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ON THE ALGORITHMIC ASPECTS OF DISCRETE AND LEXICOGRAPHIC HELLY-TYPE THEOREMS AND THE DISCRETE LP-TYPE MODEL*

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Abstract. Helly's theorem says that, if every $d+1$ elements of a given finite set of convex objects in $\mathbb{R}^d$ have a common point, there is a point common to all of the objects in the set. In discrete Helly theorems the common point should belong to an a priori given set. In lexicographic Helly theorems the common point should not be lexicographically greater than a given point. Using discrete and lexicographic Helly theorems we get linear time solutions for various optimization problems. For this, we introduce the *DLP-type* (discrete linear programming–type) model, and provide new algorithms that solve in randomized linear time fixed-dimensional DLP-type problems. For variable-dimensional DLP-type problems, our algorithms run in time subexponential in the combinatorial dimension. Finally, we use our results in order to solve in randomized linear time problems such as the discrete p-center on the real line, the discrete weighted 1-center problem in $\mathbb{R}^d$ with either l_1 or l_∞ norm, the standard (continuous) problem of finding a line transversal for a totally separable set of planar convex objects, a discrete version of the problem of finding a line transversal for a set of axis-parallel planar rectangles, and the (planar) lexicographic rectilinear p-center problem for $p = 1, 2, 3$. These are the first known linear time algorithms for these problems. Moreover, we use our algorithms to solve in randomized subexponential time various problems in game theory, improving upon the best known algorithms for these problems.

Key words. Helly-type theorems, LP-type model, design and analysis of algorithms

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1. Introduction.

1.1. Helly-type theorems. The classical theorem of Helly stands at the origin of what is known today as the combinatorial geometry of convex sets. It was discovered in 1913 and may be formulated as follows.

THEOREM 1.1 (Helly's theorem). *Let H be a family of closed convex sets in $\mathbb{R}^d$, and suppose either H is finite or at least one member of H is compact. If every $d+1$ or fewer members of H have a common point, then there is a point common to all members of H.*

A possible generalization of Helly's theorem is as follows. Let H be a family of objects, and let $\mathcal{P}$ be a predicate on subsets of H. A *Helly-type theorem* for H is of the form:

There is a constant k such that for every finite set G, $G \subseteq H$, $\mathcal{P}(G)$, if and only if, for every $F \subseteq G$ with $|F| \leq k$, $\mathcal{P}(F)$.

The minimal such constant k is called the *Helly number* of H with respect to the predicate $\mathcal{P}$. If no such constant exists, we say that the Helly number of H with

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respect to $\mathcal{P}$ is *unbounded* or *infinite* (∞). In Helly's theorem, the Helly number is $d+1$, and $\mathcal{P}$ is the predicate of having a nonempty intersection.

Over the years, a vast body of application analogues and far-reaching generalizations of Helly's theorem has been assembled in the literature (see, for instance, the excellent surveys of [10, 12, 16]).

It is possible to give lexicographic versions to some of the Helly theorems. For instance, the following theorem is a lexicographic version of Helly's theorem. (Recall that, for every $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d$, x is said to be *lexicographically* smaller than y (*lsmaller*, in short, or $x <_L y$) if either $x_1 < y_1$ or there exists $d \geq k > 1$ such that $x_i = y_i$ for $i = 1, 2, \ldots, k-1$ and $x_k < y_k$.)

THEOREM 1.2 (lexicographic Helly's theorem [26, 20]). *Let H be a finite family of convex sets in $\mathbb{R}^d$. For every $x \in \mathbb{R}^d$, if every $d+1$ or fewer members of H have a common point which is not lexicographically greater than x, then there is a point common to all members of H which is also not lexicographically greater than x.*

This theorem is folklore. It derives directly from Helly's theorem and Lemma 8.1.2 in [26] and is proved independently in [20]. $d+1$ is called the *lexicographic Helly number* of H with respect to intersection (*lex-Helly number*, in short). The following theorem is a discrete version of Helly's theorem, due to Doignon.

THEOREM 1.3 (see [11]). *Let H be a finite family of at least 2^d convex sets in $\mathbb{R}^d$. If every 2^d or fewer members of H have a common point with integer coordinates, then there is a point with integer coordinates common to all members of H.*

2^d is called the *discrete Helly number* of H with respect to intersection. Halman [20] provides discrete versions to numerous known Helly theorems. For instance, a special case of Helly's theorems is when the given convex sets are axis-parallel boxes in $\mathbb{R}^d$. In this case the Helly number is just 2 [9]. A discrete version of this Helly theorem is as follows.

THEOREM 1.4 (Theorem 2.10 in [20]). *Let S be a finite set of points in $\mathbb{R}^d$, and let D be a finite family of closed boxes in $\mathbb{R}^d$ with edges parallel to the axes. If every $2d$ or fewer members of D have a common point in S, then there is a point in S common to all members of D.*

A combined discrete-lexicographic version of this Helly theorem is as follows.

THEOREM 1.5 (Theorem 2.10 in [20]). *Let S be a finite set of points in $\mathbb{R}^d$, and let D be a finite family of closed boxes in $\mathbb{R}^d$ with edges parallel to the axes. For every $x \in \mathbb{R}^d$, if every $2d$ or fewer members of D have a common point $x' \in S$, with $x' \leq_L x$, then there is a point $x^* \in S$ common to all members of D with $x^* \leq_L x$.*

1.2. Algorithmic aspects of finite Helly numbers. In this section we discuss two optimization models and show their relations to Helly numbers.

The LP-type model. Matoušek, Sharir, and Welzl [28] defined a model which generalizes linear programming (LP) and called it the LP-type model (see definitions in section 2). Fixed-dimensional LP-type problems can be solved efficiently by *LP-type algorithms* such as the ones of Matoušek, Sharir, and Welzl [28] or Kalai [22]. The algorithm of Clarkson [8], which was originally formulated to solve LP, fits the LP-type model as well [31, 7, 15]. This provides a tool for obtaining linear time algorithms to various (continuous) optimization problems, mainly in computational geometry and location theory, as shown in [2, 28].

The DLP-type model. In *continuous* optimization models related to LP-type problems, the feasible set is defined by a finite set of constraints. In the *discrete* versions, in addition to the above, there is also a prespecified set of *relaxations*. A feasible solution is restricted to be in the set of relaxations as well as to satisfy the constraints. *Integer programming* (IP) is an example of a discrete optimization problem where the set of relaxations is the integer lattice. Another example for a discrete optimization problem is the *discrete point set width problem*, where we are given a finite set of points in the plane (i.e., constraints) and a finite set of permissible directions (i.e., relaxations). The goal is to find the minimal width of a band with a permissible direction which contains all of the points (see more detail about this problem in section 4). Many times discrete optimization problems are proved to be computationally harder to solve than their corresponding continuous versions (e.g., LP vs. IP and continuous planar Euclidean 1-center vs. the discrete version as proved in section 9). In this paper we propose the following framework for solving discrete optimization problems: We generalize integer programming by introducing the *discrete* LP-type (*DLP-type*) model. We provide randomized linear time algorithms to solve fixed-dimensional DLP-type problems satisfying a condition we call the violation condition (VC).

Helly numbers and the two optimization models. In [20] Halman defines the notion of discrete and lexicographic Helly theorems, provides lexicographic and discrete versions to numerous known Helly theorems, and studies the relations between the different types of Helly theorems. In this paper we show that discrete and lexicographic Helly theorems have interesting algorithmic aspects as well. In 1994, Amenta [2] showed that every *parameterized Helly system* satisfying a condition called the unique minimum condition (UMC) results in a fixed-dimensional LP-type problem (see definitions of the terms parameterized Helly theorems and UMC in section 2.3). In this paper we define lexicographic parameterized Helly systems and show that *every* such system results in a fixed-dimensional LP-type problem. Unlike in [2], no additional conditions are needed. Similarly to [2], this provides a framework for obtaining linear time algorithms (i.e., the LP-type algorithms mentioned above) for the optimization problems related to these Helly numbers. In this way the *existence* of finite lexicographic Helly numbers implies the *solvability* of their corresponding optimization problems by the linear time LP-type algorithms. Similarly to the above, we show that *every* lexicographic-discrete parameterized Helly system can be formulated as a fixed-dimensional DLP-type problem.

1.3. Applications. We improve upon the best known algorithms for the seven problems listed below. The problems differ in the way we solve them. The first three are solved by using the LP-type model and its connection to lexicographic Helly theorems. The next four problems are solved via the DLP-type model. While the first three of them are solved via lexicographic-discrete Helly theorems, the fourth is not. We solve in this paper the first five problems in linear time. Due to its length, we refer the reader to [17] for details of the solution of the sixth problem. The first six problems lie in the fields of research of either computational geometry or location theory. The seventh problem is solved in [19] and is different, since it lies in game theory and is solved in strongly subexponential time. We summarize the solutions we give to each of these problems in Table 1.1.

1. Planar lexicographic weighted rectilinear p-center optimization problem ($p = 1, 2, 3$). Given a finite set $H = \{h_1, \ldots, h_n\}$ of reference points in the plane and a set $W = \{w_1, \ldots, w_n\}$ of weights in $\mathbb{R}^+$, find the lexicographically smallest vector $(\lambda_1, x_1, y_1, x_2, y_2, \ldots, x_p, y_p) \in \mathbb{R}^+ \times \mathbb{R}^{2p}$ such that for every scaled square $\frac{\lambda_1}{w_i} h_i$, $h_i \in H$, centered at h_i with radius $\frac{\lambda_1}{w_i}$, there exists $1 \leq j \leq p$

Table 1.1
A comparison between the various problems solved.

Problem number	1	2	3	4	5	6	7
Running time	Linear	Linear	Linear	Linear	Linear	Linear	Subexponential
Model used	LP-type	LP-type	LP-type	DLP-type	DLP-type	DLP-type	LP/DLP-type
Type of Helly theorem used	lex	lex	lex	lex-discrete	lex-discrete	None	None

such that $\frac{\lambda_1}{w_i}h$ contains (i.e., is pierced by) point (x_j, y_j) (we call λ_1 the *radius* and $(x_1, y_1, \ldots, x_p, y_p)$ the *centers vector*).

For $p > 3$ [32] showed a lower bound of $\Omega(n \log n)$. [32, 21] solve the corresponding nonlexicographic problem in linear time.

