On the Power of Discrete and of Lexicographic Helly-Type Theorems

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Abstract

Helly's theorem says that if every $d+1$ elements of a given finite set of convex objects in $\mathbb{R}^d$ have a common point, then there is a point common to all of the objects in the set. We define three new types of Helly theorems: discrete Helly theorems - where the common point should belong to an a-priori given set, lexicographic Helly theorems - where the common point should not be lexicographically greater than a given point, and lexicographic-discrete Helly theorems. We show the relations between these new Helly theorems and their corresponding (standard) Helly theorems. We obtain several new discrete and lexicographic Helly numbers.

Using these new types of Helly theorems we get linear time solutions for various optimization problems. For this, we define a new framework, DLP-type *(Discrete Linear Programming type), and provide new algorithms that solve in randomized linear time fixed-dimensional DLP-type problems. We show that the complexity of the DLP-type class stands somewhere between Linear Programming (LP) and Integer Programming (IP).*

Finally, we use our results in order to solve in randomized linear time problems such as the discrete p-center on the real line, the discrete weighted 1-center problem in $\mathbb{R}^d$ with l_∞ norm, the standard (continuous) problem of finding a line transversal for a totally separable set of planar convex objects, a discrete version of the problem of finding a line transversal for a set of axis-parallel planar rectangles, and the (planar) lexicographic rectilinear p-center problem for $p=1,2,3$. These are the first known linear time algorithms for these problems.

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1. Introduction

1.1. Helly-Type Theorems

The classical theorem of Helly stands at the origin of what is known today as the combinatorial geometry of convex sets. It was discovered in 1913 and may be formulated as

Theorem 1 (Helly's Theorem) *Let H be a family of closed convex sets in $\mathbb{R}^d$, and suppose H is finite or each member of H is compact. If every $d+1$ or fewer members of H have a common point, then there is a point common to all members of H.*

A possible generalization is as follows. Let H be a family of objects, and let $\mathcal{P}$ be a predicate on subsets of H. A *Helly-type theorem* for H is a result of the form:

There is a constant k such that for every finite set G, $G \subseteq H$, $\mathcal{P}(G)$, if and only if, for every $F \subseteq G$ with $|F| \leq k$, $\mathcal{P}(F)$.

The least such constant k is called the *Helly number* of H with respect to the predicate $\mathcal{P}$. If no such constant exists we say that the Helly number of H with respect to $\mathcal{P}$ is *unbounded* or *infinite* ($\aleph_0$). In Helly's Theorem, the Helly number is $d+1$ and $\mathcal{P}$ is the predicate of having a non-empty intersection.

Over the years, a vast body of applications analogues and far reaching generalizations of Helly's Theorem has been assembled in the literature (see for instance the excellent surveys of [5, 7, 10]). We present in this paper three ways to generalize Helly-types theorems. For instance, suppose H is a family of objects in $\mathbb{R}^d$ which has a finite Helly number with respect to the intersection predicate. We ask whether there exists for it a *lexicographic* version, that is

Existence of lexicographic Helly-type theorem: *Does there exist a number $k(d)$ such that for any vector $x \in \mathbb{R}^d$, and for any (sufficiently large) family $G \subseteq H$, if every $k(d)$ members of G have a common point which is lex-*

icographically[1] *at most x, then all the members of G do ?*

We call such $k(d)$ the *lexicographic Helly number* (lex Helly number, in short) of H. We also ask about a *discrete* version

Existence of discrete Helly-type theorem: *Does there exist a number $k(d)$ such that for any set S of points in $\mathbb{R}^d$, and for any (sufficiently large) family $D \subseteq H$, if every $k(d)$ members of D have a common point in S, then all the members of D do ?*

We call such $k(d)$ the *discrete Helly number* of H. We ask also about the existence of a combined *lexicographic-discrete* version, that is, of a finite *lexicographic-discrete Helly number* (lex-discrete Helly number, in short).

We note in passing that the lex Helly number of Helly's theorem applied on finite families of convex sets remains $k(d) = d + 1$ (see for instance [19]) while when it is applied on infinite families of compact convex sets the lex Helly number becomes infinite [13]. We also note that the discrete Helly number of Helly's theorem is infinite (Theorem 5), and consequently its lex-discrete Helly number is infinite. By limiting the sets H and S we can get finite Helly numbers for the discrete and lex-discrete versions of Helly's Theorem (see for instance Theorem 6).

We obtain more than a dozen new such discrete, lexicographic and lex-discrete Helly numbers. We strongly believe that further research will lead to the discovery of more such Helly numbers.

1.2. Algorithmic Aspects of Finite Helly Numbers

We discuss two optimization models and show their relations to Helly numbers.

The LP-type model: Sharir and Welzl [24] defined a model which generalizes Linear Programming (LP) and called it the LP-type model (see definitions in Section 3). Fixed-dimensional LP-type problems can be solved efficiently by *LP-type algorithms* such as the ones of Clarkson [2], Matoušek, Sharir and Welzl [20] or Seidel [23]. This provides a tool for obtaining linear time algorithms to various (continuous) optimization problems, mainly in computational geometry, as shown in [1, 20].

The DLP-type model: In *continuous* optimization models related to LP-type problems the feasible set is defined by a finite set of constraints. In the *discrete* versions, in addition to the above, there is also a pre-specified set of relaxations. A feasible solution is restricted to be in the set of relaxations as well as to satisfy the constraints. Integer Programming (IP) is an example of a discrete optimization problem where the set of relaxations is the integer lattice. Many times discrete optimization problems are proved to be more time consuming to solve than their corresponding continuous versions (e.g., LP vs. IP). We propose the following framework for solving discrete optimization problems: We generalize Integer Programming by introducing the *discrete* LP-type model (*DLP-type*). We provide randomized linear time algorithms to solve fixed-dimensional DLP-type problems satisfying a condition we call the Violation Condition (VC). Comparing the complexity of the DLP-type class versus the one of LP and of IP, we get the results which are summarized in the following table (poly. stands for polynomial time. See Section 6 for more details):

dimension	*LP*	*DLP-type* s.t. VC	*IP*
fixed	linear [21]	linear	poly. [17]
variable	poly. [15]	subexponential	NP-Hard

