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The TV advertisements scheduling problem

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Abstract A TV channel has a single advertisement break of duration h and a convex continuous function $f\colon [0, h] \to \mathbb{R}^+$ representing the TV rating points within the advertisement break. Given n TV advertisements of different durations p_j that sum up to h, and willingness to pay coefficients w_j, the objective is to schedule them on the TV break in order to maximize the total revenue of the TV channel $\sum_j w_j \int_{c_j - p_j}^{c_j} f(t)dt$, where $[c_j - p_j, c_j)$ is the broadcast time interval of TV advertisement j. We show that this problem is NP-hard and propose a fully polynomial time approximation scheme, using a special dominance property of an optimal schedule and the technique of K-approximation sets and functions introduced by Halman et al. (Math Oper Res 34:674–685, 2009).

Keywords Scheduling · TV rating points · Dynamic programming · Fully polynomial time approximation scheme · K-approximation sets and functions

1 Introduction

Scheduling TV advertisements is one of the most important decisions for a TV channel, providing a vital source of revenue for the company (see e.g. [5]). In this paper, we

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study the TV advertisements scheduling problem from the following new centralized perspective. A TV channel has an advertisement break of duration $h \in \mathbb{N}$ seconds and a convex continuous function $f\colon [0, h] \rightarrow \mathbb{R}^+$ that represents the TV rating points within the advertisement break. In order to model the TV audience rating level, we assume that f is decreasing in $[0, v]$ and increasing in $(v, h]$ where $v \in \mathbb{N}$. Without loss of generality, we assume $f(v) = 0$. The assumption on the convexity of the TV rating points function follows the behavior of the TV audience rating, which tends to be higher at the start and end of a TV break than during the middle [6]. The TV channel is also given a set $\mathcal{J}$ of n TV advertisements, where each TV advertisement j has a duration of $p_j \in \mathbb{N}$ seconds and a willingness to pay coefficient $w_j \in \mathbb{Q}^+$, such that broadcasting the advertisement in time interval $[c_j - p_j, c_j)$ will result in a payment of $w_j \int_{c_j-p_j}^{c_j} f(t)dt$ dollars. We assume that $\sum p_j = h$, because TV breaks have no empty broadcast times. The TV channel goal is to define a schedule S for maximizing the total revenue from the advertisements broadcast:

$$F(S) = \sum_{j=1}^{n} w_j \int_{c_j-p_j}^{c_j} f(t)dt,$$

where $[c_j - p_j, c_j)$ is the broadcast time interval of TV advertisement j according to schedule S.

Our contribution We show that this problem is NP-hard and propose a fully polynomial time approximation scheme (FPTAS), that is, an algorithm that for every given parameter $\epsilon > 0$ returns a feasible solution with relative error up to ϵ from an optimal solution, and that runs in time polynomial in the (binary) size of the problem input and in $1/\epsilon$, Our results are based on (1) a special dominance property of an optimal schedule, which allows us to state conditions that must be satisfied for every pair of TV advertisements $i, j \in \mathcal{J}$ in any optimal schedule S^*; and (2) the technique of K-approximation sets and functions introduced by Halman et al. [14] applied on a specific dynamic programming formulation that we give for the problem.

2 Literature review

The literature related to the TV advertisements scheduling problem can be categorized into two settings: TV advertisements scheduling problems and the single machine scheduling problem with a non-monotone penalty function.

TV advertisements scheduling problems In the literature, TV advertisements scheduling problems have been studied by several authors from different perspectives, considering various parameters in its formulation, such as: advertisement/program content, viewers' interests, sponsors' preferences, program timing, program popularity and available advertisement slot [26].

Bollapragada et al. [4] aim to schedule a set of TV advertisements in a set of available TV breaks such that multiple airings of the same TV advertisement are as evenly spaced as possible, employing a branch and bound and a heuristic approach on an integer programming model. Bollapragada and Garbiras [6] address a scheduling problem that satisfies a certain percentage of total requirements on the start and end broadcast positions of TV advertisements in different TV breaks in order to obtain a higher level of TV audience rating, proposing an integer programming model and a heuristic resolution method. Mihiotis and Tsakiris [22] study the frequency and the position of TV advertisements on a set of available TV breaks in order to achieve the highest TV audience rating subject to the advertisers budget constraints, using a binary programming model and a heuristic method for its resolution. Zhang [27] models a selling time problem to the advertisers, proposing a two step hierarchical resolution method: (a) select advertisers and assign them on a specific TV break and, (b) allocate the broadcast time to the selected advertisers on the TV break. Benoist et al. [3] work on French satellite television, where the broadcast time interval of TV advertisements are sold as packages to be incorporated into a set of TV breaks. They define this optimization problem as the *TV break packing problem*, showing unary NP-hardness and some resolution approaches. Mao et al. [19] are focused on the Japanese TV advertising market, proposing an ant colony optimization algorithm to optimize the sum of products of revenue and credit information. Gassemi Tari and Alaei [11] propose a combinatorial auction to select a set of TV advertisements during the peak of viewing time in a TV channel and design a steady state genetic algorithm to find a near optimal solution. García-Villoria and Salhi [9] consider the demands of advertisers, who want multiple airings of the same brand of TV advertisement to be as spaced out as possible over a given time period, proposing two mixed integer linear programming formulations and two constructive heuristics: local search procedures and simulated annealing.

