A linear time algorithm for the weighted lexicographic rectilinear 1-center problem in the plane *

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Abstract

We present an optimal $O(n)$ time algorithm for the weighted lexicographic rectilinear 1-center problem in the plane and prove that calculating the optimal *value* of the objective function requires $\Theta(n \log n)$ time.

1 Introduction

In the *weighted rectilinear 1-center problem in the plane* we are given a set $P^1 = \{p_i^1 \mid i = 1, ..., n\}$ of n points in the plane. For each point $p_i^1 = (x_i^1, y_i^1) \in P^1$ we are given a positive real number, $\omega_i \in \mathbb{R}$. The goal is to find a point $p^1 = (x^1, y^1) \in \mathbb{R}^2$ which minimizes the maximal weighted distance (with l_1 norm) between the points in P^1 and p^1. We call such a point an *optimal center*. A possible application for this problem is finding a location for a service center, p^1, in a town in which all the streets are parallel to the axes. The location of the service center should minimize the maximal weighted distance between n customer locations given in P^1, and the service center p^1 (one can think of the weighted distance as arrival time where for each i, customer p_i^1 walks to the service center at speed $\frac{1}{\omega_i}$). This problem arises in the area of facility location in operations research, and many variants of it have been studied (see for example [FW74], [D87], [M83b], [D86] and [SW96]). The weighted rectilinear 1-center problem in the plane can be formulated as follows

$$\begin{aligned} &\textit{minimize} \quad c^1 \\ &\textit{subject to} \quad \omega_i(|x_i^1 - x^1| + |y_i^1 - y^1|) \le c^1 \qquad i = 1, ..., n. \end{aligned} \tag{1}$$

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Problem (1) is equivalent to the following 3-dimensional linear programming (LP) problem

$$\begin{array}{lll} \textit{minimize} & c^1 & \\ \textit{subject to} & \omega_i((x_i^1 - x^1) + (y_i^1 - y^1)) \leq c^1 & \\ & & i = 1, ..., n \\ & \omega_i(-(x_i^1 - x^1) + (y_i^1 - y^1)) \leq c^1 & \\ & & i = 1, ..., n \\ & \omega_i((x_i^1 - x^1) - (y_i^1 - y^1)) \leq c^1 & \\ & & i = 1, ..., n \\ & \omega_i(-(x_i^1 - x^1) - (y_i^1 - y^1)) \leq c^1 & \\ & & i = 1, ..., n. \end{array} \tag{2}$$

We note that this LP problem can be solved in linear time by algorithms such as the ones of Megiddo [M83a],[M84].

It is well known ([FW74]) that by applying a linear transformation on each point in P^1 we get a transformed set of points $P^\infty = \{p_1^\infty, ..., p_n^\infty\}$ such that the rectilinear norm l_1 in P^1 becomes the l_∞ norm in P^∞, and Problem (1) is equivalent to

$$\begin{array}{lll} \textit{minimize} & c^\infty & \\ \textit{subject to} & \omega_i |x_i^\infty - x^\infty| \leq c^\infty & \\ & & i = 1, ..., n \\ & \omega_i |y_i^\infty - y^\infty| \leq c^\infty & \\ & & i = 1, ..., n, \end{array} \tag{3}$$

which we call the *weighted planar 1-center with l_∞ norm problem.* We note that this problem can be formulated as an LP problem using similar arguments as above. We also note that the linear transformation can be done in linear time, so both 1-center problems (1) and (3) are reducible to each other in linear time. It is also possible to solve the weighted planar 1-center with l_∞ norm problem directly by decomposing it into two weighted 1-center problems on the real line ([FW74]).