2. Line transversal of axis-parallel rectangles optimization problem. Given a set B of axis-parallel rectangles, find the minimal scaling factor λ^* such that the set of scaled rectangles λ^* admits a line transversal.

For the next problem we use the following definitions. A set H of convex objects is called *totally separable* if there exists a direction such that each line in this direction intersects at most one convex object from H. We call the objects in H *simple* if they have a constant size storage description, the intersections and common tangents between any two objects can be found in constant time, and the minimal scaling factor for any 3 objects to admit a line transversal can be found in constant time.

3. Line transversal of totally separable set of convex planar objects decision problem. Given is a totally separable finite family H of simple convex objects (the direction of separation is not given). Decide whether H admits a line that intersects all of the objects in H.

Given the order in which any line transversal should meet the objects in H, the problem is solvable in linear time [13].

4. Lexicographic discrete line transversal of axis-parallel rectangles problem. Given a finite family D of axis-parallel rectangles and a finite set S of line *directions*, find a line transversal for D, $y = ax + b$, with the lexicographically smallest vector (a, b) satisfying $a \in S$.

We show in section 10 that a similar problem, where, instead of a finite family S of line directions, we are given a finite family S' of lines, and the goal is find the lexicographically smallest vector $(a, b) \in S'$ such that $y = ax + b$ is a line transversal for D, has a lower bound of $\Omega(n \log n)$ under the algebraic computation tree model.

5. Discrete weighted 1-center problem in $\mathbb{R}^d$ with an l_∞ norm. Given are sets $D = \{d_1, \ldots, d_n\}$ and $S = \{s_1, \ldots, s_m\}$ of *points* in $\mathbb{R}^d$ and a set $W = \{w_1, \ldots, w_n\}$ of *weights* in $\mathbb{R}^+$. Find a point $s \in S$ (*center*) which minimizes the real function $r(D, S) = \min_{s \in S} \max_i w_i \|s - d_i\|_\infty$ (the *optimal radius*). We solve the corresponding rectilinear problem (i.e., with an l_1 norm) in linear time as well.

It is folklore that the latter problem restricted to the case $S = D$ is solvable in $O(n \log n)$ time.

6. Discrete p-center problem on the real line. Given a finite set D of real numbers (*points*) and a finite set S of real numbers (*center locations*), find a subset $C \subseteq S$ of p points (*centers*) which minimizes the real function $r_p(D, S) = \min_{C \subseteq S, |C| \leq p} \max_{h \in D} dist(h, C)$ (the *optimal radius*). For every finite set of real numbers C and real h, $dist(h, C) = \min_{c \in C} |h - c|$. Due to space limitations we refer the interested reader to Chapter 9 in [17] for a detailed description of our linear time solution for this problem.

Assuming the order of the points on the line is given, the discrete p-center problem on the real line is solvable in linear time by the fairly involved technique of Frederickson [14].

7. Simple stochastic games and infinite games. The first strongly subexponential algorithm for binary simple stochastic games (SSGs) was given by Ludwig [25] in 1995 by using ideas from the algorithms of [22] and [31] for LP-type problems. Halman [19] gives the first strongly subexponential solution for (nonbinary) SSGs by formulating the SSG as an LP-type problem and then calculating optimal strategies for both players by the LP-type algorithm of [31]. Since several infinite games are linearly reducible to nonbinary SSGs, this gives strongly subexponential algorithms to parity games (PGs) and the first strongly subexponential algorithms to mean payoff games (MPGs) and discounted payoff games. Halman notes in [19] that nonbinary SSGs can most naturally be formulated as discrete LP-type problems, because of the essentially primal-dual nature of the two-player game. We note that, independently, Björklund, Sandberg, and Vorobyovn [6] developed a (nonstrongly) subexponential algorithm for MPGs, a strongly subexponential algorithm for PGs [5], and a (nonstrongly) subexponential algorithm to nonbinary SSGs [4]. All of their algorithms are "tailored" to the specific game solved and "adapt" ideas from the algorithms of [25, 22, 31] (see formal definitions of all of these games in [19]).

Our contribution. In this paper we define a new model for solving discrete optimization problems, the DLP-type model. We develop for it several linear time (randomized) algorithms. We study the relations between discrete Helly theorems and the DLP-type model. We study also the relations between (nondiscrete) lexicographic Helly theorems and the LP-type model. We show that *every* lexicographic parameterized Helly system results in a fixed-dimensional LP-type problem. In this case the UMC stated in the main theorem of [2] is not needed. By incorporating these "ingredients" together we provide the first linear time algorithms for the first six problems defined above. All of these problems are related to computational geometry and location theory. By solving the seventh problem we show (for the first time, to the best of our knowledge) that the LP-type and DLP-type models have applications in other fields of research, such as game theory. Moreover, we show that these two models are also useful for solving non-fixed-dimensional problems in subexponential time.

Organization of the paper. In this paper we extensively use terms which are defined in [2], two tools for establishing linear time algorithms: the LP-type framework and Helly-type results, which are reviewed in [2, 32], and the two LP-type algorithms in [8, 32]. In order to make the paper self-contained we review these terms, models, and algorithms in section 2. In section 3 we define a dual version of the LP-type model, which we use in order to define the DLP-type model in section 4. In section 5 we develop algorithms which solve (fixed-dimensional) DLP-type problems in (randomized) linear time. The rest of the paper is dedicated to show the interrelations between discrete and lexicographic Helly theorems and DLP-type and LP-type models. In section 6 we study the relations between lexicographic Helly theorems and the LP-type model. By showing that every lexicographic parameterized Helly system results in a fixed-dimensional LP-type problem, we give a partial solution for the main open problem raised by Amenta [2], who asked to characterize the parameterized Helly systems which result in fixed-dimensional LP-type problems. In section 7 we demonstrate the applicability of these relations by solving the first problem discussed in this section—the planar lexicographic weighted rectilinear p-center problem ($p = 1, 2, 3$) in linear time. In section 8 we study the relations between discrete and lex-discrete Helly theorems and the DLP-type model. In sections 9 and 10 we solve in linear time the next four problems discussed in this section.

2. Literature review. In this section we review some of the definitions and results given in Amenta [1, 2], Sharir and Welzl [31], Matoušek, Sharir, and Welzl [28], and Clarkson [8]. The term used in the first two papers is GLP (general linear programming) rather than LP-type.

2.1. LP-type problems.

DEFINITION 2.1. *An* abstract problem *is a tuple* (H, ω)*, where* H *is a finite set of elements (which we call* constraints*) and* ω *is an* objective function *from* 2^H *to some totally ordered set* Λ *which contains a special maximal (minimal) element* ∞ $(-\infty)$*, respectively. The goal is to compute* $\omega(H)$*.*

DEFINITION 2.2. *Let* (H, ω) *be an abstract problem. For any subset* $G \subseteq H$ *we say that* $F \subseteq G$ defines *the solution on* G *(*F *is a* solution-defining set *of* G*) if* $\omega(F) = \omega(G)$*.*

Clearly, for every $G \subset H$, G is a defining set for itself.

DEFINITION 2.3. *An* LP-type problem *is an abstract problem* (H, ω) *that obeys the following conditions (when we write* $<$, $\leq$, $=$ *etc., we mean under the ordered set* Λ*):*

1. Monotonicity: *For all* $F \subseteq G \subseteq H : \omega(F) \leq \omega(G)$ *(so the special element* $-\infty$ *is such that* $\omega(\emptyset) = -\infty$*).*
2. Locality: *For all* $F \subseteq G \subseteq H$*, with* $\omega(G) = \omega(F) \neq -\infty$*, and for each* $h \in H$*, if* $\omega(G \cup \{h\}) > \omega(G)$ *then* $\omega(F \cup \{h\}) > \omega(F)$*.*

Note that *lexicographic linear programming*, where the input is a finite set of closed half-planes in $\mathbb{R}^d$ and the output is the lsmallest point which lies in all half-planes, is an LP-type problem: H is the finite set consisting of these closed half-spaces, and the function $\omega(G)$ returns the coefficients of the lexicographic minimum point in $\bigcap G$. Adding half-planes to H cannot decrease the value of ω, so the monotonicity condition is satisfied. As for the locality condition, note that if $\omega(G) = \omega(F) \neq -\infty$, then $\omega(G)$ is realized in a single point $x^* \in \mathbb{R}^d$. The fact that $\omega(G \cup \{h\}) > \omega(G)$ implies that $x^* \notin h$. Therefore, $\omega(F \cup \{h\}) > \omega(F)$ as needed. An immediate consequence of the monotonicity and locality conditions is the following.

COROLLARY 2.4. *Let* (H, ω) *be an LP-type problem. For all* $F \subseteq G \subseteq H$*, with* $\omega(G) = \omega(F) \neq -\infty$*, and for each* $h \in H$*,* $\omega(G \cup \{h\}) > \omega(G)$ *if and only if* $\omega(F \cup \{h\}) > \omega(F)$*.*

We give now several definitions for every abstract problem (H, ω) which meets the monotonicity condition. Let $G \subseteq H$ be arbitrary, and let $n = |H|$. If $\omega(G) = \infty$, we say G is *infeasible*; otherwise we call G *feasible*. If $\omega(G) = -\infty$, we say G is *unbounded*; otherwise we call G *bounded*. We say a constraint $h \in H$ *violates* G when $\omega(G \cup \{h\}) > \omega(G)$. (Using this definition we note that the locality condition says that, for every bounded subset $G \subseteq H$, defining set F for G, and $h \notin G$, if h violates G, then h must violate the defining set F. Corollary 2.4 says that, for any such G, F, h, h violates G if and only if it violates its defining set F.) A *basis* B is a set $B \subseteq H$, with $\omega(B') < \omega(B)$ for all proper subsets B' of B. A *basis for* G is a basis $B \subseteq G$, with $\omega(B) = \omega(G)$. (In other words, a basis for G is a minimal (by inclusion) defining set of G.) We note that due to the monotonicity condition a basis for G, for any $G \subseteq H$, always exists. The basis for any unbounded set is the empty set.

Observation 2.5. Let (H, ω) be an LP-type problem, let $G \subset H$, and let $B \subseteq G$ be such that $\omega(B) = \omega(G)$. If $\omega(G) < \omega(H)$, then there exists a constraint $h \in H \setminus G$ which violates B.