The single machine scheduling problem with a non-monotone penalty function In a typical single machine scheduling problem, we have to order n jobs, each with given positive processing time p_j and priority weight w_j. A schedule is defined by a ranking σ, and the completion time of job j is defined as $c_j := \sum_i p_i$, where the sum ranges over all jobs i such that $\sigma_i \leq \sigma_j$. Given a penalty function g: $\mathbb{R}^{\geq 0} \mapsto \mathbb{R}^{\geq 0}$, the goal is to produce a schedule that minimizes $\sum_j w_j g(c_j)$, denoting the problem by $1||\sum_j w_j g(c_j)$.

Most of the penalty functions considered in the literature are monotone increasing. Bansal and Pruhs [2] address the more general problem $1||\sum g_j(c_j)$, where every job j is given an increasing penalty function $g_j(\cdot)$, that does not need to be of the form $w_j g(\cdot)$. They design a 16-approximation algorithm based on a geometric interpretation of the problem. The approximation factor has been improved from 16 to 4 by Mestre and Verschae [21] based on an analysis of the primal-dual approach proposed in [7]. Höhn et al. [18] give a quasi-polynomial time algorithm scheme (QPTAS) that yields an $(e + \varepsilon)$ approximation based on a certain connection of the scheduling problem $1||g_j(\cdot)$ and the *unsplittable flow problem* (UFP) on a path.

The simpler problem $1||\sum w_j g(c_j)$ was considered in [8], who provide a $(4+\varepsilon)$-approximation scheme which depends polynomially on the input size and $1/\varepsilon$, $\varepsilon > 0$ for the setting where g is an arbitrary increasing differentiable penalty function chosen by an adversary *after* the schedule has been produced. A polynomial time approximation scheme (PTAS), which runs in time $2^{O(\log(1/\varepsilon)/\varepsilon^2)} \log(\sum_j w_j)n$ has been provided by Megow and Verschae [20] for the problem $1||\sum w_j g(c_j)$, where g is an arbitrary monotone penalty function (see Vásquez [24] for complexity results of other specific penalty functions).

Höhn and Jacobs [17] derive a method to compute the tight approximation factor of the Smith-ratio-schedule for any particular monotone increasing convex or concave cost function. The method is based on an alternative interpretation of the problem, assuming a non uniform processor speed at any time t given by a nonnegative monotone function s: $\mathbb{R}^{\geq 0} \mapsto \mathbb{R}^{\geq 0}$, and the processing times (or workloads) of the jobs are given with respect to a unit speed processor. The total workload processed until time t is $G(t) := \int_0^t s(x)dx$. Conversely, if the total workload of job j and all jobs processed before it is p, then the completion time of j in the schedule is $s^{-1}(p)$ and then, the problem is equivalent to $1||\sum_j w_j G^{-1}(c_j)$, where G^{-1} is increasing concave or increasing convex defined by an increasing or decreasing speed function s, respectively.

Our problem can be viewed as the maximization variant of a new scheduling problem with a non-monotone penalty function g and a non-monotone speed function s for an alternative interpretation of the problem (G^{-1} is non-monotone).

3 A certain dominance property

Recently, dominance properties have been shown to improve the performance of exhaustive search procedures by early pruning ineffective partial solutions in problems with monotone [1,25] and non-monotone penalty function [23].

In order to reduce the search space of our problem, we study a certain dominance property that is satisfied in any optimal schedule S^* for every pair of TV advertisements $i, j \in \mathcal{J}$, when i and j have *both* a broadcast time interval before or after v.

Note that any schedule S defines three sets of TV advertisements: $\mathcal{J}_1 = \{j \in \mathcal{J} | c_j < v\}$, $\mathcal{J}_2 = \{j \in \mathcal{J} | v \leq c_j \wedge c_j - p_j \leq v\}$ and $\mathcal{J}_3 = \{j \in \mathcal{J} | v < c_j - p_j\}$, where $|\mathcal{J}_2| = 1$.

Lemma 1 *In an optimal schedule S^*, the TV advertisements in $\mathcal{J}_1$ are scheduled in a non-increasing willingness to pay order and the TV advertisements in $\mathcal{J}_3$ are scheduled in a non-decreasing willingness to pay order.*

Proof The proof of Lemma 1 is based on a simple exchange argument. We show the case for TV advertisements in $\mathcal{J}_1$ where the objective function f is decreasing, the other case is symmetric. Let $\mathcal{I}$ be an instance containing TV advertisements i, j in $\mathcal{J}_1$ and, S_1 and S_2 schedules for $\mathcal{I}$ of the form $S_1 = AijB$ and $S_2 = AjiB$, for some sets of TV advertisements A and B. Let a be the total broadcast time interval of A and $F(S)$ be the total revenue from the advertisements broadcast of schedule S. We have the following equivalences:

