa_1$ and $\kappa_{1,m}$ parameters are constants from the loglinearization that depend on the average wealth-consumption ratio and price-dividend ratio, respectively. Since $\kappa_1$ and $\kappa_{1,m}$ determine $A_1$, $A_2$, $A_{1,m}$, and $A_{2,m}$, which in turn determine the average wealth-consumption ratio and price-dividend ratio, we follow Beeler and Campbell (2012) by iterating to find the fixed points of $w - c$ and $p - d$ that allow all relations to hold.

The long-run risk VAR defined in equation (18) is defined in terms of $p_t - d_t - E(p - d)$ and $r_{f,t+1} - E(r_f)$ to simplify the equations. In practice, $E(p - d)$ and $E(r_f)$ are functions of the underlying parameters, so we perform the estimation and asset allocation on a VAR in the state variables $p_t - d_t$ and $r_{f,t+1}$. First, define

$$p_t - d_t = A_{0,m} + A_{1,m} x_t + A_{2,m} \sigma_t^2,$$  \hspace{1cm} (A16)

$$r_{f,t+1} = A_{0,f} + A_{1,f} x_t + A_{2,f} \sigma_t^2,$$  \hspace{1cm} (A17)

and note that (as shown by Constantinides and Ghosh (2011)) these equations can be inverted to get

$$x_t = \alpha_0 + \alpha_1 (p_t - d_t) + \alpha_2 r_{f,t+1},$$  \hspace{1cm} (A18)

$$\sigma_t^2 = \beta_0 + \beta_1 (p_t - d_t) + \beta_2 r_{f,t+1},$$  \hspace{1cm} (A19)
where

\[ \alpha_0 = \frac{A_{2,m}A_{0,f} - A_{0,m}A_{2,f}}{A_{1,m}A_{2,f} - A_{2,m}A_{1,f}}, \]  
(A20)

\[ \alpha_1 = \frac{A_{2,f}}{A_{1,m}A_{2,f} - A_{2,m}A_{1,f}}, \]  
(A21)

\[ \alpha_2 = \frac{-A_{2,m}}{A_{1,m}A_{2,f} - A_{2,m}A_{1,f}}, \]  
(A22)

\[ \beta_0 = \frac{A_{0,m}A_{1,f} - A_{1,m}A_{0,f}}{A_{1,m}A_{2,f} - A_{2,m}A_{1,f}}, \]  
(A23)

\[ \beta_1 = \frac{-A_{1,f}}{A_{1,m}A_{2,f} - A_{2,m}A_{1,f}}, \]  
(A24)

\[ \beta_2 = \frac{A_{1,m}}{A_{1,m}A_{2,f} - A_{2,m}A_{1,f}}. \]  
(A25)

Then the long-run risk VAR is defined by

\[
\begin{bmatrix}
    r_{f,t+1} \\
    r_{m,t+1} - r_{f,t+1} \\
    p_{t+1} - d_{t+1} \\
    r_{f,t+2} \\
    g_{t+1} \\
    g_{d,t+1}
\end{bmatrix}
= \Phi_0 + \Phi_1
\begin{bmatrix}
    r_{f,t} \\
    r_{m,t} - r_{f,t} \\
    p_{t} - d_{t} \\
    r_{f,t+1} \\
    g_{t} \\
    g_{d,t}
\end{bmatrix}
+ \nu_{t+1}, \quad \nu_{t+1} \sim N(0, \Sigma_t),
\]  
(A26)
where

\[
\Phi_0 = \begin{bmatrix}
    \mu & 0 & 0 & 0 & 0 \\
    \phi_0 & \mu_2 & 0 & 0 & 0 \\
    \phi_0 & \phi_0 & \mu_3 & 0 & 0 \\
    \phi_0 & \phi_0 & \phi_0 & \mu_4 & 0 \\
    \phi_0 & \phi_0 & \phi_0 & \phi_0 & \phi_0 \\
\end{bmatrix},
\]