To see this, it is sufficient to show that if no $h \in H \setminus G$ violates B, then $\omega(G) = \omega(H)$. We add to B and G an arbitrary constraint $h \in H \setminus G$. Since h does not violate B, the locality condition implies that h does not violate G and that $\omega(B) = \omega(G \cup \{h\})$. Repeating this argument $|H \setminus G|$ times, we get that $\omega(G) = \omega(H)$ as needed.

So B is a basis for G if and only if $B \subseteq G$ is a basis and no element in G violates it. We say that $h \in G$ is *extreme* in G if $\omega(G \setminus \{h\}) < \omega(G)$. Thus $h \in G$ is extreme in G if and only if h violates $G \setminus \{h\}$. From the minimality of a basis, every h in a basis B is extreme in B. From the monotonicity condition we get the following.

Observation 2.6. Let (H, ω) be an LP-type problem. Every $h \in G$ which is extreme in $G \subseteq H$ is contained in every basis B for G.

In other words, a basis B for G contains all of the constraints which are extreme in G. We note that not all of the elements in B are extreme in G as seen in Figure 2.1. Let G be the set of 5 lines. The two thick lines form a basis B for G, and each one of them is extreme in B. We note that the line with negative slope is extreme in G.

The terms "violates" and "extreme" are somewhat complementary: For $h \in G$ we may ask whether h is extreme in G (or, equivalently, whether it violates $G \setminus \{h\}$). Similarly, for $h \notin G$ we may test whether h violates G (or, equivalently, whether it is extreme in $G \cup \{h\}$). Using the monotonicity condition and the observation above we get the following.

Observation 2.7. Let (H, ω) be an LP-type problem. Let B be a basis for $G \subseteq H$. If $h \notin G$ violates B, then h is extreme in $G \cup \{h\}$ and is a member of every basis for $G \cup \{h\}$.

The *combinatorial dimension* d of (H, ω) is the maximum size of every basis for any feasible subset G. An abstract problem which meets the monotonicity condition and is of combinatorial dimension d, where d is independent of $|H|$, is called *fixed-dimensional*. A d-dimensional LP-type problem where the cardinality of every basis is exactly d is called a d-dimensional *basis-regular* LP-type problem. Note that if such a problem is feasible and bounded, then $\omega(H) = \max_{G \subset H, |G|=d} \omega(G)$.

For instance, in lexicographic linear programming, if $\bigcap G \neq \emptyset$, the lexicographically smallest point in $\bigcap G$ is determined by a basis of cardinality at most d (if G is unbounded, its basis is $\emptyset$). Notice that, although more than d half-spaces may have the minimum point on their boundary, a subfamily of at most d of them is sufficient to determine the minimum. In Figure 2.1 below, the thick two lines are a basis. Notice also that a subfamily G may have more than one basis.

2.2. LP-type algorithms. An LP-type algorithm takes a d-dimensional LP-type problem (H, ω) and returns a basis B for H. Several efficient randomized LP-type algorithms are known such as the ones of Clarkson [8], Matoušek, Sharir, and Welzl [28], or Kalai [22]. In the following two sections we review the first two algorithms. We develop in section 5 a DLP-type algorithm by combining these two algorithms together.

It is not clear, of course, what computational operations are possible on an abstract object such as (H, ω). We assume two computational primitives and analyze the various algorithms by counting the number of calls to these primitives. The running time for a specific LP-type problem then depends on how efficiently the primitives can be implemented. Let us now define the two primitive operations. A *basis computation* Basis(G) takes a family G of at most $d + 1$ constraints and finds a basis for

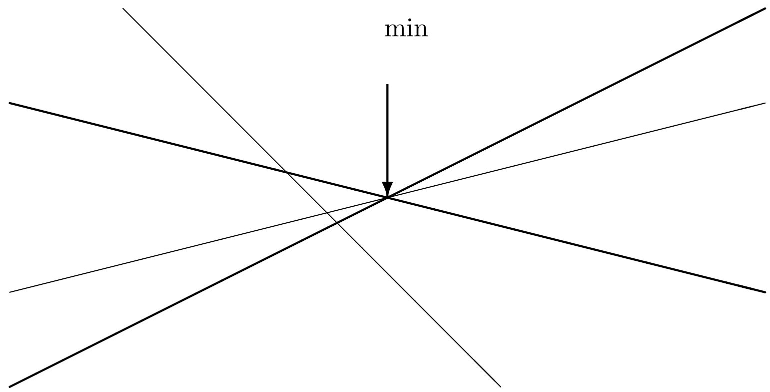

FIG. 2.1. *A basis for LP.*

G. A *violation test* Violation(B, h) takes a basis B and a constraint h and returns true if and only if h violates B (i.e., B is not a basis of $B \cup \{h\}$). Let t_b be the time required for a basis computation and t_v be the time required for a violation test.

2.2.1. Clarkson's algorithm. As originally presented, Clarkson's algorithm is aimed for linear programming. As Sharir and Welzl [31] note, the algorithm solves LP-type problems in the same time bound. We review the algorithm in the context of linear programming. Given a lexicographic linear programming problem in d variables with a set of constraints H ($|H| = n$) and objective function ω, we view it as the d-dimensional LP-type problem (ω, H).

Let x_s^* be an algorithm which gets input of size up to $9d^2$ (d is the dimension of the problem) and outputs a basis for H. The algorithm of Clarkson [8] is as follows:

Function $x_m^*(H)$ (Returns a basis for H)

1. Let $V^* := \emptyset$, let $V := H$
2. If $|H| \leq 9d^2$, then return $x_s^*(H)$
3. Else repeat the following until $V = \emptyset$:
   (a) Choose $R \subset H \setminus V^*$ uniformly at random, $|R| = d\sqrt{|H|}$
   (b) Let $B := x_i^*(R \cup V^*)$, and let $V := \{h \in H \mid \text{Violation}(B, h) = \text{TRUE}\}$
   (c) If $|V| \leq 2\sqrt{|H|}$, then let $V^* := V^* \cup V$
4. Return B

Function $x_i^*(H)$

1. Let $V := H$. For every $h \in H$ let $\nu_h := 1$
2. If $|H| \leq 9d^2$, then return $x_s^*(H)$
3. Else repeat the following until $V = \emptyset$:
   (a) Choose $R \subset H$ at random according to weights ν_h, $|R| = 9d^2$
   (b) Let $B := x_s^*(R)$.
   (c) Let $V := \{h \in H \mid \text{Violation}(B, h) = \text{TRUE}\}$
   (d) If $\nu(V) \leq 2\nu(H)/(9d - 1)$, then for every $h \in V$ let $\nu_h =: 2\nu_h$
4. Return B

As Amenta notes in [1], Clarkson's randomized algorithm for solving an LP-type problem (H, ω) improves the running time by separating the dependence on d and on n. He uses a three-level algorithm, with a "base-case" algorithm at the lowest level (x_s^*) solving subproblems of size up to $9d^2$.

The higher two levels x_m^* and x_i^* reduce the problem to smaller problems using the following idea. Take a sample $R \subseteq H$, find a basis B for R by calling the next lower level algorithm, and then find the subset $V \subseteq H$ of all of the constraints which violate B. If V is empty, Observation 2.5 tells us that B is a basis for H as well. Otherwise, by the monotonicity condition $\omega(H) > \omega(B)$. Let B' be a basis for H, and let $H' = B \cup B'$. Clearly $\omega(H) = \omega(H')$, so B' is a basis for H' as well. Applying Observation 2.5 for H' and B we get that there exists a constraint in B' which violates B. We've just proved the following lemma.

LEMMA 2.8 (Lemma 3.1 in [8]). *If the set V is nonempty, then it contains at least one constraint from every basis B for H.*

The purpose of the top level x_m^* is to get the number of constraints down so we can apply the second level (x_i^*), which is more efficient in d but less efficient in n. In the top level we take a random sample R, with $|R| = d\sqrt{n}$ so that $E[|V|] = O(\sqrt{n})$; that is, we take a big random sample which gives an expected small set of violators. This is a consequence from the following lemma.

LEMMA 2.9 (Lemma 3.2 in [8]). *Let $V^* \subset H$, and let $R \subset H \setminus V^*$ be a random subset of size r, with $|H \setminus V^*| = n$. Let $V \subset H$ be the set of constraints which violate $R \cup V^*$. Then the expected size of V is no more than $d(n - r + 1)/(r - d)$.*

We iterate, keeping the violators in a set V^* and finding a basis B' for $R \cup V^*$. At every iteration in the "repeat-until" loop of x_m^*, we add the violators to V^*, so that after d iterations V^* contains a basis B for H and $E[|V^*|] = d\sqrt{n}$. Solving the subproblem on V^* then gives the answer.

All recursive calls from the first level x_m^* call the second level algorithm x_i^*, which uses small random samples of size $9d^2$. Initially the sample R is chosen using the uniform distribution, but then we double the weights of elements in V and iterate. Since at least one basis element always ends up in V, eventually they all become so heavy that we get $B \subseteq R$. The analysis shows that the expected number of samples before $B \subseteq R$ is $O(d \log n)$. Since we need $O(n)$ work at each iteration to compare each constraint with the basis B' of R, without the first phase this algorithm alone would be $O(n \log n)$. All of the recursive calls from this reweighting algorithm are made to some "base-case" algorithm x_s^*.

Recall that t_v is the time required for a violation test, and let $t_s(n)$ be the time required for function x_s^* to run on n constraints. In his paper, Clarkson chooses x_s^* to be the simplex algorithm for linear programming on sets of $9d^2$ elements and estimates its running time by $t_s(9d^2) = d^{\frac{d}{2}+O(1)}$, using Stirling's approximation. Given a set H of n elements and a basis B (which in linear programming is equivalent to a point in $\mathbb{R}^d$), the time needed for a single call to function Violation(B, h) is d. Thus the time needed to execute the line

$$V \leftarrow \{h \in H \mid \text{Violation}(B, h) = \text{TRUE}\}$$

in the algorithm is dn, or nt_v.

In his time complexity analysis, Clarkson also uses a lemma to show that progress will be made during the execution of the algorithm. We say that an execution of the loop in x_m^* (x_i^*) is *successful* if the test $|V| \leq 2\sqrt{|H|}$ ($\nu(V) \leq 2\nu(H)/(9d-1)$) returns "true."