Helly numbers and the two optimization models: Helly numbers are obviously interesting from a combinatorial point of view, but they have algorithmic aspects as well. In 1994 Amenta [1] showed that every parameterized Helly system satisfying a condition called the Unique Minimum Condition (UMC) results an LP-type problem (see definitions in Subsection 3.2). In many cases, an LP-type problem does not satisfy the UMC, and in order to satisfy it, it is common to assume general position on the input and even change and extend the objective function itself. We suggest a different approach. We consider a lexicographic version of the original LP-type problem, which by its definition always satisfies the UMC. We note that *the* solution of the lexicographic problem is *a* solution for the original problem. We solve the lexicographic problem as follows. We show that *every* lexicographic parameterized Helly system, results an LP-type problem. (Unlike in [1], no additional conditions are needed.) Similarly to [1], this provides a framework for obtaining linear time algorithms (i.e., the LP-type algorithms mentioned above), for the optimization problems related to these Helly numbers. In this way we establish an *equivalence* between the *existence* of finite lexicographic Helly numbers and the *solvability* of their corresponding optimization problems by the linear time LP-type algorithms. Similarly to the above, we show that *every* lex-discrete parameterized Helly system can be formulated as a fixed-dimensional DLP-type problem (for more details see Chapter 5 in [13]).

To conclude, given a continuous optimization problem, we show that if one can find a finite lex-Helly number related to that problem, then the problem can be solved in linear time by LP-type algorithms (such as [2, 20, 23]).

For the discrete case, we define a new discrete optimization model (DLP-type). We present linear time algorithms for fixed-dimensional DLP-type problems in many

1 For every $x = (x_1, ..., x_d), y = (y_1, ..., y_d) \in \mathbb{R}^d$, we say that $x <_L y$ (x is *lexicographically* smaller than y (*lsmaller*, in short)) if either $x_1 < y_1$, or there exists $d \geq k > 1$ such that $x_i = y_i$ for $i = 1, 2, ..., k-1$, and $x_k < y_k$.

cases (i.e., when the VC condition applies). Given a discrete optimization problem, we show that if one can find a finite lex-discrete Helly number related to that problem, then this problem can be formulated as a fixed-dimensional DLP-type problem.

1.3. Applications

We solve in [13] the following 5 problems in randomized linear time. These are the first linear time solutions for these problems. By using lexicographic Helly numbers we solve:

Planar lexicographic rectilinear p-center problem ($p = 1, 2, 3$): Given a finite set H of axis-parallel rectangles, find the lexicographically smallest vector $(\lambda_1, x_1, y_1, x_2, y_2, ..., x_p, y_p) \in \mathbb{R}^+ \times \mathbb{R}^{2p}$ such that for every scaled rectangle $\lambda_1 h,\ h \in H$, there exists i such that $\lambda_1 h$ contains (i.e., is pierced by) point (x_i, y_i).

For $p > 3$ [25] showed a lower bound of $\Omega(n \log n)$. [25, 14] solve the corresponding non-lexicographic problem in linear time.

Lexicographic line transversal of a totally separable set of convex planar objects: Given is a totally separable finite family H of simple convex objects (H is called *totally separable* if there exists a direction such that each line in this direction intersects at most one convex object from H. We call the objects in H *simple* if they have a constant size storage description, and if the intersections and common tangents between any two objects can be found in constant time.) Among all lines that intersect all the objects in H (*line transversals*), find a line $y = ax + b$ with the lexicographically smallest vector (a, b).

Given the order in which any line transversal should meet the objects in H, the non-lexicographic problem is solvable in linear time [8].

We solve the following two problems by using lex-discrete Helly numbers:

Lexicographic discrete line transversal of axis-parallel rectangles: Given are a finite family D of axis-parallel rectangles and a finite set S of line *directions*, find a line transversal for D, $y = ax + b$, with the lexicographically smallest vector (a, b) satisfying $a \in S$.

Discrete weighted 1-center problem in $\mathbb{R}^d$ with l_∞ norm: Given are sets $D = \{d_1, ..., d_n\}$ and $S = \{s_1, ..., s_m\}$ of *points* in $\mathbb{R}^d$, and a set $W = \{w_1, ..., w_n\}$ of *weights* in $\mathbb{R}^+$. Find a point $s \in S$ (*center*) which minimizes the real function $r(D, S) = \min_{s \in S} \max_i w_i |s - d_i|$ (the *optimal radius*).

It is a folklore that the latter problem restricted to the case $S = D$ is solvable in $O(n \log n)$ time.

We solve the following problem by formulating it as a DLP-type problem:

Discrete p-center problem on the real line: Given a finite set D of real numbers (*points*) and a finite set S of real numbers (*center locations*), find a subset $C \subseteq S$ of p points (*centers*), which minimizes the real function $r_p(D, S) = \min_{C \subseteq S \text{ s.t. } |C| \leq p} \max_{h \in D} dist(h, C)$ (the *optimal radius*). (For every finite set of real numbers C, and real h, $dist(h, C) = \min_{c \in C} |h - c|$).

Assuming the order of the points on the line is given, the discrete p-center problem on the real line is solvable in linear time by the fairly involved technique of Frederickson [9].

We solve the following problem in subexponential time by either formulating it as a variable dimensional basis-regular DLP-type problem or a variable dimensional basis-regular LP-type problem (see Chapter 11 in [13])

Simple Stochastic Game (SSG): Given is a directed graph $G = (V, E)$ with the following properties. V is the disjoint union of N, X, A and two special vertices, v_0 the 0-*sink* and v_1 the 1-*sink*. We call the sets $N = \{y_1, y_2, ..., y_m\},\ X = \{x_1, x_2, ..., x_d\}$ and A the sets of *min, max* and *average* vertices, respectively. Let $n = |V| = d + m + |A| + 2$ be the number of vertices in the graph. There is one vertex in V that is called the *start* vertex. Every vertex in V has two outgoing edges, except for the sink vertices, which have no outgoing edges. The game is a contest between two players, 0 and 1. It is played in the following way. Begin by placing a token on the start vertex. When the token is on a min vertex y_j, player 0 moves it along one of the outgoing edges of y_j. When the token is on a max vertex x_i, player 1 moves it along one of the outgoing edges of x_i. When the token is on an average vertex, the edge along which the token is moved is determined by a toss of a fair coin. The game ends when one of the sink vertices is reached. The goal of player 1 is to reach the 1-sink. The goal of player 0 is to avoid the 1-sink. Before the game begins, player 1 chooses an outgoing edge from each max vertex. These edges will define a *strategy* for player 1. During the game, whenever the token is on a max vertex, player 1 will move the token along the edge that is included in his strategy. Similarly, a choice of an outgoing edge for each min vertex is a strategy for player 0. Given an SSG, we would like to determine whether player 1 wins with probability greater than $\frac{1}{2}$ when both players use their best strategies.