While the solutions of the weighted 1-center problem on the real line and the weighted planar 1-center problem with l_2 norm are unique, the solution of the weighted rectilinear 1-center in the plane, Problem (1), is not necessarily unique. In this case we are interested to choose from the set of optimal centers one that satisfies some additional properties. It is natural to look at the *weighted lexicographic rectilinear 1-center problem in the plane* which is defined in [O97]. In this problem we are given the same input as for the non-lexicographic corresponding problem. The goal is to find a point $p^1 \in \mathbb{R}^2$ which minimizes the scalar c^1 as defined in (1), as well as the *distances vector*

$$Q^1(P^1) = \text{lexicographic} \min_{p^1 \in \mathbb{R}^2} q(\{\omega_1 d^1(p_1^1, p^1), ..., \omega_n d^1(p_n^1, p^1)\}) \tag{4}$$

where $d^1 : \mathbb{R}^4 \rightarrow \mathbb{R}$ is the rectilinear distance between two points in the plane and q is the function that arranges the components of a multi-set of real elements in non-increasing order. Functions similar to q have already been used in the past (e.g., Kohlberg [K72] and Ogryczak [O97]). The location of the center should lexicographically minimize the weighted distances between n customer locations given in P^1, and the service center p^1. In this way not only the maximal weighted distance from points in P^1 to p^1 is of our concern, but also all of the remaining $n-1$ weighted distances.

We define the *weighted lexicographic planar 1-center with l_∞ norm problem* (*lex1-center problem*, in short) with the same input as for the non-lexicographic corresponding problem. The goal is to find a point $p^\infty \in \mathbb{R}^2$ which minimizes the distances vector

$$Q^\infty(P^\infty) = \text{lexicographic} \min_{p^\infty \in \mathbb{R}^2} q(\{\omega_1 d^\infty(p_1^\infty, p^\infty), ..., \omega_n d^\infty(p_n^\infty, p^\infty)\}) \quad (5)$$

where $d^\infty : \mathbb{R}^4 \rightarrow \mathbb{R}$ is the distance with l_∞ norm between two points in the plane. We note that, as above, both lex1-center problems (4) and (5) are reducible to each other in linear time. We are unaware of any previous linear time algorithms for these problems.

Example 1 *Let $p_1^\infty = (-4,1)$; $p_2^\infty = (6,-1)$; $p_3^\infty = (3,2)$; $p_4^\infty = (2,-3)$ and $p_5^\infty = (5,0)$ be 5 points in the plane. Let $P^\infty = \{p_1^\infty, ..., p_5^\infty\}$ with weights $\omega_1 = ... = \omega_5 = 1$ be the input for the lex1-center problem. The set of optimal centers for the (non-lexicographic) planer 1-center with l_∞ norm is the compact vertical interval $[(1,-3),(1,2)]$. We note that the x coordinate of the interval is determined by points p_1^∞ and p_2^∞. The optimal center of the corresponding lexicographic problem is $p^\infty = (1,-0.5)$, the white circle in Figure 1 below. The optimal distances vector is $(5,5,4,2\frac{1}{2},2\frac{1}{2})$.*

Observation 1 *Given a set P of n points on the real line, it takes $\Omega(n \log n)$ time to compute the distances vector, $Q(P)$, of the lexicographic 1-center on the real line.*

Proof: We reduce sorting to the lex1-center on the real line problem. Given a set $R = \{r_1, ..., r_n\}$ of n real numbers we find in linear time r_{max}, r_{min} the maximal and minimal numbers in R, respectively. We then construct in linear time an instance of the lex1-center problem with the set

$$P = \{r_1, r_2, ..., r_n, 2r_{max} - r_{min}\}$$

and equal weights to all its points. The optimal center is clearly $p = r_{max}$ and the distances vector $Q(P)$ yields the sorting of R. ■

Conclusion: Given a set P of n points in the plane, it takes $\Omega(n \log n)$ time to compute the distances vector, $Q(P)$, of the weighted lexicographic planar 1-center problem in either l_1, l_∞ or l_2 norm.