LEMMA 2.10 (Lemma 3.3 in [8]). *The probability that any given execution of a loop body is successful is at least 1/2, and so on average two executions are required to obtain a successful one.*

Let $T_i(n)$ ($T_m(n)$) be the expected time required by x_i^* (x_m^*) for a problem with n constraints.

THEOREM 2.11 (Theorem 3.4 in [8]). *Given an LP-type problem, the iterative algorithm x_i^* requires*

$$T_i(n) = O(d \log n(nt_v + t_s(9d^2)))$$

expected time, where the constant factors do not depend on d.

THEOREM 2.12 (Theorem 3.5 in [8]). *Given an LP-type problem, algorithm x_m^* requires*

$$T_m(n) = O(d(T_i(d\sqrt{n}) + nt_v)) = O(d^2 \log n(\sqrt{n}t_v + t_s(9d^2)) + dnt_v)$$

expected time, where the constant factors do not depend on d.

We can see Clarkson's algorithm as a tool for reducing an LP-type problem with many constraints to a collection of small problems with a few constraints.

2.2.2. Sharir and Welzl's algorithm. As Amenta notes in [1], the algorithm of Sharir and Welzl [31] for solving an LP-type problem (H, ω) is a *monotone algorithm*; i.e., the sequence of values resulted by the calls the algorithm makes to the basis calculation primitive is monotone increasing. The idea is to select a random constraint $h \in H$ and recursively find a basis B for $H \setminus \{h\}$. If h doesn't violate B, then output B; otherwise solve the problem recursively starting from a basis for $B \cup \{h\}$. Although the statement of the algorithm does not include a set of tight constraints (i.e., the set of constraints which the current minimum must satisfy), Observation 2.7 demonstrates that every basis found in the recursive call will include h. So the dimension of the problem is effectively reduced. They show that the algorithm requires expected $O(n)$ calls to the Basis primitive on subproblems with $d + 1$ constraints and $O(n)$ calls to the Violation primitive, when the constant depends exponentially on d.

For the sake of completeness we state their algorithm. Function lptype is called with an initial basis C which they call a *candidate basis*. C is not necessarily a basis for H. It can be viewed as some auxiliary information one gets for the computation of the solution. Note that C can have influence on the running time and output of the algorithm (e.g., when there are several optimal bases).

Function lptype(H, C)

1. If $H = C$, then return C
2. Else
   (a) Choose $h \in H \setminus C$ uniformly at random
   (b) Let $B :=$ lptype$(H \setminus \{h\}, C)$
   (c) If Violation$(B, h) =$ TRUE, then return lptype$(H, \text{Basis}(B \cup \{h\}))$
   (d) Else return B

Matoušek, Sharir, and Welzl [28] cite explicitly all of the properties which are needed for the correctness and time analysis of their algorithm

LEMMA 2.13 (see [28]). *Let (H, ω) be an abstract problem. The correctness and time analysis of algorithm lptype applied on (H, ω) as described in [28] are valid, if for all $F, G \subseteq H$, $F \subseteq G$, and $h \in H$:*

1. $\omega(G) \geq \omega(F)$.
2. *If* $\omega(G) = \omega(F) > -\infty$*, then* h *violates* G *if and only if* h *violates* F.
3. *If* $\omega(G) < \infty$*, then any* $F \subseteq G$ *has at most* d *extreme constraints.*
4. *If* $\omega(G) < \infty$*, then every basis* $B \subseteq G$ *for* G *has exactly* d *constraints.*

We note that a d-dimensional basis-regular LP-type problem (H, ω) satisfies all of the above properties: The monotonicity condition yields property 1. Corollary 2.4 yields property 2, the d-dimensionality of the (H, ω) together with Observation 2.6 yield property 3, and property 4 (which is needed only for the time analysis) results because (H, ω) is basis-regular.

A simple inductive argument shows that the procedure returns the required answer. This happens after a finite number of steps, since the first recursive call decreases the number of constraints, while the second call increases the value of the candidate basis (and there are only finitely many different bases).

Recall that t_v denotes the time required for a violation test and t_b denotes the time required for the Basis primitive on subproblems with $d+1$ constraints. Let n_v (n_b) be the number of violation tests (basis computations) performed throughout the execution of the algorithm. Matoušek, Sharir, and Welzl (see section 4 in [28]) show that $n_v \leq n_b n$, which implies a crude upper bound of $O(n_b(t_v n + t_b))$ for the running time of the algorithm. They [28] give a careful and complicated analysis of this algorithm for the case where n is not much larger than d (e.g., $d \leq n \leq \sqrt{d}e^{d/4}$) and show that $n_b = e^{O(\sqrt{d \ln d})}$. Hence, for this case, the algorithm of [31] runs in randomized $O(e^{O(\sqrt{d \ln d})}(t_v n + t_b))$ time, i.e., subexponential in the dimension d of the problem. (Actually they use property 4 only for showing the subexponential bound in d.) Since, for linear programming, both the violation test and the basis calculation can be performed in time polynomial in both n and d, this gives a subexponential randomized algorithm for linear programming. Using this as the base-case algorithm at the third level of Clarkson's algorithm (i.e., x_s^*) gives expected $O(e^{O(\sqrt{d \ln d})} \log n)$ basis computations and expected $O(dn + d^2 \log n e^{O(\sqrt{d \ln d})}))$ violation tests. When d is constant, the running time of the combined algorithm is $O(t_v n + t_b \log n)$. We will use this expression in the analysis of the running times of many of our applications.

2.3. Helly-type theorems and their relations to LP-type problems. The first works to systematically study the relations between Helly-type theorems and LP-type problem were those of Amenta [1, 2]. In this subsection we summarize her results.

An LP-type problem (H, ω) with combinatorial dimension k is an abstract problem with combinatorial dimension k such that ω obeys monotonicity. Therefore the theorem below implies that there is a Helly-type theorem corresponding to the constraint set of every fixed-dimensional LP-type problem.

THEOREM 2.14 (see [2]). *Let* (H, ω) *be an abstract problem with combinatorial dimension* k *such that* ω *obeys monotonicity, and let* $\lambda \in \Lambda$ *be arbitrary.* H *has the property* $\omega(H) \leq \lambda$ *if and only if every* $G \subseteq H$ *with* $|G| \leq k+1$ *has the property* $\omega(G) \leq \lambda$.

The main theorem in [2] goes in the other direction. Before stating it we need some definitions.

A *set system* is a pair (X, H), where X is a set and H is a set consisting of subsets of X. We say (X, H) is a *Helly system* if there exists a finite integer k such that H has Helly number k with respect to the intersection predicate. Most Helly theorems can be restated in terms of the intersection predicate. For example, let us consider the following Helly-type theorem.

THEOREM 2.15 (radius theorem). *A family H of points in the Euclidean d-dimensional space E^d is contained in a unit ball if and only if every $d+1$ or fewer points from H are contained in a unit ball.*

Here the family of objects is the set of points in E^d, the predicate is that a subfamily is contained in a (closed) unit ball, and the Helly number is $d+1$. In order to restate this theorem in terms of the intersection predicate, we apply the following duality transformation. We transform every point $h \in H$ into the set $\mathcal{D}(h)$ of centers of unit balls containing h. In this way $\mathcal{D}(h)$ is a unit ball centered at h. Let $\mathcal{D}(H) = \{\mathcal{D}(h) \mid h \in H\}$. From the definition of this duality transformation we get that the points in H are contained in a unit ball if and only if the unit balls in $\mathcal{D}(H)$ have a nonempty intersection (see Figure 2.2). Since balls are a special case of convex sets, the radius theorem derives directly from Helly's theorem.

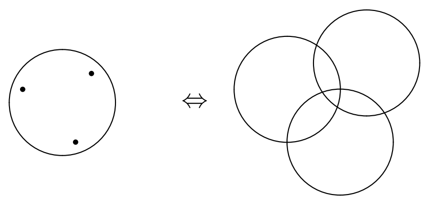

FIG. 2.2. *The 3 points on the left side are contained in a unit ball if and only if the 3 unit balls on the right side intersect.*

Recall that the range Λ of an LP-type problem can be any totally ordered set, and let (X, H) be a set system. We call $\omega' : X \to \Lambda$ a *ground set objective function.* We call $\omega : 2^H \to \Lambda$ the objective function *induced by* ω' on (X, H) if, for every $G \subseteq H$, $\omega(G)$ is the least value $\lambda^* \in \Lambda$ for which there exists $x^* \in \bigcap G$ such that $\omega'(x^*) = \lambda^*$, i.e., $\omega(G) = \min\{\omega'(x) \mid x \in \bigcap G\}$. If $\bigcap G = \emptyset$, we define $\omega(G) = \infty$. For example, when formulating lexicographic linear programming in the LP-type framework, the value of ω on a subset G of constraints is the minimum value that the ground set objective function ω' achieves on the points that are feasible with respect to G.

A *mathematical programming problem* is a triple (X, H, ω'), where X is a ground set, H is a set of subsets of X, and ω' is a ground set objective function to a totally ordered set Λ. We call the pair (H, ω), where ω is the objective function induced by ω' on (X, H), the *induced abstract problem.* If $|\{t \in \bigcap G \mid \omega'(t) = \omega(G)\}| = 1$ for all $G \subseteq H$, then we say that ω' satisfies the unique minimum condition (UMC).