Ludwig [18] solves this problem in the same time bound by giving an ad-hoc algorithm with a fairly involved analysis. Our solution derives directly from the LP-type framework.

In this paper we extensively use terms which are defined in [1] and two tools for establishing linear time algorithms: the LP-type framework and Helly-type results which are reviewed in [1, 25].

2. Helly-Type Theorems

2.1. Previous Results

In the (non-lexicographic) *rectilinear p-piercing problem* ($p \in I\!N$) we are given a finite set B of compact boxes in $I\!R^d$ with edges parallel to the coordinate axes and decide whether there exists a matrix $A = (a_1, a_2, ..., a_p)^T$ whose rows $a_1, a_2, ..., a_p$ are points in $I\!R^d$, such that for every box $b \in B$ there exists i such that box b is pierced by (i.e., contains) point a_i. The Helly number related to this problem is the least number $h = h(d, p)$ such that H is p-pierceable if each $G \subseteq H$ with $|G| \leq h$ is p-pierceable. In 1982 Danzer and Grünbaum proved

Theorem 2 ([4])

$$\begin{array}{rll} (i) & h(d,1) = 2 & \textit{for all } d \in I\!N; \\ (ii) & h(1,p) = p+1 & \textit{for all } p \in I\!N; \\ (iii) & h(d,2) = \left\{ \begin{array}{ll} 3d & \textit{for odd } d; \\ 3d-1 & \textit{for even } d; \end{array} \right. & \\ (iv) & h(2,3) = 16; & \\ (v) & h(d,p) = \aleph_0 & \textit{for } d \geq 2,\ p \geq 3 \\ & & \textit{and } (d,p) \neq (2,3). \end{array}$$

We note that case (i) in the theorem above is a special case of Helly's theorem where the Helly number is reduced from $d + 1$ to 2.

One of the most interesting Helly theorems are about line transversals for a set H of convex sets in the plane. Santaló [22] showed in 1940 that the Helly number is unbounded. Hadwiger, Debrunner and Klee [12] showed subsequently that a Helly number does not exist even if the convex sets are pairwise disjoint. It soon became clear that "Helly-type theorems with transversals" cannot be expected unless rather severe restrictions are placed on the shapes and the mutual position of the sets considered. A typical condition, for instance, is that the sets be pairwise disjoint and translates of one another. In 1989 Tverberg [26] proved a conjecture made by Grünbaum [11] some 31 years before

Theorem 3 *Let H be a set of pairwise disjoint translates of a compact set O in the plane. The minimal $k \in I\!N$ such that if any k or fewer translates from H are met by a common line, then some line meets all sets of H is:*

(i)	**([22])**	3	*when O is a segment;*
(ii)	**([3])**	5	*when O is a circle;*
(iii)	**([11])**	5	*when O is a square;*
(iv)	**([11])**	5	*when O is a rectangle;*
(v)	**([26])**	5	*when O is convex.*

As far as we are aware of, the theorem of Doignon [6] is the only example in the literature for a discrete Helly theorem.

Theorem 4 **([6])** *Let H be a finite family of at least 2^d convex sets in $I\!R^d$. If every 2^d or fewer members of H have a common point with integer coordinates, then there is a point with integer coordinates common to all members of H.*

2.2. Discrete Helly-Type Theorems

In this subsection we give a few new discrete Helly-type theorems. We prove these theorems in [13]. We start by introducing a negative result regarding a discrete version of Helly's theorem.

Theorem 5 *For every positive integer k there exists a finite family D of convex sets in $I\!R^d$ ($d \geq 2$) and a finite family S of l-flats ($0 \leq l < d$) in $I\!R^d$, such that every k or fewer members of D meet a common l-flat in S and none of the l-flats in S intersects all the members of D.*

The following theorem is a discrete version of Theorem $2(i)$

Theorem 6 *Let S be a set of points in $I\!R^d$. Let D be a family of boxes in $I\!R^d$ with edges parallel to the axes, so that either all the boxes in D are open, or all the boxes are closed. Suppose D and S are not both infinite sets. If every $2d$ or fewer members of D have a common point in S, there is a point in S common to all members of D.*

We note that even for the planar case, the discrete Helly number, 4, is greater than the corresponding (standard) Helly number, 2 (Theorem $2(i)$). To see that the discrete Helly number is at least 4 let us consider the 4 rectangles ($|D| = 4$) and 4 points ($|S| = 4$) as drawn in Figure 1. Each 3 rectangles intersect in exactly one of the black points but all 4 don't intersect in any of the 4 points. This proves that the discrete Helly number is greater than 3. This example can be extended to any dimension d to get that the discrete Helly number is greater than $2d - 1$.

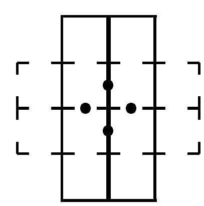

Figure 1. Every 3 rectangles intersect in a black point but all 4 don't.

A discrete version of Theorem $2(ii)$ is

Theorem 7 *Let D be a finite family of intervals on the real line and S be a finite set of points. If every $p + 1$ intervals of D are p-pierceable by points in S then D is p-pierceable by points in S.*

In the rest of this subsection we concentrate on discrete Helly theorems concerning line transversals in the plane. We start with

Theorem 8 *Let D be a totally separable finite family of convex objects, and let S be a finite set of line directions.*

If every subset of at most 4 objects admits a line transversal with direction in S then D does as well.

The corresponding continuous Helly number is 3 [16].

Theorem 9 *Let D be a finite family of parallel rectangles in the plane (i.e., all edges belong to one of two given directions), and let S be a finite set of line directions. If every subset of at most 8 rectangles admits a line transversal with direction in S then D does as well.*

The corresponding continuous Helly number is 6 [22]. We deduce the above theorem by proving

Theorem 10 *Let D be a finite family of rectangles in the plane with edges parallel to the axes and let S be a finite set of non-negative line directions. If every subset of at most 4 rectangles admits a line transversal with a slope from S then D does as well.*

The corresponding continuous Helly number is 3 [12]. Regarding the Helly number about line transversals of *translates* (Theorem 3), the following theorem applies

Theorem 11 *Let D be a finite set of pairwise disjoint translates of a compact set O in the plane, and let S be a finite set of line directions. The minimal $k \in I\!N$ such that if any k or fewer translates from D are met by a common line with direction in S, then some line with direction in S meets all sets of D is:*

(i)	4	*when O is a segment;*
(ii)	7	*when O is a square;*
(iii)	7	*when O is a rectangle.*

We believe that even when O is a general convex object the discrete Helly number remains $k = 7$.