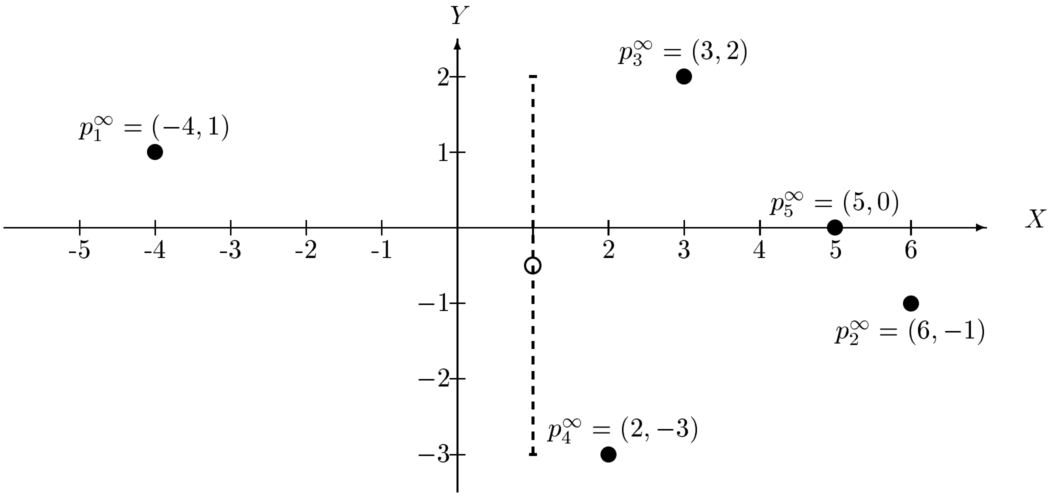

Figure 1: An instance of the planar 1-center problem with l_∞ norm.

From now on, throughout the paper, we will refer only to problems in the l_∞ norm, and will therefore omit the ∞ superscript from our notation.

Organization of the paper: In the following 3 sections we solve the planar lex1-center problem. In Section 2 we consider a restricted version of the lex1-center problem, where the x coordinate of the center must be a given value. We show that this problem is closely related to a geometric problem which we call the *min rays problem*. We give a linear time solution algorithm for the min rays problem in Section 3. In Section 4 we "put it all together" and solve the entire lex1-center problem in linear time. In the last section we formulate some generalizations of the problem, such as the lex1-center in $I\!R^d$, all of which we solve in linear time.

2 The restricted lex1-center problem vs. the min rays problem

In the *restricted lex1-center problem*, in addition to the input for the lex1-center problem (5), we are also given $x^@ \in I\!R$. The goal is to find a point which minimizes the following function of the single variable y

$$Q(P) = \text{lexicographic} \min_{p=(x^@,y)\in I\!R^2} q(\{\omega_1 d(p_1,p), ..., \omega_n d(p_n,p)\}). \qquad (6)$$

(Intuitively, in the restricted lex1-center problem the x-coordinate of the center is fixed to be $x^{@}$.) We denote by $p^{@} = (x^{@}, y^{@}) \in I\!R^2$ an optimal solution for Problem (6).

We consider now another formulation for this problem. For every pair of points $p = (x, y)$ and $p' = (x', y')$ we define $d_x(p, p') = |x - x'|$ to be the horizontal distance between p and p'. Similarly, we define $d_y(p, p') = |y - y'|$ to be the vertical distance between p and p'. Thus $d(p, p') = \max\{d_x(p, p'), d_y(p, p')\}$ is the distance between p and p' with l_∞ norm. For $i = 1, ..., n$, let

$$a_i = \omega_i |x^{@} - x_i|$$

be the weighted horizontal distance from point p_i to $p^{@}$ ($\omega_i d_x(p_i, p^{@})$). Let

$$g_i(y) = \max\{a_i, \omega_i |y - y_i|\}$$

be the weighted distance between point p_i and the restricted center $(x^{@}, y)$. Thus $Q(P)$ in Problem (6) is equivalent to

$$Q(P) = \text{lexicographic} \min_{y \in I\!R} q(\{g_1(y), ..., g_n(y)\}). \tag{7}$$