Let $(X \times \Lambda, \bar{H})$ be a set system where Λ is a totally ordered set which contains a maximal element ∞. We call a ground set objective function ω' a *natural ground set objective function* if, for all $(x, \lambda) \in X \times \Lambda$, $\omega'(x, \lambda) = \lambda$. We call an objective function ω *natural* if it is induced by a natural ground set objective function. For every particular constraint $\bar{h} \in \bar{H}$ and $\lambda \in \Lambda$ we write $h_\lambda = \{x \in X \mid \exists \nu \leq \lambda \text{ s.t. } (x, \nu) \in \bar{h}\}$ for the projection into X of the part of $\bar{h}$ with Λ-coordinate no greater than λ. Also, for a subfamily of constraints $\bar{G} \subseteq \bar{H}$, we write G_λ as shorthand for $\{h_\lambda \mid \bar{h} \in \bar{G}\}$. We call an indexed family of subsets $\{h_\lambda \mid \bar{h} \in \bar{G}\}$, such that $h_\alpha \subseteq h_\beta$, for all $\alpha, \beta \in \Lambda$ with $\alpha < \beta$, a *nested family.*

Figure 2.3 (based upon Figure 1 in [2]) is a schematic diagram of a parameterized Helly system. The whole stack represents $X \times \Lambda$, and each of the cones represents a set $\bar{h} \in \bar{H}$. Each $\bar{h}$ is a subset of $X \times \Lambda$. Since all of the $\bar{h}$ are indexed with respect to Λ, the cross section at λ (represented by one of the planes) is equivalent to the Helly system (X, H_λ). Notice that if $\bar{G} \subseteq \bar{H}$ does not intersect at some value λ_2, then

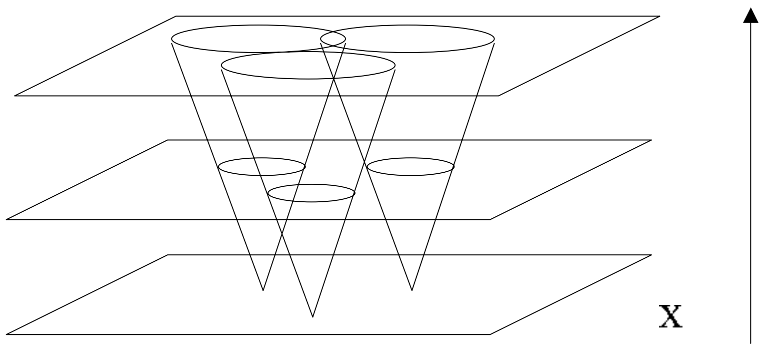

FIG. 2.3. *A parameterized Helly system.*

$\bar{G}$ also fails to intersects at all $\lambda_1 < \lambda_2$, and if $\bar{G} \subseteq \bar{H}$ intersects at λ_1, then $\bar{G}$ also intersects at all $\lambda_2 > \lambda_1$.

In her paper, Amenta [2] relates Helly-type theorems and LP-type problems by parameterization (a similar parameterization appears in [27] under the name "concrete LP-type problem").

DEFINITION 2.16. *A set system* $(X \times \Lambda, \bar{H})$ *is a* parameterized Helly system *with Helly number k, when*

1. $\{h_\lambda \mid \lambda \in \Lambda\}$ *is a nested family for all* $\bar{h} \in \bar{H}$*;*
2. (X, H_λ) *is a Helly system, with Helly number k for all* λ*.*

So the function ω' is just the projection into the Λ coordinate, and, for $\bar{G} \subseteq \bar{H}$, $\omega(\bar{G}) = \min\{\lambda \mid \bigcap G_\lambda \neq \emptyset\}$, or $\omega(\bar{G}) = \infty$ if $\bar{G}$ does not intersect at any value of λ.

Amenta [1] notes that it is almost always useful to think of Λ as time, so that a subfamily G_λ is a "snapshot" of the situation at time λ. Usually we can think of some initial time 0 at which G_0 does not intersect and then envision the h_λ growing greater with time, so that $\lambda^* = \omega(\bar{G})$ is the first "moment" at which G_λ intersects.

As an example, let us consider how the Helly system (X, H) for the radius theorem can be extended to a parameterized Helly system. (Recall that the ground set X of the Helly system representing the radius theorem is the set of centers of unit balls in E^d (which is equivalent to $\mathbb{R}^d$) and that each $h = h(p) \in H$ is the set of centers of unit balls which contain point p; i.e., $h(p)$ is a unit ball centered at p.) We define a parameterized Helly system $(X \times \Lambda, \bar{H})$, where $\Lambda = \mathbb{R}^+$ is the set radii, and each $h_\lambda = h(p)_\lambda \in H_\lambda$ is the set of centers at which a ball of radius at most λ contains a particular point p. The nested family $\bar{h} = \bar{h}(p)$ is the set of *all* balls containing p. The ground set $X \times \Lambda$ is the set of all balls in E^d, and $\bar{H}$ is the family of nested families for all points (see Figure 2.3).

The natural objective function for this parameterized Helly system $\omega(\bar{G})$ returns the smallest radius at which there is a ball containing all of the points corresponding to constraints $\bar{h} \in \bar{G}$. So $(X \times \Lambda, \bar{H}, \omega')$ is the following mathematical programming problem:

Problem: Smallest enclosing ball
Input: *A finite family* H *of points in* E^d*.*
Output: *The smallest ball enclosing* H*.*

In Figure 2.3 we see the parameterized Helly system corresponding to an instance H of the smallest enclosing ball problem consisting of 3 points. Each nested family $\bar{h}$ is a cone whose base is a point from H.

We say a ball is *realized* by points of H if it is the smallest volume ball enclosing the points on its boundary. Assuming that the points in H are at general positions, such that no two different congruent balls are realized by points of H, the theorem below implies that the smallest enclosing ball in E^d problem can be formulated as a d-dimensional LP-type problem $(\bar{H}, \omega)$.

THEOREM 2.17 (main theorem in [2]). *Let $(X \times \Lambda, \bar{H})$ be a parameterized Helly system with Helly number k, natural ground set function ω', and natural objective function ω. If ω' meets the UMC, then $(\bar{H}, \omega)$ is an LP-type problem of combinatorial dimension k.*

Amenta showed that, without requiring the UMC, the theorem is not correct by giving an example of a Helly system with no fixed combinatorial dimension [2]. The theorem above is applied in [2] to get linear time solution algorithms for various geometric problems.

In her paper [2], Amenta investigates lexicographic objective functions. Let $(X \times \Lambda, \bar{H})$ be a parameterized Helly system with Helly number k and natural objective function ω. For all $\lambda \in \Lambda$, we assume a function $\nu_\lambda : 2^{H_\lambda} \to \Lambda'$, where Λ' is a totally ordered set containing a maximal element ∞, such that (H_λ, ν_λ) is an LP-type problem of combinatorial dimension at most d. The functions ν_λ may themselves be lexicographic. Amenta [2] imposes a lexicographic order on $\Lambda \times \Lambda'$ with $(\lambda, \kappa) > (\lambda', \kappa')$ if $\lambda > \lambda'$ or if $\lambda = \lambda'$ and $\kappa > \kappa'$. She defines a lexicographic objective function $\nu : 2^{\bar{H}} \to \Lambda \times \Lambda'$ in terms of ω and the functions ν_λ as seen in the following.

THEOREM 2.18 (see [2]). *Let Λ' be a totally ordered set. If $(X \times \Lambda, \bar{H})$ is a parameterized Helly system with Helly number k and natural objective function ω, and if, for every λ, (H_λ, ν_λ) is an LP-type problem of combinatorial dimension d, where $\nu_\lambda : 2^{H_\lambda} \to \Lambda'$, then $(\bar{H}, \nu)$ is an LP-type problem of combinatorial dimension $\leq k+d$, where $\nu : 2^{\bar{H}} \to \Lambda \times \Lambda'$ is defined as $\nu(\bar{G}) = (\omega(\bar{G}), \nu_{\omega(\bar{G})}(G_{\omega(\bar{G})}))$ for all $\bar{G} \subseteq \bar{H}$.*

Certainly, this bound on the combinatorial dimension is not always tight. For d-dimensional linear programming, for instance, this theorem gives an upper bound of $2d-1$ on the combinatorial dimension, since each H_λ is the constraint set of a $(d-1)$-dimensional linear program, and $(E^d, \bar{H})$ is a parameterized Helly system with Helly number d. Nonetheless, the theorem provides the best general bound as shown in [2].

3. Dual LP-type problems.

DEFINITION 3.1. *A* dual LP-type problem *is an abstract problem (H, ω) that obeys the following conditions (when we write $<$, $\leq$, $=$ etc., we mean under the totally ordered set Λ):*

1. Monotonicity: *For all $F \subseteq G \subseteq H : \omega(F) \geq \omega(G)$ (so the special element ∞ is such that $\omega(\emptyset) = \infty$).*
2. Locality: *For all $F \subseteq G \subseteq H$, with $\omega(G) = \omega(F) \neq \infty$, and for each $h \in H$, if $\omega(G \cup \{h\}) < \omega(G)$, then $\omega(F \cup \{h\}) < \omega(F)$.*

Let $G \subseteq H$ be arbitrary. If $\omega(G) = \infty$, we say G is *infeasible*; otherwise we call G *feasible*. If $\omega(G) = -\infty$, we say G is *unbounded*; otherwise we call G *bounded*. A *basis* B is a set $B \subseteq H$, with $\omega(B') > \omega(B)$ for all proper subsets B' of B. A *basis for* G is a basis $B \subseteq G$, with $\omega(B) = \omega(G)$. We note that due to the monotonicity condition a basis for G, for every $G \subseteq H$, always exists.

The *combinatorial dimension* d of a dual LP-type problem is the maximum cardinality of every basis for any *bounded* subfamily G. We note that the basis for every infeasible set is the empty set. A dual LP-type problem of combinatorial dimension d, where d is independent of $|H|$, is called *fixed-dimensional*. We choose the term dual LP-type (which should not be confused with the term dual in linear programming) because of the following.

Observation 3.2. The abstract problem (H, ω) is a dual LP-type problem if and only if $(H, -\omega)$ is an LP-type problem.

Looking at (H, ω), in order to prevent confusion between LP-type problems and their dual versions, we will denote by (D, ω) LP-type problems and by (S, ω) dual LP-type problems. The motivation for the choice of the letters "D" and "S" is as follows. We use the letter "D" in the LP-type problem (D, ω) since we look at D as a set of *demand elements (d-elements)*, or constraints on the feasible region, on which the minimum value is $\omega(D)$. Adding demand elements to D may increase the minimum solution of the problem and will never decrease its value. We use the letter "S" in the dual LP-type problem (S, ω) since we look at S as a set of *supply elements (s-elements)*, or relaxations on the feasible region, on which the minimum value is $\omega(S)$. Adding supply elements to S may decrease the minimum solution of the problem and will never increase its value. In the next section we define *discrete LP-type problems* by using the *same* ω in a primal and a dual LP-type problem.