\[\Phi_1 = \begin{bmatrix}
    1 & 0 & 0 & 0 & 0 \\
    0 & \beta_1 & 0 & 0 & 0 \\
    0 & \rho_1 & \beta_2 & 0 & 0 \\
    0 & \rho_1 & \beta_2 & \rho_2 & 0 \\
    0 & \rho_1 & \beta_2 & \rho_2 & \rho_3 \\
\end{bmatrix} \quad \text{and} \quad \sigma_t^2 = \beta_0 + \beta_1 (p_t - d_t) + \beta_2 r_{f,t+1}.
\]

\[\Sigma_t = \begin{bmatrix}
    (\beta_{m,\eta}^2 + \beta_{m,\nu}^2) \sigma_t^2 + \beta_{m,\nu}^2 \sigma_t^2 & A_{1,m} \varphi \beta_{m,\eta} \sigma_t^2 + A_{2,m} \beta_{m,\nu} \sigma_t^2 & A_{1,f} \varphi \beta_{m,\eta} \nu_t \sigma_t^2 + A_{2,f} \beta_{m,\nu} \nu_t \sigma_t^2 & \beta_{m,\eta} \nu_t \alpha_t \sigma_t^2 & \beta_{m,\eta} \varphi^2 \sigma_t^2 + \beta_{m,\nu} \varphi^2 \sigma_t^2 \\
    A_{1,m} \varphi \beta_{m,\eta} \sigma_t^2 + A_{2,m} \beta_{m,\nu} \sigma_t^2 & A_{1,m} \varphi \beta_{m,\nu} \sigma_t^2 + A_{2,m} \beta_{m,\nu} \sigma_t^2 & A_{1,m} \beta_{m,\eta} \varphi \sigma_t^2 + A_{2,m} \beta_{m,\nu} \varphi \sigma_t^2 & \beta_{m,\eta} \varphi \nu_t \sigma_t^2 + \beta_{m,\nu} \nu_t \sigma_t^2 & \beta_{m,\eta} \beta_{m,\nu} \varphi \nu_t \sigma_t^2 + \beta_{m,\nu} \beta_{m,\nu} \nu_t \sigma_t^2 \\
    A_{1,f} \varphi \beta_{m,\eta} \sigma_t^2 + A_{2,f} \beta_{m,\nu} \sigma_t^2 & A_{1,f} \varphi \beta_{m,\nu} \sigma_t^2 + A_{2,f} \beta_{m,\nu} \sigma_t^2 & A_{1,f} \beta_{m,\eta} \varphi \sigma_t^2 + A_{2,f} \beta_{m,\nu} \varphi \sigma_t^2 & \beta_{m,\eta} \beta_{m,\nu} \varphi \nu_t \sigma_t^2 + \beta_{m,\nu} \beta_{m,\nu} \nu_t \sigma_t^2 & \beta_{m,\eta} \beta_{m,\nu} \beta_{m,\nu} \varphi \nu_t \sigma_t^2 + \beta_{m,\nu} \beta_{m,\nu} \beta_{m,\nu} \nu_t \sigma_t^2 \\
    \beta_{m,\eta} \sigma_t^2 & \beta_{m,\eta} \sigma_t^2 & \beta_{m,\eta} \sigma_t^2 & \beta_{m,\eta} \sigma_t^2 & \beta_{m,\eta} \sigma_t^2 \\
    \varphi \beta_{m,\eta} \sigma_t^2 + \beta_{m,\nu} \nu_t \sigma_t^2 & \varphi \beta_{m,\eta} \sigma_t^2 + \beta_{m,\nu} \nu_t \sigma_t^2 & \varphi \beta_{m,\eta} \sigma_t^2 + \beta_{m,\nu} \nu_t \sigma_t^2 & \varphi \beta_{m,\eta} \sigma_t^2 + \beta_{m,\nu} \nu_t \sigma_t^2 & \varphi \beta_{m,\eta} \sigma_t^2 + \beta_{m,\nu} \nu_t \sigma_t^2 \\
\end{bmatrix}.
\]