Looking at a variant of the discrete line transversal problem, where S is a finite set of lines (and not of line directions) we prove

Theorem 12 *For every positive integer k there exists a compact convex set O in the plane, a finite set D of at least k pairwise disjoint translates of O and a set S of lines, such that every k translates from D are met by a common line in S and no line in S meets all translates of D.*

Comparing between the two last theorems, which are both discrete versions of Theorem 3, we note that in Theorem 11 there is a bounded Helly number while in Theorem 12 there isn't. A possible intuitive explanation for this is as follows. In the original (continuous) Theorem 3, the line transversal is chosen from the infinite set of all lines. In Theorem 11, even though the set of allowed lines is a proper subset of the original one, it is still infinite. In Theorem 12 the set of allowed lines is finite. In this sense Theorem 11 resembles the continuous Helly theorem, Theorem 3, more than Theorem 12. We can think of Theorem 11 as a "weak" discrete version of Theorem 3 and of Theorem 12 as a "strong" discrete version. We give more details about the relations between continuous and discrete Helly numbers in Subsection 2.4.

2.3. Lexicographic Helly-Type Theorems

In this subsection we give a few new lexicographic (continuous) Helly-type theorems, and lex-discrete Helly-type theorems. The lexicographic version of Theorem 2 is

Theorem 13

$$\begin{array}{rll} (i) & h^*(d,1) = \max(2,d) & \textit{for all } d \in I\!N; \\ (ii) & h^*(1,p) = p+1 & \textit{for all } p \in I\!N; \\ (iii) & h^*(2,2) = 6; & \\ (iv) & 16 \le h^*(2,3) \le 34; & \\ (v) & h^*(d,p) = \aleph_0 & \textit{for } d \ge 2,\ p \ge 3 \\ & & \textit{and } (d,p) \ne (2,3). \end{array}$$

To see that $h^*(3,1) > h(3,1)$ we look at the 3 axis-parallel cubes centered at $(0,0,1)$; $(0,1,0)$ and $(1,0,0)$ with radius 1 (i.e., the edges are of length 2). Choosing $x = (0,0,-0.5)$, we get that any pair of cubes have a common point which is lexicographically smaller than x, but the lexicographically smallest point common to all 3 is $(0,0,0)$.

Other examples for lexicographic Helly theorems are about the existence of line transversals. The lexicographic Helly numbers of the lexicographic continuous versions of theorems 8, 9 and 10 remain the same. For example, the lexicographic continuous version of Theorem 8 is

Theorem 14 *Let H be a totally separable family of convex objects. For every pair of reals a' and b', if every subset of at most 3 objects admit a line transversal $y = ax + b$ with $(a,b) \le_L (a',b')$ then H does as well.*

It is also possible to "lexify" discrete Helly theorems. The Helly numbers of the lexicographic versions of theorems 6, 8, 9, 10 and 11 remain the same.

2.4. The Relations between the Different Types of Helly Theorems

We are interested in a particular subclass of Helly-type theorems, in which the objects in H are subsets of a set X which we call a *ground set*, and $\mathcal{P}(G)$ is the intersection predicate, i.e., $\forall\ G \subseteq H,\ \ \mathcal{P}(G)$ is that the sets in G intersect in a common point. We write $\bigcap G$ for $\{x \in X \mid x \in h,\ \forall h \in G\}$, and we say that the family of sets *intersects* when $\bigcap G \ne \emptyset$, that is, when $\mathcal{P}(G)$ is true.

A *set system* is a pair (X, H), where X is a set and H is a family of subsets of X. We say (X, H) is a *Helly system* if there exists a finite integer k such that H has Helly number k with respect to the intersection predicate $\mathcal{P}$. The natural computational problem associated with a Helly system is, given a subset $G \subseteq H$, return a point $x \in \bigcap G$, or show that $\bigcap G = \emptyset$. Most Helly theorems can be restated in terms of the intersection predicate (see [1] for more detail).

Let Λ be a totally ordered set (e.g., $\Lambda = I\!R$). For every $X \subseteq \Lambda^d$ and $x \in \Lambda^d$ let $X_x = \{x' \in X \mid x' \le_L x\}$ be the set of points in X which are lexicographically at most

x. We note that if X is a convex set then for every $x \in X$, X_x is a convex set as well. Let (X, H) be a set system. If for every $x \in X$, $(X, \{h \cap X_x \mid h \in H\})$ is a Helly system with Helly number l, then we say (X, H) is a *Helly system with lexicographic Helly number* l. Regarding the relations between lexicographic and non-lexicographic Helly theorems, we observe the following monotone structure

Observation 1 *The lexicographic Helly number of each Helly system is not smaller than its corresponding (non-lexicographic) Helly number.*

A *discrete set system* is a triple (X, D, S), where X is a set and D, S are families of subsets of X. For $D' \subseteq D$, we write $\bigcap D'$ for $\{x \in X \mid \forall h \in D',\ x \in h\}$. For $S' \subseteq S$, we write $\bigcup S'$ for $\{x \in X \mid \exists h \in S',\ x \in h\}$. Let $G = (D', S') \in 2^D \times 2^S$. We write $\bigcap G$ for $\bigcap D' \cap (\bigcup S')$. We say that G satisfies the intersection predicate, or simply *intersects*, when $\bigcap G \neq \emptyset$. One can look at a discrete set system (X, D, S) as a set system (X, D) on the grid $\bigcup S'$, or as the set system $(X \cap (\bigcup S'), D \cap (\bigcup S'))$. However, our definition of a discrete set system will enable us to use LP-type algorithms to solve DLP-type problems efficiently (see definitions of these terms below).