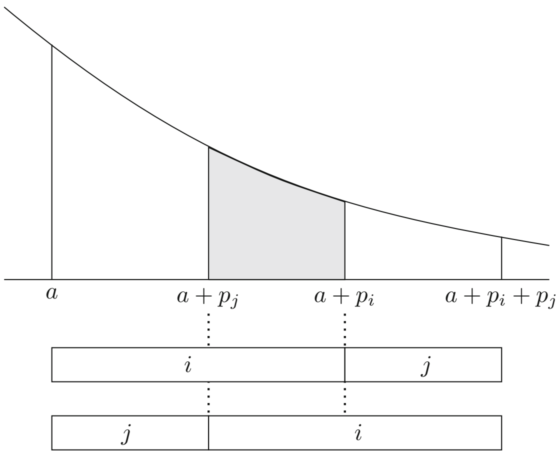


Fig. 1 Illustration of Eq. (1) for $p_i > p_j$

$$
\begin{aligned}
F(S_1) - F(S_2) &= w_i \int_a^{a+p_i} f(t)dt + w_j \int_{a+p_i}^{a+p_i+p_j} f(t)dt \\
&\quad - \left(w_j \int_a^{a+p_j} f(t)dt + w_i \int_{a+p_j}^{a+p_j+p_i} f(t)dt \right) \\
&= (w_i - w_j) \left(\int_a^{a+\min\{p_i,p_j\}} f(t)dt - \int_{a+\max\{p_i,p_j\}}^{a+p_i+p_j} f(t)dt \right) \qquad (1)
\end{aligned}
$$

Figure 1 illustrates Eq. (1) for $p_i > p_j$. The shaded area is the contribution of TV advertisement i on the total revenue from both schedules S_1 and S_2.

From the mean value theorem, we have:

$$
\int_a^{a+\min\{p_i,p_j\}} f(t)dt = \min\{p_i, p_j\} f(\epsilon), \epsilon \in [a, a + \min\{p_i, p_j\}] \qquad (2)
$$

and

$$
\begin{aligned}
\int_{a+\max\{p_i,p_j\}}^{a+p_i+p_j} f(t)dt &= (p_i + p_j - \max\{p_i, p_j\}) f(\epsilon'), \epsilon' \in [a + \max\{p_i, p_j\}, \\
&\qquad a + p_i + p_j] \\
&= \min\{p_i, p_j\} f(\epsilon'). \qquad (3)
\end{aligned}
$$

Thus, we replace the expression (2) and the expression (3) in expression (1) and we have

$$
(w_i - w_j) \min\{p_i, p_j\}(f(\epsilon) - f(\epsilon'))
$$

We have $f(\epsilon) - f(\epsilon') > 0$ follows since f is strictly decreasing in $[0, c]$, $\min\{p_i, p_j\} > 0$ and then, Eq. (1) ≥ 0 is equivalent to $w_i - w_j \geq 0$. To end, we note that Eq. (1) is independent on a, implying that the TV advertisement order resulting from the exchange argument directly defines the optimal schedule S^* among TV advertisements in $\mathcal{J}_1$. □

Definition 1 Given a TV advertisements scheduling S, an advertisement j such that all advertisements preceding it are scheduled in a non-increasing willingness to pay order, and all advertisements succeeding it are scheduled in a non-decreasing willingness to pay order is called a *pivot* advertisement. Such a schedule S is called a *pivotal schedule around* j.

We get from Lemma 1 the following corollary:

Corollary 1 *Any optimal schedule is a pivotal schedule.*

Corollary 1 tells us that we can restrict the search space to pivotal solutions, i.e., an optimal solution must be an optimal pivotal solution around some advertisement j.

4 Computational complexity

Theorem 1 *The TV advertisements scheduling problem is NP-hard.*

Proof The TV advertisements scheduling problem is clearly in NP, as the conditions on the completion times for a feasible solution and the scheduling value from it can be checked in polynomial time.

To show that the TV advertisements scheduling problem is NP-hard, we consider the 2-PARTITION problem [10]:

Instance: n numbers $p_1, \ldots, p_n \in \mathbb{N}$ such that $\sum_j p_j = 2A$.

Question: Is there a subset S such that $\sum_{j \in S} p_j = A$?, and we reduce an instance of 2-PARTITION to the decision version of the TV advertisements scheduling problem, asking for a solution with an objective value equal to a given threshold k.

We construct an instance $\mathcal{I}$ of TV advertisements scheduling problem as follows: a set $\mathcal{J}$ of n TV advertisements with $w_j = w \quad \forall j = 1, \ldots, n$ and $p_j \in \mathbb{N} \quad \forall j = 1, \ldots, n$ such that $\sum_j p_j = 2A$; a TV advertisement $n+1$ with $0 \leq w_{n+1} < w$ and $p_{n+1} = A$; a break TV of duration $h = 3A$ and a TV rating points function $f(t)$ strictly convex with $f(A) = f(2A)$ and $f(v) = 0, \quad v \in [A, 2A]$. The threshold k is $w \int_0^{3A} f(t)dt + (w_{n+1} - w) \int_A^{2A} f(t)dt$.

We claim that this instance $\mathcal{I}$ has a solution S with at least k if and only if there exists a solution to the 2-PARTITION instance.

For the easy direction, given a solution to the 2-PARTITION instance we construct a schedule consisting of TV advertisement of one partition before time A, then TV advertisement $n+1$ and the remaining TV advertisements from time $2A$ to $3A$. Straightforward verification shows that the resulting schedule has the required value k.

For the hard direction, given the instance $\mathcal{I}$ we consider a solution to the TV advertisements scheduling problem of total revenue k. Its revenue cannot be smaller.