Next, we define a problem, called the *min rays problem*, and show it is equivalent to the restricted lex1-center problem. In the min rays problem we are given the same input as for the restricted lex1-center problem. The goal is to find

$$\begin{array}{lll} \textit{minimize} & c & \\ \textit{subject to} & |y_i - y| > |x_i - x^{@}| \rightarrow \omega_i |y_i - y| \le c & \\ & & i = 1, ..., n. \end{array} \tag{8}$$

The variables in this problem are c and y. For $i = 1, ..., n$, let $r_i^- : I\!R \rightarrow I\!R \cup \{-\infty\}$

$$r_i^-(y) = \begin{cases} \omega_i(y_i - y) & \text{if } y < y_i - |x_i - x^{@}| \\ -\infty & \text{otherwise} \end{cases}$$

be the restriction of the function g_i to $(-\infty, y_i - |x_i - x^{@}|)$ concatenated with the constant function $-\infty$ on the interval $[y_i - |x_i - x^{@}|, \infty)$. We call r_i^- a *decreasing ray* and call the point $(y_i - |x_i - x^{@}|, \omega_i |x_i - x^{@}|)$ the *root* of r_i^-. Similarly, let

$$r_i^+(y) = \begin{cases} \omega_i(y - y_i) & \text{if } y > y_i + |x_i - x^{@}| \\ -\infty & \text{otherwise} \end{cases}$$

be the restriction of the function g_i to $(y_i + |x_i - x^{@}|, \infty)$ concatenated with the constant function $-\infty$ on $(-\infty, y_i + |x_i - x^{@}|]$. We call r_i^+ an *increasing ray* and call the point $(y_i + |x_i - x^{@}|, \omega_i(\omega_i |x_i - x^{@}|)$ the *root* of r_i^+. We define the *upper envelope* of the rays to be the function $u : I\!R \rightarrow I\!R \cup \{-\infty\}$ such that for all $y \in I\!R$

$$u(y) = \max_{i=1,...,n} \max\{r_i^-(y), r_i^+(y)\}.$$

Thus, the min rays problem is equivalent to calculating

$$c = \min_{y \in \mathbb{R}} u(y). \tag{9}$$

Example 2 *The min rays problem corresponding to the lex1-center problem given in Example 1 is shown in Figure 2. The rays r_3^- and r_4^+, corresponding respectively to points p_3^∞ and p_4^∞ determine the optimal solution $c = 2\frac{1}{2}$ and $y = -\frac{1}{2}$.*

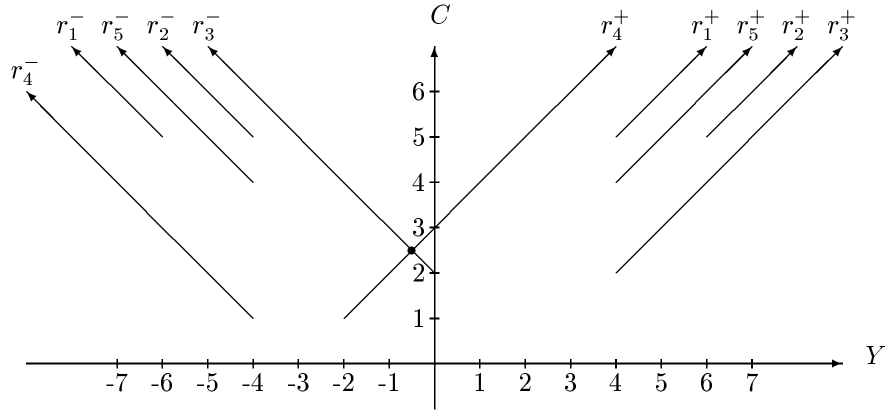

Figure 2: An instance of the min rays problem.