4. Discrete LP-type problems.

DEFINITION 4.1. *A* discrete abstract problem *is a triple* (D, S, ω)*, where* D *and* S *are finite sets of elements and* ω *is an* objective function *from* $2^D \times 2^S \setminus \{(\emptyset, \emptyset)\}$ *to some totally ordered set* Λ *which contains a special maximal (minimal) elements* ∞ $(-\infty)$*. The goal is to compute* $\omega(D, S)$*.*

DEFINITION 4.2. *Let* (D, S, ω) *be a discrete abstract problem. For every* $D', D'' \subseteq D$ *and* $S', S'' \subseteq S$ *let* $\alpha_{S'}(D'') = \omega(D'', S')$*, and let* $\beta_{D'}(S'') = \omega(D', S'')$*. We say that* (D, S, ω) *is a* discrete LP-type problem *(DLP-type, in short) when* $(D, \alpha_{S'})$ *is an LP-type problem and* $(S, \beta_{D'})$ *is a dual LP-type problem for all* $D' \subseteq D$ *and* $S' \subseteq S$*. We say that* $(D, \alpha_{S'})$ $((S, \beta_{D'}))$ *is an* induced *LP-type (dual LP-type) problem of* (D, S, ω)*.*

We note that we do not include $(\emptyset, \emptyset)$ in the domain of ω since this will result in the trivial ordered set $\Lambda = \{-\infty, \infty\}$, where $-\infty = \infty$: To see this, recall that the definition of LP-type problems implies that $\alpha_\emptyset(\emptyset) = -\infty$, the definition of dual LP-type problems implies that $\beta_\emptyset(\emptyset) = \infty$, and the definition of DLP-type problems implies that $\alpha_\emptyset(\emptyset) = \omega(\emptyset, \emptyset) = \beta_\emptyset(\emptyset)$.

Throughout this paper, whenever we call (D, α) $((S, \beta))$ *the induced* LP-type (dual LP-type) problem of (D, S, ω), we mean that $\alpha = \alpha_S$ $(\beta = \beta_D)$. It is easy to see that the following definition for a DLP-type problem is equivalent to the former one.

DEFINITION 4.3. *A* DLP-type problem *is a discrete abstract problem* (D, S, ω) *which for all* $S' \subseteq S$ *and for all* $D' \subseteq D$ *obeys the following conditions (when we write* $<, \leq, =$*, etc., we mean under the ordered set* Λ*):*

1. Monotonicity of demand: *For all* $D'' \subseteq D' \subseteq D : \omega((D'', S')) \leq \omega((D', S'))$*.*
2. Monotonicity of supply: *For all* $S'' \subseteq S' \subseteq S : \omega((D', S'')) \geq \omega((D', S'))$*.*
3. Locality of demand: *For all* $D'' \subseteq D' \subseteq D$ *such that* $\omega((D', S')) = \omega((D'', S')) > -\infty$ *and for each* $h \in D$*, if* $\omega((D' \cup \{h\}, S')) > \omega((D', S'))$*, then* $\omega((D'' \cup \{h\}, S')) > \omega((D'', S'))$*.*
4. Locality of supply: *For all* $S'' \subseteq S' \subseteq S$ *such that* $\omega((D', S')) = \omega((D', S'')) < \infty$ *and for each* $h \in S$*, if* $\omega((D', S' \cup \{h\})) < \omega((D', S'))$*, then* $\omega((D', S'' \cup \{h\})) < \omega((D', S''))$*.*

Before continuing any further, we give an example of a DLP-type problem.

Problem: Discrete point set width

Input: *A finite set* D *of points in* E^d *and a finite set* S *of permissible directions.*

Output: *The minimal width of the set in the permissible directions (i.e., the minimal width of a band with a permissible direction which contains all the points in D).*

We assume general positions of the points and directions; that is, all $|S|\ \binom{|D|}{2}$ distances (in each one of the $|S|$ permissible directions) between pairs of points are different. For every set D of points and set S of permissible directions we define $\omega(D,S)$ to be the minimal width of the points in D in the permissible directions from S. Clearly (D,S,ω) is a discrete abstract problem. Let $S' \subseteq S$. We show now that $(D,\alpha_{S'})$ is an LP-type problem for every choice of S'. Since adding points to a set can only increase its width, $(D,\alpha_{S'})$ meets the monotonicity condition. Let $D'' \subset D' \subseteq D$ be such that $\alpha_{S'}(D'') = \alpha_{S'}(D')$. Due to the general position assumption there are unique $d_1, d_2 \in D''$ and $s \in S'$ such that the width of (D'',S') and of (D',S') is the distance between d_1 and d_2 in direction s. (In other words, the width of (D',S') and of (D'',S') is the width of the band in direction s between d_1 and d_2 in which all of the points of D' lie.) If for $h \notin D'$ $\alpha_{S'}(D' \cup \{h\}) > \alpha_{S'}(D')$, then point h is not inside this band, so there must be another triple of two points and one direction which realizes the width $\alpha_{S'}(D'' \cup \{h\})$. Due to the monotonicity condition, $\alpha_{S'}(D'' \cup \{h\}) \geq \alpha_{S'}(D'')$, and from the general position assumption we get that $\alpha_{S'}(D'' \cup \{h\}) > \alpha_{S'}(D'')$, so $(D,\alpha_{S'})$ meets the locality condition as well and thus is an LP-type problem.

Let $D' \subseteq D$. It remains to show that $(S,\beta_{D'})$ is a dual LP-type problem for every choice of D'. $(S,\beta_{D'})$ satisfies the monotonicity condition since adding directions to the set of permissible directions can only decrease the width. Let $S'' \subset S' \subseteq S$ be such that $\beta_{D'}(S'') = \beta_{D'}(S')$, and let $h \notin S'$. If $\beta_{D'}(S' \cup \{h\}) < \beta_{D'}(S')$, then the width $\beta_{D'}(S' \cup \{h\})$ must be realized by a band in direction h, that is, $\beta_{D'}(S' \cup \{h\}) = \beta_{D'}(\{h\})$. Hence we must have $\beta_{D'}(S'' \cup \{h\}) = \beta_{D'}(\{h\}) < \beta_{D'}(S') = \beta_{D'}(S'')$, so $(S,\beta_{D'})$ satisfies the locality condition as well and thus is a dual LP-type problem.

We now give more definitions. Let $G = (D',S') \in 2^D \times 2^S$ be arbitrary. Throughout this paper, if not explicitly specified otherwise, we choose G such that ω is defined on G, i.e., $G \neq (\emptyset,\emptyset)$. If $\omega(G) = \infty$, we say G is *infeasible*; otherwise we call G *feasible*. If $\omega(G) = -\infty$, we say G is *unbounded*; otherwise we call G *bounded*. We extend the terms "violates" and "extreme" in a natural way: We say that a d-element $h \in D \setminus D'$ (s-element $h \in S \setminus S'$) *violates* G if h violates D' (S') in the induced LP-type problem $(D,\alpha_{S'})$ (induced dual LP-type problem $(S,\beta_{D'})$). A d-element $h \in D'$ (s-element $h \in S'$) is *extreme* in G if h is extreme in D' (in S') in its induced LP-type problem $(D',\alpha_{S'})$ (induced dual LP-type problem $(S',\beta_{D'})$). We define bases in the following natural way.

DEFINITION 4.4. *Let (D,S,ω) be a DLP-type problem, let α and β be as defined in Definition* 4.2*, and let $G = (D',S') \in 2^D \times 2^S$. $B = (B_D,B_S) \in 2^{D'} \times 2^{S'}$ is a* basis *for G in (D,S,ω) if B_D is a basis for D' in its induced LP-type problem $(D,\alpha_{S'})$, and B_S is a basis for S' in its induced dual LP-type problem $(S,\beta_{D'})$.*

We note that there always exists a basis $B = (B_D,B_S)$ for any G.

Observation 4.5. Let (D,S,ω) be a DLP-type problem, and let $G = (D',S') \in 2^D \times 2^S$. If B is a basis for G, then $\omega(B) = \omega(G)$, and no $h \in D' \cup S'$ violates B.

This follows from both monotonicity conditions. $\omega(B) = \omega(G)$ since $\omega(G) = \omega(B_D,S') \leq \omega(B_D,B_S) \leq \omega(D',B_S) = \omega(G)$. $h \in D'$ doesn't violate B since $\omega(G) = \omega(B_D,B_S) \leq \omega(B_D \cup \{h\},B_S) \leq \omega(D',B_S) = \omega(G)$, and in a similar way $h \in S'$ doesn't violate B. In order to illustrate the term "basis" let us consider the following instance of the discrete point set width problem.

Example 4.6. Let $G = (D,S)$, where $D = \{(0,0);(2,1);(1,5)\}$ and $S = \{\text{horizontal, vertical}\}$, be an instance of the discrete point set width problem. The minimal width is achieved by a vertical strip of width 2. Let (D,α) and (S,β) be its induced LP-type and induced dual LP-type problems, respectively. At first glance one may be tempted to suggest $B=(B_D,B_S)=(\{(0,0);(2,1)\},\{\text{vertical}\})$ as a basis for G, since $\omega(B)=\omega(G)$. This B is not a basis for G, since B_D is not a basis for (D,α) (because of the horizontal direction: $\alpha(B_D)=\omega(B_D,\{\text{horizontal}\})=1\neq\omega(B_D\cup\{(1,5)\},\{\text{horizontal, vertical}\}))$. The other subsets of (D,S) on which the value of ω is 2 are (D,S) and $(D,\{\text{vertical}\})$. (D,S) fails to be a basis for G because S is not a basis in (S,β) $(\beta(S\setminus\{\text{horizontal}\})=\beta(S))$. It is easy to verify that D is a basis for D in (D,α) and $\{\text{vertical}\}$ is a basis for S in (S,β). Thus, $(D,\{\text{vertical}\})$ is a basis for G.

A "discrete" version of Observation 2.5 is as follows.

Observation 4.7. Let (D,S,ω) be a DLP-type problem. Let $G=(D',S')\in 2^D\times 2^S$, and let $B=(B_D,B_S)\in 2^{D'}\times 2^{S'}$ be such that $\omega(B)=\omega(G)$. If $\omega(B)\neq\omega(D,S)$, then there exists an element in either $D\setminus D'$ or $S\setminus S'$ which violates B.