By Lemma 1, any optimal solution S^* has the TV advertisement $n+1$ scheduled in the last position among TV advertisements in $\mathcal{J}_1$, in the first position among TV

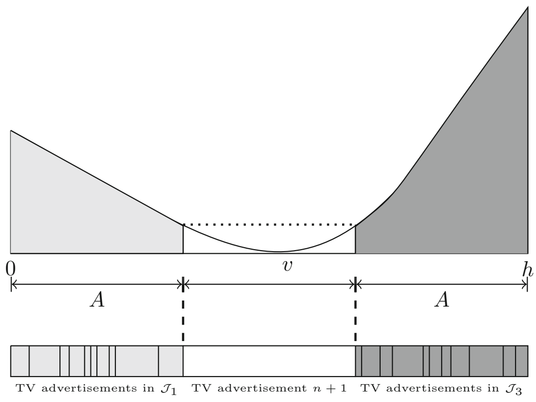


Fig. 2 Feasible solution S of 2-PARTITION instance

advertisements in $\mathcal{J}_3$ or it is the single TV advertisement in $\mathcal{J}_2$. Thus, the total revenue of schedule where TV advertisement $n+1$ starting at x, with $0 \le v - A \le x \le v$ is defined as follows:

$$\begin{aligned} & w \int_0^x f(t)dt + w_{n+1} \int_x^{x+A} f(t)dt + w \int_{x+A}^h f(t)dt \\ & = w \int_0^h f(t)dt - w \int_x^{x+A} f(t)dt + w_{n+1} \int_x^{A+x} f(t)dt \\ & = w \int_0^{3A} f(t)dt + (w_{n+1} - w) \int_x^{x+A} f(t)dt \end{aligned} \tag{4}$$

Now, we compute the first derivative in x and have

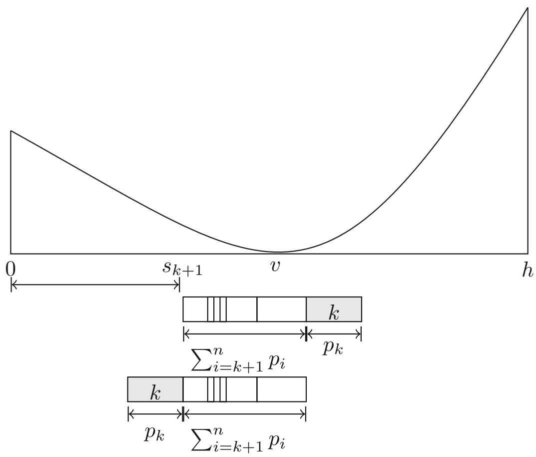
$$(w_{n+1} - w)(f(A+x) - f(x)), \tag{5}$$

with $(w_{n+1} - w) < 0$ by case assumption, $f(A+x) - f(x)$ strictly increasing in $x \in [v-A, v]$ by f strictly convex, $f(v) - f(v-A) < 0$ and $f(A+v) - f(v) > 0$. Therefore, the expression in (4) is strictly concave in x and then, the expression (5) equal to 0 is a sufficient condition to define an unique optimal value x^*. Thus, it suffices to find $f(A+x^*) = f(x^*)$. We have $f(A) = f(2A)$ by case assumption and then $x^* = A$. Therefore, the optimal solution S^* corresponds to the solution S of 2-PARTITION instance, such as shown in Fig. 2, concluding the proof. □

5 A dynamic programming formulation

In this section, we develop a dynamic programming (DP) formulation to our problem. For each advertisement j we will build a feasible solution in which j serves as a pivot advertisement. Let the pivot advertisement be fixed. We re-index the remaining advertisements from 1 to $n-1$ in non-increasing willingness to pay order and re-index the pivot advertisement to n. For every $0 \le k \le n$ and $0 \le \ell \le h - \sum_{i=k}^{n} p_i$ we define a partial solution $F_k(\ell)$, where the assigned TV advertisements are $k, \dots, n$ and $F_k(\ell)$

Fig. 3 Illustration of the possible broadcast time intervals for TV advertisement k



is the best value of the objective function for broadcasting these advertisements in the time interval $\left[\ell, \ell + \sum_{i=k}^{n} p_i\right]$, such that advertisement n is a pivot advertisement. The DP formulation reads as follows.

$$
\begin{aligned}
&F_n(\ell) = w_n \int_{\ell}^{\ell+p_n} f(t)dt, && \ell = 0, \ldots, h - p_n,\\
&F_k(\ell) = \max \begin{cases} w_k \int_{\ell}^{\ell+p_k} f(t)dt + F_{k+1}(\ell + p_k), \\ \\ w_k \int_{\ell+\sum_{i=k+1}^{n} p_i}^{\ell+\sum_{i=k}^{n} p_i} f(t)dt + F_{k+1}(\ell) \end{cases} && \ell = 0, \ldots, h - \textstyle\sum_{i=k}^{n} p_n,\ k = n-1, \ldots, 1.
\end{aligned}
\tag{6}
$$

The first equation in DP formulation (6) deals with the boundary case of a single advertisement that is scheduled in the time interval $[\ell, \ell + p_n]$. The upper (resp. lower) expression in the second equation deals with the case in which advertisement k is scheduled some time before (resp. after) the pivot advertisement. The value of an optimal solution is $F_1(0)$.

Theorem 2 *For any given pivot advertisement j, DP* (6) *yields an optimal pivotal schedule around j.*

Proof Recall that the remaining TV advertisements are re-indexed in non-increasing willingness to pay order and the pivot advertisement is re-indexed to n. We denote by s_{k+1} the starting broadcast time of the subset of TV advertisements $\{k+1, k+2, \ldots, n\}$, $k = 1, \ldots, n-1$, where advertisement n is a given pivot advertisement. Indeed, in an optimal pivotal schedule around n each TV advertisement $k \in \{1, \ldots, n-1\}$ can only be scheduled on either the broadcast time interval $\left[s_{k+1} - p_k, s_{k+1}\right]$ or $\left[s_{k+1} + \sum_{i=k+1}^{n} p_i, s_{k+1} + \sum_{i=k}^{n} p_i\right]$, as shown in Fig. 3. $\square$

6 An overview of K-approximation sets and functions

In this section we provide an overview of the technique of K-approximation sets and functions. In the next section we use this technique in order to construct an FPTAS for the TV advertisements scheduling problem.