A discrete set system (X, D, S) is a *discrete Helly system* if there exists a finite integer k such that the intersection of every k or less d-elements of D have a common element in $\bigcup S$ implies that $\bigcap D \cap (\bigcup S) \neq \emptyset$. Regarding the relations between discrete and continuous (standard) Helly theorems we have

Observation 2 *If* $(\mathbb{R}^d, D, S_1), (\mathbb{R}^d, D, S_2)$ *are discrete Helly systems with Helly numbers* k_D^1, k_D^2*, respectively, and* $S_2 \subseteq S_1$ *then* $k_D^2 \geq \max\{k, k'\}$ *where*

- *k is the largest possible integer such that there exists a k-membered subset $D' \subset D$, $\bigcap D' \cap (\bigcup S_1) = \emptyset$ and $\forall D'' \subset D',\ \bigcap D'' \cap (\bigcup S_2) \neq \emptyset$. If no such k exists, let $k = 1$.*
- *k' is the largest possible integer such that the boundary of $\bigcap D$ consists of at least k' different sets $d_1, ..., d_{k'} \in D$ and there is a k'-membered subset $S' = \{s_1, ..., s_{k'}\} \subseteq S_2$ such that $\forall i\ \bigcap (D \setminus \{d_i\}) \cap s_i \neq \emptyset$ and $\bigcap D \cap (\bigcup S_2) = \emptyset$. If no such k' exists, let $k' = 1$.*

If any of k, k' is infinitely large, then $k_D^2 = \aleph_0$.

3. LP-Type Problems

In this section we review some of the definitions given in Amenta [1] and Matoušek, Sharir and Welzl [20]. The term used in the former paper is GLP (General Linear Programming) rather than LP-type.

3.1. Preliminaries

Definition 1 *An* abstract problem *is a tuple* (H, ω) *where* H *is a finite set of elements (which we call* constraints*) and* ω *is an* objective function *from* 2^H *to some totally ordered set* Λ *which contains a special maximal (minimal) element* ∞ *(*$-\infty$*), respectively. The goal is to compute* $\omega(H)$*.*

Definition 2 *An* LP-type problem *is an abstract problem* (H, ω) *that obeys the following conditions (when we write* $<, \leq, =$ *etc. we mean under the ordered set* Λ*)*

Monotonicity: *For all* $F \subseteq G \subseteq H : \omega(F) \leq \omega(G)$*.*

Locality: *For all* $F \subseteq G \subseteq H$ *with* $\omega(G) = \omega(F) \neq -\infty$ *and for each* $h \in H$*, if* $\omega(G \cup \{h\}) > \omega(G)$ *then* $\omega(F \cup \{h\}) > \omega(F)$*.*

Let $G \subseteq H$ be arbitrary and let $n = |H|$. If $\omega(G) = \infty$, we say G is *infeasible*; otherwise we call G *feasible*. If $\omega(G) = -\infty$, we say G is *unbounded*; otherwise we call G *bounded*. A *basis* B is a set $B \subseteq H$ with $\omega(B') < \omega(B)$ for all proper subsets B' of B. A *basis for* G is a basis $B \subseteq G$ with $\omega(B) = \omega(G)$. The *combinatorial dimension* d of an LP-type problem is the maximum size of any basis for any feasible subfamily G. An LP-type problem of combinatorial dimension d, where d is independent of $|H|$, is called *fixed dimensional*. A d-dimensional LP-type problem where the cardinality of every basis is exactly d is called a d-dimensional *basis-regular* LP-type problem.

An LP-type algorithm takes a d dimensional LP-type problem (H, ω) and returns a basis B for H. The randomized algorithm of [20] uses two primitive operations. A *basis computation* takes a family G of at most $d+1$ constraints and finds a basis for G. A *violation test* takes a basis B and a constraint h, and returns false if and only if B is a basis of $B \cup \{h\}$. Let t_b be the time required for a basis computation and t_v be the time required for a violation test. The algorithm of [20] runs in randomized $O(n(t_v + t_b))$ time. A combined algorithm of [2] and [20] gives a randomized running time of $O(t_v n + t_b \log n)$. In both algorithms the constant is $e^{\sqrt{d \log d}}$, i.e., depends subexponentially on the dimension d which is assumed to be a constant.

3.2. Helly-Type Theorems and their Relations to LP-Type Problems

Let $(X \times \Lambda, \bar{H})$ be a set system where Λ is a totally ordered set which contains a maximal element ∞. For any particular constraint $\bar{h} \in \bar{H}$ and $\lambda \in \Lambda$ we write $h_\lambda = \{x \in X \mid \exists \nu \leq \lambda\ s.t.\ (x, \nu) \in \bar{h}\}$ for the projection into X of the part of $\bar{h}$ with Λ-coordinate no greater than λ. Also for a subfamily of constraints $\bar{G} \subseteq \bar{H}$, we write G_λ as shorthand for $\{h_\lambda \mid \bar{h} \in \bar{G}\}$.

We say that a function $f : A \rightarrow B$ is well defined if it is defined on every $a \in A$ and $f(a) \in B$. We say that

$\omega' : X \times \Lambda \to \Lambda$ is *a ground set objective function* if ω' is a well defined function on points $(x, \lambda) \in X \times \Lambda$. We call a ground set objective function ω' a *natural ground set objective function* if $\forall (x, \lambda) \in X \times \Lambda;\ \omega'(x, \lambda) = \lambda$. The *natural objective function* $\omega : 2^{\bar{H}} \to \Lambda$ *induced by* ω' on subfamilies of constraints returns the least value of Λ at which the subfamily intersects, so that $\omega(\bar{G}) = \inf\{\lambda \in \Lambda \mid \bigcap G_\lambda \neq \emptyset\}$. If $\bigcap \bar{G} = \emptyset$ we define $\omega(\bar{G}) = \infty$. For example, when formulating linear programming in the LP-type framework, the value of ω on subfamily $\bar{G}$ of constraints is the minimum value that a linear ground set objective function ω' achieves on the points that are feasible with respect to $\bar{G}$.

We represent a *mathematical programming* as a triple $(X \times \Lambda, \bar{H}, \omega')$, and we call the pair $(\bar{H}, \omega)$ the *abstract problem*. If $|\{t \in \bigcap \bar{G} \mid \omega'(t) = \omega(\bar{G})\}| = 1$ for all $\bar{G} \subseteq \bar{H}$ then we say that ω' satisfies the *Unique Minimum Condition* on $(X \times \Lambda, \bar{H})$. We call an indexed family of subsets $\{h_\lambda \mid \bar{h} \in \bar{G}\}$, such that $h_\alpha \subseteq h_\beta$, for all $\alpha, \beta \in \Lambda$ with $\alpha < \beta$, a *nested family*.