We let $c^\$$ and $y^\$$ be an optimal solution of the min rays problem ($c^\$$ can be $-\infty$ in such a solution). Such an optimal solution exists since the lower-level sets of the functions $\max\{r_i^-(y), r_i^+(y)\}$ are compact sets. We will use the following observation whose proof is straight forward.

Observation 2 *If $V = \{v_1, ..., v_n\}$ and $V' = \{v'_1, ..., v'_n\}$ are sets of n real numbers each satisfying that $v_i \geq v'_i$ for $i = 1, ..., n$ then $q(V) \geq_L q(V')$.*

We are now ready to prove

Theorem 1 *Let $y^\$, c^\$$ be an optimal solution of the min rays problem. Then $(x^@, y^\$)$ is an optimal center in the corresponding restricted lex1-center problem.*

Proof: If $c^\$ = -\infty$ then the line $y = y^\$$ intersects all the rays at $-\infty$ and hence for $i = 1, ..., n,\ g_i(y^\$) = a_i$. Suppose y^0 is an optimal solution of the restricted lex1-center problem. From the definition of functions g_i, for $i = 1, ..., n,\ g_i(y^0) \geq a_i$, so due to Observation 2

$$q(\{g_1(y^0), ..., g_n(y^0)\}) \geq_L q(\{g_1(y^\$), ..., g_n(y^\$)\}).$$

Else $c^\$ \geq 0$ and there are $i, j \in \{1, ..., n\}$ such that $c^\$$ is resulted by the intersection of an increasing ray r_i^+ and a decreasing ray r_j^-. Hence either $g_i(y^\$) = c^\$$ or $g_j(y^\$) = c^\$$ (or both) is satisfied. Let $K = \{k \mid a_k \geq c^\$\}$. Since the point $(y^\$, c^\$)$ lies on the upper envelope of the rays, we get that

$$\forall k \in K \;\; g_k(y^\$) = a_k \geq c^\$. \tag{10}$$

Now, suppose that $y^0 \neq y^\$$ is an optimal solution of the restricted lex1-center problem. One of $g_i(y^0), g_j(y^0)$ is strictly greater than $c^\$$ while

$$\forall k \in K \;\; g_k(y^0) \geq a_k \geq c^\$.$$

Hence, by (10) we get

$$q(\{g_k(y^0) \mid k \in K\} \cup \{g_i(y^0), g_j(y^0)\}\}) >_L q(\{g_k(y^\$) \mid k \in K\} \cup \{g_i(y^\$), g_j(y^\$)\}\}). \tag{11}$$

We define $K' = \{1, 2, ..., n\} \setminus \{K \cup \{i, j\}\}$. We have $\forall k \in K'$ $g_k(y^\$) \leq c^\$$ so by (11) we conclude that

$$q(\{g_1(y^0), ..., g_n(y^0)\}) >_L q(\{g_1(y^\$), ..., g_n(y^\$)\})$$

and that the optimal center is unique, for this case where $c^\$$ is finite. ■

3 A linear time solution algorithm for the min rays problem

As in the previous section, let $c^\$$ and $y^\$$ be an optimal solution for the min rays problem. By using Theorem 1, $p^@ = (x^@, y^\$)$ is an optimal center for the restricted lex1-center problem. We define a *test oracle* as follows. The input of the test oracle is the same as for the min rays problem with an additional point $p = (x^@, y)$. The output is one of the following:

- p is an optimal center.
- p is not an optimal center. All optimal centers lie to the right of p (i.e., $y^\$ > y$).
- p is not an optimal center. All optimal centers lie to the left of p (i.e., $y^\$ < y$).