Notation For a function $\varphi\colon \{A,\ldots,B\}\to\mathbb{R}$ that is not identically zero we denote $\varphi^{\min} := \min_{A\le x\le B}\{|\varphi(x)|\colon \varphi(x)\ne 0\}$, and $\varphi^{\max} := \max_{A\le x\le B}\{|\varphi(x)|\}$. We define $\sigma_\varphi(x) := \varphi(x+1)-\varphi(x)$ as the slope of φ at x for any integer $A\le x<B$. We define $\sigma_\varphi^{\max} := \max_{A\le x<B}\{|\sigma_\varphi(x)|\}$ and $\sigma_\varphi^{\min} := \min_{A\le x<B}\{|\sigma_\varphi(x)|\colon |\sigma_\varphi(x)|>0\}$. Let t_φ be the time needed to calculate $\varphi(x)$, for any x.

Halman et al. [14] have introduced the technique of K-approximation sets and functions, and used it to develop an FPTAS for a certain stochastic inventory control problem. Halman et al. [13] have applied this tool to develop a framework for constructing FPTASs for three general classes of DPs: when the single-period cost functions are nondecreasing (resp. nonincreasing) in the state variable, the DP is called *nondecreasing* (resp. *nonincreasing*) and when the single-period cost functions have a certain convex structure and the transition function is affine, the DP is called *convex*. This technique has been applied to yield FPTASs to various optimization problems, see [12,13,16] and the references therein. Halman et al. [15] have accelerated the FPTAS running time for convex DPs.

K-approximation sets and functions As the TV advertisements scheduling problem has a convex structure, to simplify the discussion, we concentrate on Halman et al.'s definitions for K-approximation sets and functions specialized for convex functions, as in [15]. Let $K\ge 1$, and let $\varphi,\tilde\varphi\colon \{A,\ldots,B\}\to\mathbb{R}^+$ be arbitrary functions. We say that $\tilde\varphi$ is a *K-approximation function of φ* if $\varphi(x)\le\tilde\varphi(x)\le K\varphi(x)$ for all $x=A,\ldots,B$.

The following property of K-approximation functions is extracted from Proposition 5.1 of [13], which provides a set of general computational rules of K-approximation functions. Its validity follows directly from the definition of K-approximation functions.

Property 1 (*Calculus of K-approximation functions*) For $i=1,2$ let $K_i\ge 1$, let $\varphi_i,\tilde\varphi_i\colon \{A,\ldots,B\}\to\mathbb{R}^+$ and let $\tilde\varphi_i$ be a K_i-approximation of φ_i. Let $\psi_1\colon \{A',\ldots,B'\}\to\{A,\ldots,B\}$ be an arbitrary function. The following properties hold:

Summation of approximation:	$\tilde\varphi_1+\tilde\varphi_2$ is a $\max\{K_1,K_2\}$-approximation function of $\varphi_1+\varphi_2$.
Composition of approximation:	$\tilde\varphi_1(\psi_1)$ is a K_1-approximation of $\varphi(\psi_1)$.
Maximization of approximation:	$\max\{\tilde\varphi_1,\tilde\varphi_2\}$ is a $\max\{K_1,K_2\}$-approximation of $\max\{\varphi_1,\varphi_2\}$.
Approximation of approximation:	If $\varphi_2=\tilde\varphi_1$ then $\tilde\varphi_2$ is a K_1K_2-approximation function of φ_1.

We next turn to defining K-approximation sets. The idea behind such approximation sets is to keep a small (i.e. polynomially bounded size) set of points in the domain of

Algorithm 1 Function COMPRESS(φ, D, K) returns a convex K-approximation of φ: $D \to \mathbb{R}^+$

1: **Function Compress**(φ, D, K)
2: obtain a K-approximation set W of φ over domain D
3: **return** the convex extension of φ induced by W as an array $\{(x, \tilde{\varphi}) \mid x \in W\}$ sorted in increasing order of x.

a function, ensuring that linear interpolation between the function's values on this set guarantees rigorous error bounds.

Definition 2 [13] Let φ: $\{A, \ldots, B\} \to \mathbb{R}$ be a convex function. $\forall E \subseteq \{A, \ldots, B\}$, the *convex extension of φ induced by E* is the function $\hat{\varphi}$ defined as the lower envelope of the convex hull of $\{(x, \varphi(x))\colon x \in E\}$.

Definition 3 [15, Def. 3.1] Let $K \geq 1$ and let φ: $\{A, \ldots, B\} \to \mathbb{R}^+$ be a convex function. Let $W \subseteq \{A, \ldots, B\}$ and let $\hat{\varphi}$ be the convex extension of φ induced by W. We say that W is a *K-approximation set of φ* if: (i) $A, B \in W$; (ii) For every $x \in \{A, \ldots, B\}$, $\hat{\varphi}(x) \leq K\varphi(x)$.

Definition 3 tell us that the convex extension of φ induced by a K-approximation set of φ is a K-approximation function of φ.