**B Maximum Likelihood Estimation**

We use maximum likelihood estimation to obtain the parameter estimates for the VAR

\[
\begin{bmatrix}
    p_{t+1} - d_{t+1} \\
    r_{f,t+2} \\
    g_{t+1} \\
    g_{d,t+1} \\
\end{bmatrix} = \Phi_0 + \Phi_1 \begin{bmatrix}
    p_t - d_t \\
    r_{f,t+1} \\
    g_t \\
    g_{d,t} \\
\end{bmatrix} + \nu_{t+1}, \quad \nu_{t+1} \sim N(0, \Sigma_t),
\]

where \( \Phi_0 \) and \( \Phi_1 \) are matrices defined above.
where

$$
\Phi_0 = \begin{bmatrix}
A_{0,m} + A_{1,m}\rho\alpha_0 + A_{2,m}(1 - \nu_1)\sigma^2 + \nu_1\beta_0 \\
A_{0,f} + A_{1,f}\rho\alpha_0 + A_{2,f}(1 - \nu_1)\sigma^2 + \nu_1\beta_0 \\
\mu + \alpha_0 \\
\mu_d + \phi\alpha_0
\end{bmatrix},
$$

(B31)

$$
\Phi_1 = \begin{bmatrix}
\rho\alpha_1 A_{1,m} + \nu_1\beta_1 A_{2,m} & \rho\alpha_2 A_{1,m} + \nu_1\beta_2 A_{2,m} & 0 & 0 \\
\rho\alpha_1 A_{1,f} + \nu_1\beta_1 A_{2,f} & \rho\alpha_2 A_{1,f} + \nu_1\beta_2 A_{2,f} & 0 & 0 \\
\alpha_1 & \alpha_2 & 0 & 0 \\
\phi\alpha_1 & \phi\alpha_2 & 0 & 0
\end{bmatrix},
$$

(B32)

$$
\Sigma_t = \begin{bmatrix}
A^2_{1,m}\varphi^2_\epsilon\sigma^2_t + A^2_{2,m}\sigma^2_w & A_{1,m}A_{1,f}\varphi^2_\epsilon\sigma^2_t + A_{2,m}A_{2,f}\sigma^2_w & 0 & 0 \\
A_{1,m}A_{1,f}\varphi^2_\epsilon\sigma^2_t + A_{2,m}A_{2,f}\sigma^2_w & A^2_{1,f}\varphi^2_\epsilon\sigma^2_t + A^2_{2,f}\sigma^2_w & 0 & 0 \\
0 & 0 & \sigma^2_t & \pi_d\sigma^2_t \\
0 & 0 & \pi_d\sigma^2_t & (\varphi_d^2 + \pi_d^2)\sigma^2_t
\end{bmatrix},
$$

(B33)

and $\sigma^2_t = \beta_0 + \beta_1(p_t - d_t) + \beta_2r_{f,t+1}$. One issue with the estimation is that the linear function for $\sigma^2_t$ does not restrict the variance to be positive. We therefore replace negative $\sigma^2_t$ estimates with a small positive number (0.00001). The likelihood function is highly nonlinear with several local maxima. To address this issue, we use a combination of simulated annealing and local maximization techniques to find local maxima. We then restart the optimization many times from different points in the parameter space to ensure that the maximum likelihood estimates achieve the global maximum of the likelihood function.
References


Table I: Long-Run Risk Parameter Estimates.

This table shows the estimated parameter values for the long-run risk model as well as the calibrated values from Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2007a, 2012). The parameters are estimated over the period 1947 to 2009 using maximum likelihood estimation on the long-run risk VAR in equation (18). Panel A shows the preference parameters for the representative investor. Panel B contains the parameters of the processes for consumption growth, dividend growth, the persistent component of growth, and the level of economic uncertainty. These processes of the long-run risk economy appear in Section 2.1.

<table>
<thead>
<tr>
<th>Panel A: Representative Investor Preference Parameters</th>
</tr>
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<tbody>
<tr>
<td>γ</td>
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<tr>
<td>----</td>
</tr>
<tr>
<td>Maximum Likelihood</td>
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<tr>
<td>BY Calibration</td>
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<tr>
<td>BKY Calibration</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Economic Parameters</th>
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</thead>
<tbody>
<tr>
<td>µ</td>
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<tr>
<td>----</td>
</tr>
<tr>
<td>Maximum Likelihood</td>
</tr>
<tr>
<td>BY Calibration</td>
</tr>
<tr>
<td>BKY Calibration</td>
</tr>
</tbody>
</table>
Table II: Vector Autoregression Estimates.