We first show that

Lemma 1 *The test oracle can be implemented in linear time.*

Proof: We first note that, $u(y)$, the upper envelope of the rays in the min rays problem, is a unimodal function, i.e., it has either a unique finite local minimum which is also its global minimum, or the minimum is $-\infty$ and the source of $-\infty$, $u^{-1}(-\infty)$, is a compact interval. (See [JKP72] pp. 52 for a definition of unimodal one variable functions).

We compute the value of each of the $2n$ rays at y. If all the values are $-\infty$ we conclude that p is an optimal center. Else there is a finite maximal value among the $2n$ calculated values. We look at the rays attaining this maximal value on y. If all of them are decreasing we conclude that p is not an optimal center and $y^{\$} > y$. If all of them are increasing we conclude that p is not an optimal center and $y^{\$} < y$. Else, there are rays of both types and we conclude that p is an optimal center. Clearly, the time consumed in the algorithm given above is linear in n. ∎

We are now ready to prove

Theorem 2 *The min rays problem is solvable in $\Theta(n)$ time.*

Proof: The input consists of $2n$ rays of which we arbitrary form n distinct pairs of rays. We partition the pairs of rays into 12 types as follows. If one ray in the pair $\{r, r'\}$ is increasing and the other is decreasing, we call it a *mixed pair* and partition it into the 4 types (m1),(m2),(m3) and (m4) as in Figure 3. If both rays in the pair are increasing, we call it an *increasing pair* and partition it into the 4 types (i1),(i2),(i3),(i4) as in Figure 4. Finally, if both rays in the pair are decreasing, we call it a *decreasing pair* and partition it into 4 types (d1), (d2), (d3), (d4), in a symmetric way to the partition we make with the increasing pairs. Altogether there are 12 types of pairs.

We group all pairs of rays into 12 distinct sets, each set contains pairs of the same type. We look at each of these sets separately, and drop a constant fraction of its rays in a manner to be explained below. Then we arbitrary form pairs of the remaining rays, partition the pairs of rays into 12 types as before and reiterate.

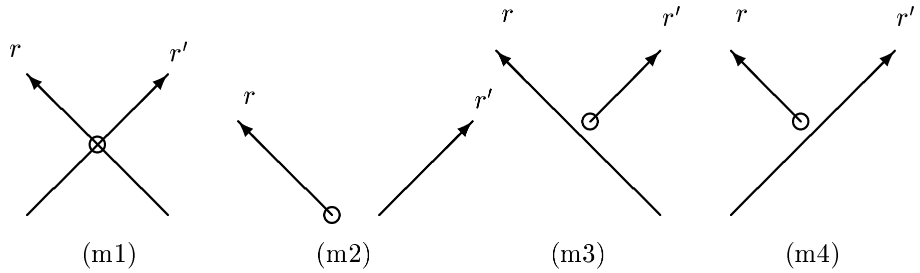

Figure 3: The 4 types of mixed pairs.

We process the set of pairs of type (i1) as follows. Since in each pair the ray r' dominates the ray r, we drop the ray r from each pair. In this case we drop

$\frac{1}{2}$ of the rays.

We process each one of the sets of pairs of type (m1),(m2),(m3),(m4),(i2) and (i3) as follows. For each pair of rays we define a *critical* real value which divides the real line into two distinct intervals. This value is the projection on the y axis of either the intersection point of the two rays, or the root of one of the rays. For each pair of rays we define the critical value such that on each one of the resulting intervals, one ray dominates the other ray, as indicated by a circle in Figures 3 and 4. Now, let y^m denote the median of the critical values. The value y^m can be found in linear time due to Blum, Floyd, Pratt, Rivest and Tarjan [B73] (see alternatively [CLR90]). At this time we call the test oracle with $p = (x^@, y^m)$. This test runs in linear time (Lemma 1). Suppose first that the test oracle returns that p is not an optimal center and that all optimal centers lie to the right of p. The critical values of at least half of the pairs are to the left of y^m and hence to the left of any optimal center. Thus we drop one of the rays in each one of these pairs, namely r. If the test oracle returns that p is not an optimal center and that all optimal centers lie to the left of p we act similarly and drop the ray r' of at least half of the pairs. In this case we drop at least $\frac{1}{4}$ of the rays.