Proposition 1 [15, Thm. 3.2] *Let φ: $\{A, \ldots, B\} \to \mathbb{R}^+$ be a convex function. Then for every $K > 1$, it is possible to compute a K-approximation set of φ of size $O\left(\log_K \min\left\{\frac{\sigma_\varphi^{\max}}{\sigma_\varphi^{\min}}, \frac{\varphi^{\max}}{\varphi^{\min}}\right\}\right)$ in $O\left(t_\varphi \log_K \min\left\{\frac{\sigma_\varphi^{\max}}{\sigma_\varphi^{\min}}, \frac{\varphi^{\max}}{\varphi^{\min}}\right\} \log(B-A)\right)$ time.*

A procedure for the construction of a K-approximation function $\breve{\varphi}$ for φ is stated as Algorithm 1. By applying the calculus of approximation (approximation of approximation) and the discussion above we get the following result (see also [13, Prop. 4.5]).

Proposition 2 *Let $K_1, K_2 \geq 1$ be real numbers and let φ: $\{A, \ldots, B\} \to \mathbb{R}^+$ be a convex function. Let $\bar{\varphi}$ be a convex K_2-approximation function of φ. Then Algorithm 1 (Function* COMPRESS($\bar{\varphi}$, $\{A, \ldots, B\}$, K_1)*) returns in $O\left(t_\varphi \log_K \min\left\{\frac{\sigma_\varphi^{\max}}{\sigma_\varphi^{\min}}, \frac{\varphi^{\max}}{\varphi^{\min}}\right\} \log(B-A)\right)$ time a piecewise linear convex function $\breve{\varphi}$ with $O\left(\log_K \min\left\{\frac{\sigma_\varphi^{\max}}{\sigma_\varphi^{\min}}, \frac{\varphi^{\max}}{\varphi^{\min}}\right\}\right)$ pieces that K_1K_2-approximates φ, and of which the query time is $t_{\breve{\varphi}} = O\left(\log\log_K \min\left\{\frac{\sigma_\varphi^{\max}}{\sigma_\varphi^{\min}}, \frac{\varphi^{\max}}{\varphi^{\min}}\right\}\right)$.*

7 An FPTAS

We are ready to state and analyze our FPTAS for the TV advertisements scheduling problem, see Algorithm 2. The algorithm has two "for" loops. In the outer "for" loop (with index j), the algorithm considers pivotal schedules around advertisement j. In the inner "for" loop it constructs a $\frac{1}{1+\epsilon}$-approximation for an optimal pivotal schedule

Algorithm 2 Function TV(ϵ) returns a $\left(\frac{1}{1+\epsilon}\right)$-approximation for the TV scheduling problem

1: **Function TV**(ϵ)
2: Let $K \leftarrow \sqrt[n]{1+\epsilon}$
3: **for** $j := 1$ to n **do**
4: Re-index the advertisements in $\{1, \ldots, n\} \setminus \{j\}$ from 1 to $n-1$ in non-increasing willingness to pay order and re-index the pivot advertisement to n
5: Let $\breve{F}_n^j(\cdot) \leftarrow \text{Compress}\left(w_t \int_{\cdot}^{\cdot+p_t} f(t)dt, \{0, \ldots, h-p_n\}, K\right)$
6: **for** $t := n-1$ downto 1 **do**
7: $\breve{f}_t(\cdot) \leftarrow \text{Compress}\left(w_t \int_{\cdot}^{\cdot+p_t} f(t)dt, \{0, \ldots, h-p_t\}, K^{n-t}\right)$
8: Define $\bar{F}_t^j(x) := \max\left\{\breve{f}_t(x) + \breve{F}_{t+1}^j(x+p_t), \breve{f}_t\left(x + \sum_{i=t+1}^n p_i\right) + \breve{F}_{t+1}^j(x)\right\}$
9: $\breve{F}_t^j(\cdot) \leftarrow \text{Compress}\left(\bar{F}_t^j(\cdot), \left\{0, \ldots, h - \sum_{i=t}^n p_i\right\}, K\right)$ /* $\bar{F}_t^j(\cdot)$ as defined in line 8 */
10: **end for**
11: **end for**
12: **return** $max_{j=1,\ldots,n} \frac{\breve{F}_1^j(\cdot)}{1+\epsilon}$ and the corresponding schedule by performing backtracking

around advertisement j. In its last step, the algorithm returns the best approximated pivotal schedule among the n pivotal schedules it constructed. As any optimal solution is a pivotal solution around some pivot advertisement, see Corollary 1, we get that the solution returned by the algorithm is a $\frac{1}{1+\epsilon}$-approximation for an optimal solution. We need some more notation first. From hereon after we use the notation $z(\cdot)$, where the "$\cdot$" stands for the argument of function z. E.g., the value of $z(\cdot - w)$ for variable value 2 is $z(2-w)$. Put it differently, the function z is shifted by $-w$. $\breve{z}$ stands for a succinct representation of a convex approximation of z, i.e., a sorted array $\{(x, z(x)) \mid x \in W\}$, where W is an approximation set of z. $\bar{z}$ stands for an approximate oracle (black box) to z. Let $f_t(\cdot) := w_t \int_{\cdot}^{\cdot+p_t} f(t)dt$. Let $U_f := \frac{\max_{t=1,\ldots,n} f_t^{\max}}{\min_{t=1,\ldots,n} f_t^{\min}}$ and $U_\sigma := \frac{\max_{t=1,\ldots,n} \sigma_{f_t}^{\max}}{\min_{t=1,\ldots,n} \sigma_{f_t}^{\min}}$. In Sect. 6, we defined a K-approximation function $\breve{\varphi}$ for φ as a function that is "sandwiched" between φ and K-times φ, for any $K > 1$. This works out for minimization problems in which each feasible solution lies not below the optimal solution. For maximization problems, as is the case with the TV advertisements scheduling problem, we want to approximate from below, that is, to construct a function that is sandwiched between φ and φ/K, for any $K > 1$. Note that if $\breve{\varphi}$ is a K-approximation of φ (in the ordinary sense, as defined in Sect. 6) then $\breve{\varphi}/K$ is a $1/K$-approximation function of φ from below.