This table shows the parameter estimates for the long-run risk and reduced-form VARs. The long-run risk VAR parameters are those implied by the underlying parameter estimates in Table I, while the reduced-form parameters are unrestricted. The VAR state variables are the log price-dividend ratio \((p - d)\) and log risk-free rate \((r_f)\). The sample period is 1947 to 2009.

| Panel A: Long-Run Risk VAR |  |  |
|-----------------------------|-------------------|-------------------|-------------------|
| Intercept  |  |  |
|  \(r_{m,t} - r_{f,t}\) | 0.038  | 0.012  | -0.958  |
|  \(p_t - d_t\) | 0.000  | 1.000  | -0.012  |
|  \(r_{f,t+1}\) | -0.004  | 0.001  | 0.898  |

| Covariance Matrix \((\times 100^2)\) |  |  |
|-----------------------------|-------------------|-------------------|-------------------|
|  \(r_{m,t} - r_{f,t}\)  | 948.739  | 144.266  | -2.873  |
|  \(p_t - d_t\)  | 144.266  | 517.633  | 8.888  |
|  \(r_{f,t+1}\)  | -2.873  | 8.888  | 19.514  |

| Panel B: Reduced-Form VAR |  |  |
|-----------------------------|-------------------|-------------------|-------------------|
| Intercept  |  |  |
|  \(r_{m,t} - r_{f,t}\) | 0.482  | -0.124  | 0.248  |
|  \(p_t - d_t\) | 0.295  | 0.913  | 1.444  |
|  \(r_{f,t+1}\) | 0.013  | -0.003  | 0.562  |

| Covariance Matrix \((\times 100^2)\) |  |  |
|-----------------------------|-------------------|-------------------|-------------------|
|  \(r_{m,t} - r_{f,t}\)  | 258.841  | 233.470  | 2.475  |
|  \(p_t - d_t\)  | 233.470  | 243.597  | 1.838  |
|  \(r_{f,t+1}\)  | 2.475  | 1.838  | 3.729  |
This table shows the attribution of return variance to discount-rate shocks, cash-flow shocks, and covariance between the two shocks. The decomposition is performed on the long-run risk and reduced-form VARs. We follow Campbell (1991) to decompose the return variance.

<table>
<thead>
<tr>
<th>Panel A: Long-Run Risk VAR – MLE</th>
<th>Contribution to Total Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discount-Rate Risk</td>
<td>0.044</td>
</tr>
<tr>
<td>Cash-Flow Risk</td>
<td>1.068</td>
</tr>
<tr>
<td>Covariance</td>
<td>-0.112</td>
</tr>
<tr>
<td>Total Variance</td>
<td>1.000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Reduced-Form VAR, 1947–2009</th>
<th>Contribution to Total Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discount-Rate Risk</td>
<td>1.061</td>
</tr>
<tr>
<td>Cash-Flow Risk</td>
<td>0.217</td>
</tr>
<tr>
<td>Covariance</td>
<td>-0.278</td>
</tr>
<tr>
<td>Total Variance</td>
<td>1.000</td>
</tr>
</tbody>
</table>
### Table IV: Portfolio Weights and Realized Returns

This table shows statistics for the portfolio weights and returns of the long-run risk and reduced-form investment strategies. The investors vary their allocation to stocks over time as the state variables (the price-dividend ratio and risk-free rate) change. The market timing coefficient in the Henriksson–Merton model measures the amount by which beta is lower during periods of negative market returns relative to periods with positive market returns. The investors have coefficients of relative risk aversion of 10, and the investment period is 1948 to 2009.