We process the set of pairs of type (i4) as follows. As before we define for each pair a critical real value. This value is the projection on the y axis of the intersection point of the rays composing the pair, as indicated by the big white circle in Figure 4 (i4). Now, let y^m denote the median of the critical values. At this time we call the test oracle with $p = (x^@, y^m)$. Suppose first that the test oracle returns that p is not an optimal center and that all optimal centers lie to the right of p. The critical values of at least half of the pairs are to the left of y^m and hence to the left of the optimal center. Thus we drop one of the rays in each one of these pairs, namely r. Now suppose that the test oracle returns that p is not an optimal center and that all optimal centers lie to the left of p. The critical values of at least half of the pairs are to the right of y^m and hence to the right of the optimal center. We construct a new critical value for each of these pairs as indicated by the small black circle in Figure 4 (i4). Now, let $y^{m'}$ denote the median of the new critical values. At this time we call the test oracle with $p = (x^@, y^{m'})$. Suppose first that the test oracle returns that p is not an optimal center and that all optimal centers lie to the right of p. The critical values of at least half of the pairs are to the left of $y^{m'}$ and hence to the left of any optimal center. Thus we drop one of the rays in each one of these pairs, namely r'. If the test oracle returns that p is not an optimal center and that all optimal centers lie to the left of p we act similarly and drop the ray r of each one of at least half of the pairs. In this case we drop at least $\frac{1}{8}$ of the rays.

We process the sets of decreasing pairs (d1), (d2), (d3) and (d4) in a symmetric way.

If during the iteration the test oracle returns that p is an optimal center we exit the algorithm and return this point. Otherwise we are left with no more than $\frac{7}{8}$ of the original rays. We arbitrary form pairs of the remaining rays, partition the pairs of rays into the 12 types as before and reiterate until we are left with a constant number of rays. The critical values we associate with these constant number of rays are the projections on the y axis of their roots and of their intersection points. There is a constant number of such critical values. We perform a test oracle for each of these values. Since there exists an optimal center whose projection on the y axis is either a root of a ray or an intersection point of two rays, we are assured to get a "p is an optimal center" reply from the test oracle at least once.

It follows that during this process we drop at each iteration one ray per pair, for at least $\frac{1}{4}$ of our pairs of rays. In other words, at least $\frac{1}{8}$ rays will be dropped in each iteration with an $O(n)$ effort. It thus follows that the entire process runs in linear time. ∎

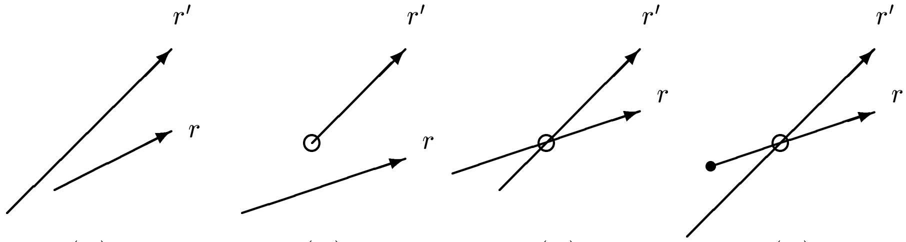

Figure 4: The 4 types of increasing pairs.

4 A linear time solution algorithm for the lex1-center problem

Let $p^* = (x^*, y^*)$ be an optimal center of the lex1-center problem (5). Let c^∞ be the solution value of the (non lexicographic) weighted planar 1-center problem (3) and $p^\infty = (x^\infty, y^\infty)$ be a center realizing this solution.