In her paper, Amenta [1] relates Helly-type theorems and LP-type problems by parameterization

Definition 3 *A set system* $(X \times \Lambda, \bar{H})$ *is a* parameterized Helly system *with Helly number k, when*

1. $\{h_\lambda \mid \lambda \in \Lambda\}$ *is a nested family, for all* $\bar{h} \in \bar{H}$.
2. (X, H_λ) *is a Helly system, with Helly number* k, $\forall \lambda$.

Theorem 15 ([1]) *Let* (H, ω) *be an abstract problem with combinatorial dimension* k *such that* ω *obeys monotonicity and let* $\lambda \in \Lambda$ *be arbitrary.* H *has the property* $\omega(H) \leq \lambda$ *if and only if every* $G \subseteq H$ *with* $|G| \leq k + 1$ *has the property* $\omega(G) \leq \lambda$.

An LP type problem (H, ω) with combinatorial dimension k is an abstract problem with combinatorial dimension k such that ω obeys monotonicity. Therefore the above theorem implies that there is a Helly-type theorem corresponding to the constraint set of every fixed dimensional LP-type problem. The Main Theorem in [1] goes in the other direction

Theorem 16 (Main Theorem in [1]) *Let* $(X \times \Lambda, \bar{H})$ *be a parameterized Helly system with Helly number k, natural ground set function* ω'*, and natural objective function* ω*. If* ω' *meets the Unique Minimum Condition on* $(X \times \Lambda, \bar{H})$ *and* ω *is well defined then* $(\bar{H}, \omega)$ *is an LP-type problem of combinatorial dimension k.*

Amenta showed that without requiring the Unique Minimum Condition the theorem is not correct by giving an example of a Helly system with no fixed combinatorial dimension [1]. The theorem above is applied in [1] to get linear time solution algorithms for various geometric problems.

4. Dual LP-Type Problems

Definition 4 *A* dual LP-type problem *is an abstract problem* (H, ω) *that obeys the following conditions (when we write* $<, \leq, =$ *etc., we mean under the totally ordered set* Λ*)*

Monotonicity: *For all* $F \subseteq G \subseteq H : \omega(F) \geq \omega(G)$ *(so the special element* ∞ *is such that* $\omega(\emptyset) = \infty$*).*

Locality: *For all* $F \subseteq G \subseteq H$ *with* $\omega(G) = \omega(F) \neq \infty$ *and for each* $h \in H$*, if* $\omega(G \cup \{h\}) < \omega(G)$ *then* $\omega(F \cup \{h\}) < \omega(F)$.

We choose the term dual LP-type (which should not be confused with the term dual in linear programming) because of the following

Observation 3 *The abstract problem* (H, ω) *is an LP-type problem if and only if* $(H, -\omega)$ *is a dual LP-type problem.*

We note in passing that there is much more meaning in dual LP-type problems than just a multiplication of the objective function of an LP-type problem by -1. We will see its importance in the next section.

Looking at (H, ω), in order to prevent confusion between LP-type problems and their dual versions we will denote by (D, ω) LP-type problems, and by (S, ω) dual LP-type problems. The motivation for the choice of the letters "D" and "S" is as follows. We use the letter "D" in the LP-type problem (D, ω) since we look at D as a set of *demand elements (d-elements)*, or constraints on the feasible region, on which the minimum value is $\omega(D)$. The more there are demand elements, the minimum solution of the problem increases. We use the letter "S" in the dual LP-type problem (S, ω) since we look at S as a set of *supply elements (s-elements)*, or relaxations on the feasible region, on which the minimum value is $\omega(S)$. The more there are supply elements, the minimum solution of the problem decreases. In the next section we define *discrete LP-type problems* by combining LP-type and dual LP-type problems together.

5. Discrete LP-Type Problems

Definition 5 *A* discrete abstract problem *is a triple* (D, S, ω) *where* D *and* S *are finite sets of elements and* ω *is an* objective function *from* $2^D \times 2^S \setminus \{(\emptyset, \emptyset)\}$ *to some totally ordered set* Λ *which contains a special maximal (minimal) elements* ∞ $(-\infty)$*, respectively. The goal is to compute* $\omega(D, S)$.

Definition 6 *Let* (D, S, ω) *be a discrete abstract problem. For any* $D' \subseteq D$ *let* $\alpha(D') = \omega(D', S)$*. For any* $S' \subseteq S$ *let* $\beta(S') = \omega(D, S')$*. We say that* (D, S, ω) *is a* discrete LP-type problem (DLP-type) *when* (D, α) *is an LP-type problem and* (S, β) *is a dual LP-type problem. We call*

(D,α) $((S,\beta))$ *the* induced *LP-type (dual LP-type) problem of* (D,S,ω)*, respectively.*

The following condition is used in the proof of correctness of our DLP-type algorithms

Definition 7 *We say that the discrete LP-type problem* (D,S,ω) *satisfies the* Violation Condition (VC) *if for every* $(D',S') \in 2^D \times 2^S$ *and* $(D'',S'') \in 2^{D'} \times 2^{S'}$ *with* $\omega(D',S') = \omega(D'',S'')$ *the following properties hold:*

1. *For every* $h \in D$*, if* $\omega(D'' \cup \{h\}, S'') > \omega(D'',S'')$ *then* $\omega(D' \cup \{h\}, S') > \omega(D',S')$
2. *For every* $h \in S$*, if* $\omega(D'', S'' \cup \{h\}) < \omega(D'',S'')$ *then* $\omega(D', S' \cup \{h\}) < \omega(D',S')$*.*

Let $G \in 2^D \times 2^S$. G is *feasible* when $\omega(G) \neq \infty$. G is *bounded* when $\omega(G) \neq -\infty$. We define bases in the following natural way

Definition 8 *Let* (D,S,ω) *be a discrete LP-type problem and let* α *and* β *be as defined in Definition 6. Let* $G = (D',S') \in 2^D \times 2^S$ *be such that* $G \neq (\emptyset,\emptyset)$*.* $B = (B_D,B_S) \in 2^{D'} \times 2^{S'}$ *is a* basis for G in (D,S,ω) *if* $\omega(B_D,B_S) = \omega(D',S')$*,* B_D *is a basis for* D' *in* (D,α)*, and* B_S *is a basis for* S' *in* (S,β)*.*

The *demand combinatorial dimension* k_D (d-dimension, in short) of (D,S,ω) is the maximum cardinality $|B_D|$ of *any* basis $B=(B_D,B_S)$ for any feasible subset G. A discrete LP-type problem of d-dimension k_D, where k_D is independent of $|D|+|S|$, is called *fixed d-dimensional*. We define the terms *supply combinatorial dimension* (s-dimension, in short) and *fixed s-dimensionality* analogously. We call a discrete LP-type problem which is both fixed s-dimensional (of dimension k_S) and fixed d-dimensional (of dimension k_D) (k_D,k_S)-dimensional. A (k_D,k_S)-dimensional DLP-type problem where the cardinality of B_D, B_S of every basis $B=(B_D,B_S)$ is exactly k_D, k_S, respectively, is called a (k_D,k_S)-dimensional *basis-regular* DLP-type problem.