Theorem 3 *For any given parameter* $\epsilon > 0$, *Algorithm* 2 *returns in* $O\left(\frac{n^3}{\epsilon} \log \frac{n}{\epsilon} \log (\min\{U_\sigma, nU_f\}) \log h \log\log(\min\{U_\sigma, nU_f\})\right)$ *time a* $\frac{1}{1+\epsilon}$*-approximation of the TV advertisements scheduling problem.*

Proof Due to Corollary 1, an optimal schedule for the TV advertisement problem has advertisement j as a pivot advertisement for a certain j. We will show that the solution that the algorithm constructs when the value of the index of the outer "for" loop is j, is a $\frac{1}{1+\epsilon}$-approximation for the TV advertisement problem. For simplicity we omit the superscript j from $F_t^j(\cdot)$ and use instead $F_t(\cdot)$.

We note first that all calls to Compress are well defined: regarding lines 5 and 7, $f_t(\cdot) = w_t \int_{\cdot}^{\cdot+p_t} f(t)dt$ is a convex function because $f(\cdot)$ is convex. Regarding line 9, $\bar{F}_t(\cdot)$ is convex as a maximization of convex functions.

We start by analyzing the error bound. We show by backward induction that $\breve{F}_t(\cdot)$ is a K^{n-t+1}-approximation of $F_t(\cdot)$. The base case for $t = n$ holds true due to the fact that there is a single advertisement to broadcast and the definition in line 5. We assume by induction correctness for $t + 1$, i.e., $\breve{F}_{t+1}(\cdot)$ is a K^{n-t}-approximation of $F_{t+1}(\cdot)$, and prove that $\breve{F}_t(\cdot)$ is a K^{n-t+1}-approximation of $F_t(\cdot)$. By the parameters set in line 7 and Proposition 2 with the parameter $K_2 = 1$, we have that $\breve{f}_t(\cdot)$ is K^{n-t}-approximation of $f_t(\cdot)$. In line 8, for a fixed value of x we define the value of $\bar{F}_t(x)$ according to DP formulation (6). By the induction hypothesis and the calculus of approximation (summation, composition and maximization of approximation in Proposition 1) we get that $\bar{F}_t(x)$ is a K^{n-t}-approximation of $F_t(\cdot)$. We remark that no actual computation is involved in this step because we did not fix a value of x yet (it will be determined in the next step), but including line 8 in the algorithm helps us for the analysis. In line 9, by Proposition 2 we get that $\breve{F}_t(\cdot)$ is a K^{n-t+1}-approximation of $F_t(\cdot)$ as needed.

We turn to analyzing the running time. By Proposition 2, the query time of $\breve{F}_t(\cdot)$ is $t_{\breve{F}_t} = O(\log\log_K \min\{U_\sigma, nU_f\})$. The running time of Compress in line 7 is dominated by the one in line 9, which is by the same proposition $O(t_{\breve{F}_t} \log_K \min\{U_\sigma, nU_f\} \log h)$. Moving to base 2 logarithm, using the equation $\log K = \log \sqrt[n]{1+\epsilon} = O(\frac{\epsilon}{n})$, substituting $t_{\breve{F}_t}$ for its value and taking into account that there are n iterations in each one of the outer and inner "for" loops, the claimed running time follows. □

We note that the FPTAS for convex DPs in general, and function Compress in particular, have been implemented in practice with excellent computational performance, see [15, Section 4]. That paper includes an extensive computational evaluation based on randomly generated problem instances coming from applications in supply chain management and finance, and shows that the FPTAS runs faster than an exact algorithm even for small problem instances and small approximation factors, becoming orders of magnitude faster as the problem size increases, see [15, Section 6]. Moreover, that paper shows that with careful algorithm design, the errors introduced by floating point computations can be bounded, so a guarantee on the approximation factor over an exact infinite-precision solution can be provided, see [15, Section 4.3]. Furthermore, to reduce the optimality gap, the paper suggests an adaptive selection strategy of the approximation factors in each iteration, see [15, Section 6.5]. A posteriori error analysis is given in [15, Appendix B.3]. More detail on the computational efficiency of the FPTAS can be found in [15].

8 Concluding remarks

In this paper, we study the TV advertisements scheduling problem from a new centralized perspective, considering a TV channel with a TV rating points function and willingness to pay coefficients, in order to define a schedule that maximizes the total revenue from the advertisements broadcasted on a single TV break. This problem corresponds to the maximization variant of a new scheduling problem with a non-monotone penalty function g and a non-monotone speed function s for an alternative interpretation of the problem (G^{-1} is non-monotone) in a scheduling general setting. We prove NP-hardness of the problem and develop for it an FPTAS.