#### Panel A: Weight in Stocks

<table>
<thead>
<tr>
<th></th>
<th>Long-Run Risk</th>
<th>Reduced-Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.180</td>
<td>1.462</td>
</tr>
<tr>
<td>Std Dev</td>
<td>0.097</td>
<td>0.541</td>
</tr>
<tr>
<td>Min</td>
<td>0.033</td>
<td>0.230</td>
</tr>
<tr>
<td>Max</td>
<td>0.311</td>
<td>2.469</td>
</tr>
</tbody>
</table>

#### Panel B: Portfolio Returns

<table>
<thead>
<tr>
<th></th>
<th>Long-Run Risk</th>
<th>Reduced-Form</th>
<th>Market</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.024</td>
<td>0.123</td>
<td>0.068</td>
</tr>
<tr>
<td>Std Dev</td>
<td>0.039</td>
<td>0.254</td>
<td>0.172</td>
</tr>
<tr>
<td>Min</td>
<td>-0.052</td>
<td>-0.559</td>
<td>-0.554</td>
</tr>
<tr>
<td>Max</td>
<td>0.098</td>
<td>0.743</td>
<td>0.350</td>
</tr>
<tr>
<td>Sharpe Ratio</td>
<td>0.453</td>
<td>0.446</td>
<td>0.342</td>
</tr>
</tbody>
</table>

#### Panel C: CAPM Regressions

<table>
<thead>
<tr>
<th></th>
<th>Long-Run Risk</th>
<th>Reduced-Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>0.005</td>
<td>0.032</td>
</tr>
<tr>
<td>Beta</td>
<td>0.146</td>
<td>1.391</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.657</td>
<td>0.868</td>
</tr>
</tbody>
</table>

#### Panel D: Henriksson–Merton Model

<table>
<thead>
<tr>
<th></th>
<th>Long-Run Risk</th>
<th>Reduced-Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>-0.008</td>
<td>-0.042</td>
</tr>
<tr>
<td>Beta</td>
<td>0.235</td>
<td>1.889</td>
</tr>
<tr>
<td>Market Timing</td>
<td>0.171</td>
<td>0.960</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.723</td>
<td>0.898</td>
</tr>
</tbody>
</table>
Figure 1: Optimal Portfolio Choice

This figure shows the optimal allocation to the stock market portfolio for long-run risk and reduced-form investors. Investors have an elasticity of intertemporal substitution of 1.5, a time-preference parameter of 0.9989, and the coefficient of relative risk aversion, $\gamma$, indicated in the figure. The optimal allocation is also decomposed into a hedging motive component (dashed line) and a mean-variance component (dotted line).
Figure 2: Portfolio Allocation Responses to Changing State Variables

This figure shows the adjustments to the optimal allocations to the stock market portfolio when the state variables (the price-dividend ratio and risk-free rate) deviate from their long-run means. The price-dividend ratio and risk-free rate are increased or decreased by one standard deviation. Allocations are shown for long-run risk and reduced-form investors. The investors have an elasticity of intertemporal substitution of 1.5, a time-preference parameter of 0.9989, and the coefficient of relative risk aversion, $\gamma$, indicated in the figure.
Figure 3: Long-Run Risk and Reduced-Form Allocations to Stocks

This figure shows the time series of optimal allocations to the stock market portfolio for long-run risk and reduced-form investors. The investment period is 1948 to 2009. Investors have an elasticity of intertemporal substitution of 1.5, a time-preference parameter of 0.9989, and the coefficient of relative risk aversion, γ, indicated in the figure. The optimal allocation is also decomposed into a hedging motive component (dashed line) and a mean-variance component (dotted line).
Figure 4: Realized Portfolio Returns for the Long-Run Risk and Reduced-Form Investors

This figure shows the time series of portfolio returns for long-run risk and reduced-form investors as well as the market portfolio return. To plot the returns with similar scale, we normalize the annual returns by the time-series standard deviation of returns. The investment period is 1948 to 2009. Investors have an elasticity of intertemporal substitution of 1.5, a time-preference parameter of 0.9989, and the coefficient of relative risk aversion, $\gamma$, indicated in the figure. The optimal allocation is also decomposed into a hedging motive component (dashed line) and a mean-variance component (dotted line).