To see this suppose first that the inequality is $\omega(B)<\omega(D,S)$. Considering the induced LP-type problem (D,α_{B_S}), and since $\omega(D,S)\leq\omega(D,B_S)$, this implies that $\alpha_{B_S}(B_D)<\alpha_{B_S}(D)$. Applying Observation 2.5 on (D,α_{B_S}), D', and B_D, we get that there exists $h\in D\setminus D'$ that violates B_D in (D,α_{B_S}). Hence h violates B. The case where the inequality is $\omega(B)>\omega(D,S)$ is treated similarly by considering the induced LP-type problem (S,β_{B_D}).

Corollary 4.8. *Let (D,S,ω) be a DLP-type problem. Let $G=(D',S')\in 2^D\times 2^S$, and let $B\in 2^{D'}\times 2^{S'}$ be a basis for G. If no $h\in(D\setminus D')\cup(S\setminus S')$ violates B, then B is a basis for (D,S) as well.*

Proof. We need to prove that B_D is a basis for D in the induced problem (D,α) and that B_S is a basis for S in the induced problem (S,β). We prove the first part. The proof of the second part is similar. We first show that B_D is a basis in (D,α). Let B'_D be a proper subset of B_D.

$$\alpha(B'_D)=\omega(B'_D,S)\leq\omega(B'_D,S')<\omega(B_D,S')=\omega(B_D,B_S)=\omega(B_D,S)=\alpha(B_D). \tag{4.1}$$

The first inequality follows from monotonicity of supply, the second (strict) inequality follows from the fact that B is a basis for G, and therefore B_D is a basis in $(D',\alpha_{S'})$, the following equality is due to the fact that B is a basis for G, and the next equality is due to Observation 2.5 applied on (S,β_{B_D}) (B_S is a basis for S' in this dual LP-type problem). It remains to show that $\alpha(B_D)=\alpha(D)$. From (4.1) we have $\alpha(B_D)=\omega(B_D,S)=\omega(B_D,B_S)$. We conclude by deriving from Observation 4.7 that $\omega(B_D,B_S)=\omega(D,S)=\alpha(D)$. ☐

We now define a condition sufficient for a DLP-type problem (D,S,ω) to satisfy a discrete version of Observation 2.7. This condition is used in the proof of correctness of our DLP-type algorithms.

Definition 4.9. *We say that the DLP-type problem (D,S,ω) satisfies the violation condition (VC) if for every $(D',S')\in 2^D\times 2^S$ and $(D'',S'')\in 2^{D'}\times 2^{S'}$ with $\omega(D',S')=\omega(D'',S'')$ the following properties hold:*

1. *For every $h\in D$, if $\omega(D''\cup\{h\},S'')>\omega(D'',S'')$, then $\omega(D'\cup\{h\},S')>\omega(D',S')$;*
2. *for every $h\in S$, if $\omega(D'',S''\cup\{h\})<\omega(D'',S'')$, then $\omega(D',S'\cup\{h\})<\omega(D',S')$.*

Note that due to Corollary 2.4 this condition is always satisfied whenever either $S'=S''$ or $D'=D''$. The lemma below is a discrete version of Observation 2.7.

LEMMA 4.10. *Let (D, S, ω) be a DLP-type problem which satisfies the violation condition. Let $G = (D', S') \in 2^D \times 2^S$, and let $B = (B_D, B_S) \in 2^{D'} \times 2^{S'}$ be a basis for G. If $h \in D$ ($h \in S$) violates B, then h is extreme in $(D' \cup \{h\}, S')$ ($(D', S' \cup \{h\})$) and is a member of every basis for this set.*

Proof. We will prove the case where $h \in D$. The proof for $h \in S$ is similar. If $h \in D$ violates B, i.e., $\omega(B_D \cup \{h\}, B_S) > \omega(B)$, then due to the VC

$$\omega(D' \cup \{h\}, S') > \omega(B) = \omega(D', S'), \tag{4.2}$$

so h is extreme in $(D' \cup \{h\}, S')$. To see that h is a member for every basis $B' = (B'_D, B'_S)$ for $(D' \cup \{h\}, S')$, we use the fact that B'_D is a basis for the induced LP-type problem of $(D' \cup \{h\}, S')$ (so $\omega(B') = \omega(B'_D, S')$) and (4.2) to get

$$\omega(B'_D, S') = \omega(D' \cup \{h\}, S') > \omega(D', S'). \tag{4.3}$$

We conclude the proof by noting that if h is not a member in B'_D, then $B'_D \subseteq D'$ and due to monotonicity of demand $\omega(B'_D, S') \leq \omega(D', S')$, in contradiction to (4.3). ∎

The *demand combinatorial dimension* k_D (d-dimension, in short) of (D, S, ω) is the combinatorial dimension of its induced LP-type problem. A DLP-type problem of d-dimension k_D, where k_D is independent of $|D| + |S|$, is called *fixed d-dimensional*. We define the terms *supply combinatorial dimension* (s-dimension, in short) and *fixed s-dimensionality* analogously. We call a DLP-type problem which is both fixed s-dimensional (of dimension k_S) and fixed d-dimensional (of dimension k_D) (k_D, k_S)-dimensional. A (k_D, k_S)-dimensional DLP-type problem where both its induced LP-type problem and induced dual LP-type problem are basis regular is called a (k_D, k_S)-dimensional *basis-regular* DLP-type problem.

We note that the discrete point set width problem is not fixed-dimensional. To see this, suppose by negation that it is k-d-dimensional. Consider an instance of the problem with $n = 2k$ d-elements, consisting of k pairs of antipodal points which are located on a unit circle. Let the s-elements be the n directions perpendicular to the one-unit length segments connecting the antipodal points. Clearly, each proper subset of the d-elements admits a width of less than one unit, whereas the width of the whole set is one unit. Therefore, the number of d-elements in any basis is at least $2k$, in contradiction to our assumption that the problem is k-d-dimensional.

If the problem were fixed-dimensional, the DLP algorithms stated in the next section would solve the problem in (randomized) linear time. The variable dimensionality of the problem is not surprising, since the problem admits an $\Omega(n \log n)$ (deterministic) lower bound under the algebraic computation tree model due to a linear time reduction from:

Problem: Set equality
Input: *Sets A and B of n real numbers each.*
Output: *"true" if and only if $A = B$.*

LEMMA 4.11 (see [3]). *Solving set equality requires $\Omega(n \log n)$ operations under the algebraic computation tree model.*

LEMMA 4.12. *Solving discrete point set width requires $\Omega(n \log n)$ operations under the algebraic computation tree model.*

Proof. Consider without loss of generality two sets A and B of n positive numbers each and a unit circle with center at the origin. We construct from A and B an instance (D, S) of discrete point set width. The numbers of A are transformed into points in D, and the numbers of B are transformed into directions in S as follows. We transform each number $a \in A$ into the two intersection points of the unit circle with the line with slope a that passes through the origin. We transform each number $b \in B$ into the direction vertical to a line with slope b. It is easy to see that the solution of the instance (D, S) of the discrete point set width is 1 if and only if $A \equiv B$. □

We now define a condition sufficient for a DLP-type problem (D, S, ω) to be fixed s-dimensional.

DEFINITION 4.13. *Let* (D, S, ω) *be a discrete abstract problem, and let* $p \in \mathbb{N}$. *We say that* (D, S, ω) *is a* p-supply problem *if for every* $G = (D', S') \in 2^D \times 2^S$ *there exists* $S'' \subseteq S'$ *such that* $|S''| \leq p$ *and* $\omega(D', S'') = \omega(G)$.

LEMMA 4.14. *A DLP-type problem* (D, S, ω) *which is a* p*-supply problem is* p*-*s*-dimensional.*

Proof. Let (S, β) be its induced dual LP-type problem. Suppose by negation that there exists a bounded $S' \in 2^S$ and a basis B for S' with $|B| > p$. From the definition of a basis in dual LP-type problems, for every proper subset $B' \subset B$, $\beta(B') > \beta(B) = \beta(S')$. This contradicts the fact that (D, S, ω) is a p-supply problem. □

Integer programming can be formulated as a DLP-type problem where D is a set of half-hyperplanes and $S = \mathbb{Z}^k$. There is one problem with this formulation: The set S is not finite. We can overcome this by noting that, when given an instance of an IP problem, it is always possible to bound the integer lattice by a big box (whose radius depends exponentially on the input size), such that the solution of the IP problem, if it exists, is found inside the bounding box (see, for example, Theorem 17.2 in [30]). Because of the above, solving IP by the DLP-type model is not efficient.

5. DLP-type algorithms. Given an instance (D, S, ω) of a (k_D, k_S)-dimensional DLP-type problem, let $n = |D|$ and $m = |S|$. Similarly to the assumptions made with the LP-type model, we assume two primitive operations. A *basis computation* Basis(D', S') takes an ordered pair $G = (D', S')$, with $|D'| \leq k_D + 1$ and $|S'| \leq 9k_S^2$, and finds a basis for G. A *violation test* Violation(B, h) takes a basis B and a constraint h and returns true if and only if h violates B. Let t_b be the time required for a basis computation and t_v be the time required for a violation test.

We observe that, when changing (by deleting or adding elements) the set D (S) while keeping the set S (D) unchanged, the problem behaves like an LP-type (dual LP-type) problem. Thus, while "fixing" the set S (D) one can use LP-type algorithms in order to solve the induced LP-type (dual LP-type) problem on D (S).

In Chapter 6 in [17] we have developed several randomized algorithms that solve fixed-dimensional DLP-type problems that satisfy the VC in linear time. The algorithms differ in the choice of the LP-type algorithms used to solve the induced LP-type and dual LP-type problems and in the decision rules when and with which input to call these algorithms.

The 4-layer algorithm given below uses this observation. In the first layer, i.e., in Function DLP, the set of s-elements does not change, so Function DLP resembles Function x_m^* in Clarkson's algorithm [8] applied on the induced LP-type problem. In the second layer, i.e., in Function M, the set of d-elements does not change, so Function M (as well as its name) resembles Function x_m^* in Clarkson's algorithm [8] applied on the induced dual LP-type problem. The purpose of Function DLP (Function M) is to get the number of constraints (relaxations) down, so we can apply the third level Function I, which resembles (as well as its name) Function x_i^* in [8] and is more efficient in k_S but less efficient in $|D|$ and $|S|$. The fourth layer Function Demand is called only when the cardinality of the s-element set is bounded by $9k_S^2$ and it behaves similarly to Sharir and Welzl's algorithm [31], applied on the induced LP-type problem.