Integer Programming can be formulated as a DLP-type problem where $S = \mathbb{Z}^k$. There is one problem with this formulation: the set S is not finite. We can overcome this by noting that when given an instance of an IP problem, it is always possible to bound the integer lattice by a big box (whose radius depends exponentially on the input size), such that the solution of the IP problem, if exists, is found inside the bounding box. Because of the above, solving IP by the DLP-type model is not efficient.

6. DLP-Type Algorithms

Given an instance (D,S,ω) of a (d_D,d_S)-dimensional DLP-type problem, we observe that when changing (by deleting or adding elements) the set D (S) while keeping the set S (D) unchanged, the problem behaves like an LP-type (dual LP-type) problem, respectively. Thus, while "fixing" the set S (D) one can use LP-type algorithms in order to solve the induced LP-type (dual LP-type) problem on D (S), respectively.

We have developed several randomized algorithms that solve fixed-dimensional DLP-type problems that satisfy the Violation Condition in linear time. The algorithms differ in the choice of the LP-type algorithms used to solve the induced LP-type and dual LP-type problems, and in the decision rules when and with which input to call these algorithms (see Chapter 6 in [13]).

We give below an example of such an algorithm. The algorithm consists of 4 layers. In the first (second) layer, i.e., Function DLP (Function R), the set of s-elements (d-elements) does not change, so it resembles Function x^*_m in Clarkson's algorithm [2], applied on the induced LP-type (dual LP-type) problem, respectively. The purpose of Function DLP (Function R) is to get the number of constraints (relaxations) down, so we can apply the next level, Function I, which resembles Function x^*_i in [2], and is more efficient in d_S but less efficient in $|D|$ and $|S|$. The 4th layer, Function Demand, is called only when the cardinality of the s-element set is bounded by a constant and it behaves similarly to Sharir and Welzl's algorithm [20], applied on the induced LP-type problem.

Function DLP(D,S)

1. Let $V^* := \emptyset$, let $V := D$ and find a candidate basis C_D for D in the induced LP-type problem of (D,S,ω).
2. If $|D| \le 9d_D^2$ then return R(D,S,C_D).
3. Else repeat the following until $V=\emptyset$:
   (a) Choose $R \subset D \setminus V^*$ at random, $|R| = d_D\sqrt{|D|}$.
   (b) Find a candidate basis C_D for $R \cup V^*$ in the induced LP-type problem of $(R \cup V^*, S, \omega)$.
   (c) Let B :=R$(R \cup V^*, S, C_D)$ and let $V := \{d \in D \mid$ Violation(B,d)=TRUE$\}$.
   (d) If $|V| \le 2\sqrt{|D|}$ then let $V^* := V^* \cup V$.
4. Return B.

Function R(D,S,C_D)

1. Let $V^* := \emptyset$, let $V := S$ and let $C =: 9d_S^2$.
2. If $|S| \le C$ then return Demand(D,S,C_D).
3. Else repeat the following until $V=\emptyset$:
   (a) Choose $R \subset S \setminus V^*$ at random, $|R| = d_S\sqrt{|S|}$.
   (b) Let $B :=$ I$(D, R \cup V^*, C_D)$ and let $V := \{s \in S \mid$ Violation(B,s)=TRUE$\}$.
   (c) If $|V| \le 2\sqrt{|S|}$ then let $V^* := V^* \cup V$.
4. Return B.

Function I(D, S, C_D)

1. Let $C := 9d_S^2$ and let $V := S$.
2. For every $s \in S$ let $\nu_s := 1$.
3. If $|S| \leq C$ then return Demand(D, S, C_D).
4. Else repeat the following until $V = \emptyset$:
   (a) Choose $R \subset S$ at random, $|R| = C$.
   (b) Let $B :=$ Demand(D, R, C_D).
   (c) Let $V := \{s \in S \mid$ Violation(B, s)=TRUE$\}$.
   (d) If $\nu(V) \leq 2\nu(S)/(9d_S - 1)$ then for every $s \in V$ let $\nu_s =: 2\nu_s$.
5. Return B.

Function Demand(D, S, C_D)

1. If $D = C_D$ then return Basis(C_D, S).
2. Else
   (a) Choose a random $d \in D \setminus C_D$.
   (b) Let $B = (B_D, B_S) :=$ Demand$(D \setminus \{d\}, S, C_D)$.
   (c) If Violation(B, d)=TRUE then return Demand$(D, S,$ the first coordinate of Basis$(B_D \cup \{d\}, S))$.
   (d) Else return B.

Theorem 17 ([13]) *If a (k_D, k_S)-dimensional discrete LP-type problem (D, S, ω) meets the Violation Condition, then it can be solved in $O((|S| + |D|)t_v + \min\{\log|S|, \log|D|\}t_b)$ randomized time where t_v (t_b) is the time needed for the violation test (basis calculation) primitive, respectively. If (D, S, ω) is basis regular then the constants depend subexponentially on k_D and k_S.*

7. Discrete Helly-Type Theorems and their Relations to Discrete LP-Type Problems

Let $(X \times \Lambda, \bar{D}, \bar{S})$ be a discrete set system where Λ is a totally ordered set which contains a maximal element ∞, that is, the constraints $\bar{d} \in \bar{D}$ and the relaxations $\bar{s} \in \bar{S}$ are subsets of $X \times \Lambda$. For every $\lambda \in \Lambda$ and $\bar{h} \in \bar{D} \cup \bar{S}$, we write $h_\lambda = \{x \in X \mid \exists \nu \leq \lambda \; s.t. \; (x, \nu) \in \bar{h}\}$, for the projection into X of the part of $\bar{h}$ with Λ-coordinate no greater than λ. Also, for $G \in 2^D \times 2^S$, we write G_λ as a shorthand for $\{h_\lambda \mid \bar{h} \in \bar{G}\}$.