For future research, we propose to address another TV advertisement problem where the duration of the single TV break is less than the total duration of the TV advertisements, and then an optimal solution with idle times in the break is possible. The above problem can be interpreted as a special knapsack problem, where the contribution of each item is determined by its position in the knapsack via a specific function (in the classic version this function is a constant, i.e., the profit coefficient of the item).

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References

1. Bansal, N., Dürr, C., Thang, N.K., Vásquez, Ó.C.: The local–global conjecture for scheduling with non-linear cost. J. Sched. **20**(3), 239–254 (2017)
2. Bansal, N., Pruhs, K.: The geometry of scheduling. SIAM J. Comput. **43**(5), 1684–1698 (2014)
3. Benoist, T., Bourreau, E., Rottembourg, B.: The TV-break packing problem. Eur. J. Oper. Res. **176**(3), 1371–1386 (2007)
4. Bollapragada, S., Bussieck, M.R., Mallik, S.: Scheduling commercial videotapes in broadcast television. Oper. Res. **52**(5), 679–689 (2004)
5. Bollapragada, S., Cheng, H., Phillips, M., Garbiras, M., Scholes, M., Gibbs, T., Humphreville, M.: NBC's optimization systems increase revenues and productivity. Interfaces **32**(1), 47–60 (2002)
6. Bollapragada, S., Garbiras, M.: Scheduling commercials on broadcast television. Oper. Res. **52**(3), 337–345 (2004)
7. Cheung, M., Shmoys, D.: A primal–dual approximation algorithm for min-sum single-machine scheduling problems. In: Proceedings of the 14th International Workshop APPROX and 15th International Workshop RANDOM, pp. 135–146 (2011)
8. Epstein, L., Levin, A., Marchetti-Spaccamela, A., Megow, N., Mestre, J., Skutella, M., Stougie, L.: Universal sequencing on an unreliable machine. SIAM J. Comput. **41**(3), 565–586 (2012)
9. García-Villoria, A., Salhi, S.: Scheduling commercial advertisements for television. Int. J. Prod. Res. (2014). https://doi.org/10.1080/00207543.2014.951095
10. Garey, M.R., Johnson, D.S.: Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman, San Francisco (1979)
11. Ghassemi, F., Tari, Alaei, R.: Scheduling TV commercials using genetic algorithms. Int. J. Prod. Res. **51**(16), 4921–4929 (2013)
12. Halman, N.: A deterministic fully polynomial time approximation scheme for counting integer knapsack solutions made easy. Theor. Comput. Sci. **645**, 41–47 (2016)
13. Halman, N., Klabjan, D., Li, C.-L., Orlin, J., Simchi-Levi, D.: Fully polynomial time approximation schemes for stochastic dynamic programs. SIAM J. Discrete Math. **28**, 1725–1796 (2014)
14. Halman, N., Klabjan, D., Mostagir, M., Orlin, J., Simchi-Levi, D.: A fully polynomial time approximation scheme for single-item stochastic inventory control with discrete demand. Math. Oper. Res. **34**, 674–685 (2009)
15. Halman, N., Nannicini, G., Orlin, J.: A computationally efficient FPTAS for convex stochastic dynamic programs. SIAM J. Optim. **25**, 317–350 (2015)
16. Halman, N., Orlin, J.B., Simchi-Levi, D.: Approximating the nonlinear newsvendor and single-item stochastic lot-sizing problems when data is given by an oracle. Oper. Res. **60**, 429–446 (2012)

17. Höhn, W., Jacobs, T.: On the performance of Smith's rule in single-machine scheduling with nonlinear cost. ACM Trans. Algorithms **11**(4), 25:1–25:30 (2015)
18. Höhn, W., Mestre, J., Wiese, A.: How unsplittable-flow-covering helps scheduling with job-dependent cost functions. In: International Colloquium on Automata, Languages, and Programming. Springer, pp. 625–636 (2014)
19. Mao, J., Shi, J., Wanitwattanakosol, J., Watanabe, S.: An ACO-based algorithm for optimising the revenue of TV advertisement using credit information. Int. J. Revenue Manag. **5**(2), 109–120 (2011)
20. Megow, N., Verschae, J.: Dual techniques for scheduling on a machine with varying speed. In: Proceedings of the 40th International Colloquium on Automata, Languages and Programming (ICALP), pp. 745–756 (2013)
21. Mestre, J., Verschae, J.: A 4-approximation for scheduling on a single machine with general cost function. CoRR, arXiv:1403.0298 (2014)
22. Mihiotis, A., Tsakiris, I.: A mathematical programming study of advertising allocation problem. Appl. Math. Comput. **148**(2), 373–379 (2004)
23. Pereira, J., Vásquez, O.C.: The single machine weighted mean squared deviation problem. Eur. J. Oper. Res. **261**(2), 515–529 (2017)
24. Vásquez, Ó.C.: On the complexity of the single machine scheduling problem minimizing total weighted delay penalty. Oper. Res. Lett. **42**(5), 343–347 (2014)
25. Vásquez, O.C.: For the airplane refueling problem local precedence implies global precedence. Optim. Lett. **9**(4), 663–675 (2015)
26. Velusamy, S., Gopal, L., Bhatnagar, S., Varadarajan, S.: An efficient ad recommendation system for TV programs. Multimed. Syst. **14**(2), 73–87 (2008)
27. Zhang, X.: Mathematical models for the television advertising allocation problem. Int. J. Oper. Res. **1**(3), 302–322 (2006)