We observe that if the maximal distance c^∞ is realized by an horizontal distance then there exists i such that $c^\infty = \omega_i d_x(p^\infty, p_i) \geq \omega_i d_y(p^\infty, p_i)$ and $x^* = x^\infty$. We find x^∞ by solving Problem (3) in linear time as demonstrated in Section 1. Next we let $x^@ = x^\infty$ and find an optimal center $p^@ = (x^@, y^@)$ for the restricted lex1-center problem by solving the min rays problem in linear time by the algorithm presented in the previous section. Due to Theorem 1 the solution is an optimal solution for the restricted lex1-center problem.

If the maximal distance c^∞ is not realized by an horizontal distance (i.e., it is realized by a vertical distance), we rotate the points in P by $\frac{\pi}{2}$ radians around the origin and repeat the above process. We proved

Theorem 3 *The lexicographic rectilinear 1-center problem in the plane is solvable in* $\Theta(n)$ *optimal time.*

5 Concluding remarks

We note that our algorithm can be modified in order to solve, in the same time bound, the following generalizations of the lex1-center problem (with l_∞ norm).

- The lex1-center problem in $\mathbb{R}^d$. We describe in short the solution algorithm for this problem. As in the planar problem, we first solve the corresponding non-lexicographic 1-center problem. This fixes one of the d coordinates. Then we successively solve $d-1$ restricted problems, of decreasing dimensions $d-1, d-2, ..., 1$ by decomposing each restricted problem of dimension d' into d' one-dimensional min rays problems. It thus follows that the total running time is linear in n where the constant is quadratic in d.

- In the *general lexicographic planar 1-center problem* we are given a set, $P^g = \{p_1^g, ..., p_n^g\}$, of n points in the plane as well as $2n$ positive real numbers ω_i^x, ω_i^y for $i = 1, ..., n$. The goal is to find a point $p \in \mathbb{R}^2$ which minimizes the distance vector

$$Q(P^g) = \text{lexicographic} \min_{p \in \mathbb{R}^2} q(\{d^g(p_1^g, p), ..., d^g(p_n^g, p)\}) \qquad (12)$$

  where $d^g : \mathbb{R}^4 \to \mathbb{R}$ is the following general distance function. For every $i = 1, ..., n$ and for every $p \in \mathbb{R}^2$, $d^g(p_i^g, p) = \max\{\omega_i^x d_x(p_i^g, p), \omega_i^y d_y(p_i^g, p)\}$.

- In the *all-centers lex1-center problem* we are given the same input as for the lex1-center problem. Instead of finding *an* optimal center, one has to find *all* optimal centers.

We now consider another version of the lex1-center problem, the *discrete lex1-center problem*. In this problem the center must be one of the input points.

The discrete lex1-center problem on the real line is solved in $O(n \log n)$ time as follows. Let p^* be the optimal center of the (non-lexicographic) 1-center problem on the real line. We next find the two neighboring points $p_l, p_r \in P$ of p^*. Due to the convexity of the objective function one of p_l, p_r is an optimal solution. We calculate the lexicographic distances vector from each of p_l, p_r to the remaining points by first calculating the $n-1$ distances and then sorting it. We then compare the two vectors. The center realizing the lexicographically smaller vector is an optimal solution. The above operations can be done in $O(n \log n)$ total time. We observe that $\Omega(n \log n)$ is a lower bound for the all-centers discrete non-weighted lex1-center problem on the real line by a simple reduction from set equality problem [R72].

Tamir [T02] observes that the discrete lex1-center problem in $\mathbb{R}^d$ ($d \geq 2$) is solved in $O(n^2)$ time (where the constant depends exponentially on d) as follows. First construct the n sorted weighted distances vectors from each point in P to all the remaining $n-1$ points. This is done by a reduction to the $O(n^2)$ algorithm of Lee and Ching [LC85] which sorts the n sets of directions from each point in an n-membered set of points in the plane, to the remaining points. The optimal center is the point which realizes the lexicographically smallest vector among the resulting n sorted vectors.

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