We say that $\omega' : X \times \Lambda \rightarrow \Lambda$ is *a ground set objective function* if ω' is a well defined function on points $(x, \lambda) \in X \times \Lambda$. We call a ground set objective function ω' a *natural ground set objective function* if $\forall (x, \lambda) \in X \times \Lambda;\ \omega'(x, \lambda) = \lambda$. The *natural objective function* $\omega : 2^{\bar{D} \times \bar{S}} \rightarrow \Lambda$ *induced by* ω' on subfamilies of constraints returns the least value of Λ at which the subfamily intersects, so that $\omega(\bar{H}) = \inf\{\lambda \in \Lambda \mid \bigcap H_\lambda \neq \emptyset\}$. If $\bigcap \bar{H} = \emptyset$ we define $\omega(\bar{H}) = \infty$.

We represent a *discrete mathematical programming* as a quadruple $(X \times \Lambda, \bar{D}, \bar{S}, \omega')$, and we call the triple $(\bar{D}, \bar{S}, \omega)$ the *abstract discrete problem*. If $|\{t \in \bigcap \bar{G} \mid \omega'(t) = \omega(\bar{G})\}| = 1$ for all $\bar{G} \subseteq \bar{D} \times \bar{S}$ then we say that $(X \times \Lambda, \bar{D}, \bar{S}, \omega')$ satisfies the *Unique Minimum Condition*. An indexed family of subsets $\{h_\lambda \mid \bar{h} \in \bar{H}\}$, such that $h_\alpha \subseteq h_\beta$, for all $\alpha, \beta \in \Lambda$ with $\alpha < \beta$, is a *nested family*.

Definition 9 *A discrete set system $(X \times \Lambda, \bar{D}, \bar{S})$ is a* discrete parameterized Helly system *with Helly number k, when*

1. *$\{h_\lambda \mid \lambda \in \Lambda\}$ is a nested family, for all $\bar{h} \in \bar{D} \cup \bar{S}$, and*
2. *$(X, D_\lambda, S_\lambda)$ is a discrete Helly system, with Helly number k, $\forall \lambda$.*

We now expose the relations between discrete Helly-type theorems and DLP-type problems. We extend the Main Theorem in [1] to the discrete case and get

Theorem 18 ([13]) *Let $(X \times \Lambda, \bar{D}, \bar{S})$ be a discrete parameterized Helly system with Helly number k, natural ground set function ω', and natural objective function ω. If ω' meets the Unique Minimum Condition on $(X \times \Lambda, \bar{D}, \bar{S})$ and ω is well defined, then $(\bar{D}, \bar{S}, \omega)$ is a discrete LP-type problem of combinatorial dimension $(k_D, 1)$.*

8. Solving the Discrete Weighted 1-Center Problem in $I\!R^d$ with l_∞ Norm in Linear Time

In this section we briefly show how to solve the discrete weighted 1-center problem in $I\!R^d$ with l_∞ norm (1-center problem, in short) in linear time, by formulating it as a fixed-dimensional DLP-type problem which satisfies the Violation Condition.

Given an instance D, S, W of the 1-center problem, for every $G = (D', S') \in 2^D \times 2^S$ let $r(D', S')$ be the optimal radius of the 1-center problem on D', S', W, realized by making $s^*(D', S') \in S'$ the center.

Considering the set of boxes $r(D', S')D' = \{r(D', S')d_i \mid d_i \in D'\}$, where $r(D', S')d_i$ is the box with center at d_i and radius $\frac{r(D', S')}{w_i}$, we note that $s^*(D', S')$ intersects all the boxes of $r(D', S')D'$. The lexicographic version of Theorem 6 applied on the set of boxes $r(D', S')D'$ and the set of points S', tells us that $2d$ boxes "define" the optimal solution.

For every $i = 1, ..., d$, let $L_i(D', S') \in D'$ be a d-element $d_{j(i)} \in D'$, such that the projection of box $r(D', S')d_{j(i)}$ on the i'th coordinate results an interval $[l_i, r_i]$ with the smallest r_i. Let $l_i(D', S') \in I\!R$ be the right endpoint of the projection of $r(D',S')d_{j(i)}$ on the i'th coordinate.

For every $i = 1,...,d$, let $G_i(D',S') \in D'$ be a d-element $d_{j(i)} \in D'$, such that the projection of box $r(D',S')d_{j(i)}$ on the i'th coordinate results an interval $[l'_i, r'_i]$ with the greatest l'_i. For every $i = 1,...,d$, let $g_i(D',S') \in I\!\!R$ be the left endpoint of the projection of $r(D',S')d_{j(i)}$ on the i'th coordinate.

We also define $C(D',S') \in S'$ to be the lexicographically smallest optimal center. We let the range of the objective function be $I\!\!R^+ \times I\!\!R^{2(d+1)}$ and define the objective function $\omega(D',S') = \omega(G)$ to be $(r(G), C(G), l_1(G), -g_1(G), ..., l_d(G), -g_d(G))$. Clearly (D,S,ω) is a discrete abstract problem.

We show now that (D,S,ω) is a $(2d,1)$-dimensional DLP-type problem. (D,S,ω) obeys the Monotonicity of Demand Condition since adding a new element h to D' cannot lexicographically decrease the value, i.e. $\omega(D',S') \leq_L \omega(D' \cup \{h\}, S)$. Similarly, (D,S,ω) obeys the Monotonicity of Supply Condition since adding a new point h cannot lexicographically increase the objective function value.

We show in Section 9.3 in [13] that (D,S,ω) obeys both Locality Conditions and that it obeys the Violation Condition. It is easy to verify that $B(D',S') = B(G) = (\{L_1(G), G_1(G), ..., L_d(G), G_d(G)\}, \{C(G)\})$ is a basis of a feasible and bounded (D',S') and that the problem is of d-dimension $2d$ and s-dimension 1. It is easy to implement the primitives "violation test" and "basis calculation" to run in constant time (see Section 9.3 in [13]). We conclude with

Theorem 19 *The discrete weighted* 1*-center problem in* $I\!\!R^d$ *with* l_∞ *norm is solvable in randomized linear time.*

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