Function DLP(D,S)

1. Let $V^* := \emptyset$, let $V := D$, and find a candidate basis C_D for D in the induced LP-type problem of (D,S,ω)
2. If $|D| \le 9k_D^2$, then return $\mathrm{M}(D,S,C_D)$
3. Else repeat the following until $V=\emptyset$:
   (a) Choose $R \subset D \setminus V^*$ uniformly at random, $|R| = k_D\sqrt{|D|}$
   (b) Find a candidate basis C_D for $R \cup V^*$ in the induced LP-type problem of $(R \cup V^*, S, \omega)$
   (c) Let $B := \mathrm{M}(R \cup V^*, S, C_D)$, and let $V := \{d \in D \mid \mathrm{Violation}(B,d) = \mathrm{TRUE}\}$
   (d) If $|V| \le 2\sqrt{|D|}$, then let $V^* := V^* \cup V$
4. Return B

Function M(D,S,C_D)

1. Let $V^* := \emptyset$, let $V := S$
2. If $|S| \le 9k_S^2$, then return $\mathrm{Demand}(D,S,C_D)$
3. Else repeat the following until $V=\emptyset$:
   (a) Choose $R \subset S \setminus V^*$ uniformly at random, $|R| = k_S\sqrt{|S|}$
   (b) Let $B := \mathrm{I}(D, R \cup V^*, C_D)$, and let $V := \{s \in S \mid \mathrm{Violation}(B,s) = \mathrm{TRUE}\}$
   (c) If $|V| \le 2\sqrt{|S|}$, then let $V^* := V^* \cup V$
4. Return B

Function I(D,S,C_D)

1. For every $s \in S$ let $\nu_s := 1$
2. If $|S| \le 9k_S^2$, then return $\mathrm{Demand}(D,S,C_D)$
3. Else repeat the following until $V=\emptyset$:
   (a) Choose $R \subset S$ at random according to weights ν_s, $|R| = 9k_S^2$
   (b) Let $B := \mathrm{Demand}(D,R,C_D)$
   (c) Let $V := \{s \in S \mid \mathrm{Violation}(B,s) = \mathrm{TRUE}\}$
   (d) If $\nu(V) \le 2\nu(S)/(9k_S - 1)$, then for every $s \in V$ let $\nu_s =: 2\nu_s$
4. Return B

Function Demand(D,S,C_D)

1. If $D = C_D$, then return $\mathrm{Basis}(C_D,S)$
2. Else
   (a) Choose a random $d \in D \setminus C_D$
   (b) Let $B = (B_D, B_S) := \mathrm{Demand}(D \setminus \{d\}, S, C_D)$
   (c) If $\mathrm{Violation}(B,d) = \mathrm{TRUE}$, then return $\mathrm{Demand}(D, S,$ the first coordinate of $\mathrm{Basis}(B_D \cup \{d\}, S))$
   (d) Else return B

We will first show that Function Demand returns the required answer by showing that all of the arguments of [28] apply here as well. We can view Function Demand applied on the DLP-type problem (D, S, ω) as a function applied on its induced LP-type problem (D, α). This is true since the s-elements set S does not change throughout the execution of Function Demand. In this way all of the conditions stated in Lemma 2.13 are satisfied.

Function Demand is similar, but not identical, to Function lptype of Sharir and Welzl, only because of line 2(c). Due to Corollary 2.4, the violation test in Function lptype, Violation(B, h), returns true if and only if B is not a basis for H. If in Function Demand we called Function Violation$((B_D, S), d)$ instead of calling Function Violation(B, d), then we would get *exactly* Function lptype applied on the induced LP-type problem (D, α) (but the running time would increase by a big constant depending on k_S). Because of this difference we need to prove Lemma 5.1.

LEMMA 5.1. *Let (D, S, ω) be a (k_D, k_S)-dimensional DLP-type problem which meets the VC, and suppose $\omega(D, S) = \omega(B) > -\infty$. Let $B = (B_D, B_S)$ be a basis for $(D \setminus \{d\}, S)$. Let (D, α) be the induced LP-type problem of (D, S, ω). The violation test Violation(B, d) in Function Demand applied on (D, S, ω) returns true if and only if B_D is not a basis for D in (D, α).*

Proof. If d does not violate B, then, due to Corollary 4.8, B is a basis for (D, S). Hence, B_D is a basis for D in (D, α). If d does violate B, then due to the VC we get that $\alpha(D) > \alpha(D \setminus \{d\}) = \alpha(B_D)$, which implies that B_D is not a basis for D. ∎

So this lemma implies that Function Demand correctly computes a basis for (D, S) whenever (D, S, ω) meets the VC. We now compute t_D, the time needed for Function Demand to run. Let t_{vD} (t_{bDS}) be the time required for the violation test Violation(B, d) (the basis calculation Basis$_{DS}$). Using the analysis in [28], Function Demand calls Functions Basis$_{DS}$ and Violation $O(|D|)$ times where the constant depends (exponentially) on k_D, so the running time of Function Demand is

$$t_D = O(|D|(t_{vD} + t_{bDS})). \tag{5.1}$$

If the violation test and basis calculation are done in constant time, Function Demand runs in $O(n)$ time.

We next show that Functions M and I return the required value. In order to prove this we need to show that Lemmas 2.8, 2.9, and 2.10 and Theorem 2.11 can be modified for the DLP-type framework. We also rely, of course, on the correctness of Function Demand. Lemma 2.10 and Theorem 2.11 are straightforwardly adapted to the DLP-type case. We provide proofs for the first 2 lemmas.

LEMMA 5.2 (adaptation of Lemma 3.1 in [8]). *In Functions M and I, if the set V is nonempty, and if (D, S, ω) satisfies the VC, then V contains an element from B'_S, where $B' = (B'_D, B'_S)$ is any basis of (D, S).*

Proof. We prove the correctness of Function M. The proof for Function I is similar. Let $S^* = R \cup V^*$, and let $B = (B_D, B_S)$ be a basis for (D, S^*). Let NV be the set of s-elements in $S \setminus S^*$ that do not violate B; i.e., S decomposes into $S = S^* \uplus NV \uplus V$. If V is not empty, then there exists $s \in V$ such that $\omega(B_D, B_S \cup \{s\}) < \omega(B)$. Hence, since (D, S, ω) satisfies the VC and s violates B, it also violates (D, S^*), that is, $\omega(D, S^* \cup \{s\}) < \omega(B)$. From the monotonicity of supply condition we get

$$\omega(D, S) < \omega(B). \tag{5.2}$$

None of $s \in NV$ violates B, so, by Corollary 4.8, B is a basis for $(D, S^* \cup NV)$. Let us consider an arbitrary basis $B' = (B'_D, B'_S)$ for (D, S). If B'_S does not contain an element from V, then $B'_S \subseteq S^* \cup NV$, so by the monotonicity of supply condition we get

$$\omega(D,S) = \omega(B') = \omega(D,B'_S) \geq \omega(D,S^* \cup NV) = \omega(B),$$

in contradiction to (5.2). □

Note that the above lemma is the sole reason for which the VC is required to derive a linear time solution. (The discussion in the paragraph preceding Lemma 5.1 implies that, if the VC were not satisfied, it would still be possible to modify Function Demand to work correctly at an additional constant cost.)

LEMMA 5.3 (adaptation of Lemma 3.2 in [8]). *Let $R \subset S$ be a random subset of size r, where $|S| = m$. If $V \subset S$ is the set of elements violating a basis of (D,R), then its expected size is no more than $k_S(m-r)/(r+1)$.*

Proof. The probability that a random element $s \in S \setminus R$ violates a basis of (D,R) is not more than $k_S/(r+1)$, since $|B_S| \leq k_S$ for every basis (B_D, B_S) and the total size of the sample R with the element s is $r+1$. From the linearity of expectation the expected size of V is not more than $k_S(m-r)/(r+1)$. □

We now compute the complexity of Functions M and I. Theorem 2.12 tells us that Function M calls Function I $O(k_S)$ times (with an s-element set of size $O(\sqrt{|S|})$) and calls Function Violation $O(k_S|S|)$ times. Function I (when called with $|S|$ elements) calls Function Demand $O(k_S \log|S|)$ times and calls Function Violation $O(k_S|S|\log|S|)$ times.

If Function Demand runs in t_D time, then the total running time of Function M is $O(k_S|S|t_{vS} + k_S^2 \log|S|t_D))$, where the constant factors do not depend on k_D and k_S. Using (5.1), we get that the total running time of Function M is $O(k_S|S|t_{vS} + k_S^2(\log|S|)|D|(t_{vD} + t_{bDS}))$, where the constant depends exponentially on k_D.

After proving that Functions M, I, and Demand are correct and calculating their running times, it remains to consider Function DLP. In order to prove that Function DLP works correctly, we need to show that Lemmas 2.8, 2.9, and 2.10 and Theorem 2.11 can be modified for the DLP-type framework. This is done similarly to the way it was proved for Functions M and I.

It remains to consider the running time of Function DLP. Due to Theorem 2.12, Function DLP calls Function M (with a d-element set of size $k_D\sqrt{|D|}$) $O(k_D)$ times and calls Function Violation $O(k_D|D|)$ times. In this way Function Demand is called $O(k_D k_S^2 \log|S|)$ times, with an s-element set of constant size C and a d-element set of size $O(k_D\sqrt{|D|})$. If Function Demand is implemented in t_D time, then the total running time of Function DLP is $O(k_D(|D|t_{vD} + k_S|S|t_{vS} + k_S^2 \log|S|t_D))$, where the constant factors do not depend on k_D and k_S. Using (5.1), we get that the total running time of Function DLP is $O(k_D(|D|t_{vD} + k_S|S|t_{vS} + k_S^2 \log|S|\sqrt{|D|}(t_{vD} + t_{bDS})))$, where the constant depends exponentially on k_D. If the violation tests and basis calculations are done in constant time, this algorithm runs in $O(|D|+|S|)$ time. We have proved the following.

THEOREM 5.4. *Let (D,S,ω) be a (k_D,k_S)-dimensional DLP-type problem which meets the VC. Function DLP solves it in $O((|S|+|D|)t_v + \sqrt{|D|}\log|S|t_b)$ randomized time, where t_v (t_b) is the time needed for the violation test (basis calculation of a set consisting of k_D+1 d-elements and $9k_S^2$ s-elements) primitive.*

We summarize the structure of our algorithm in the following table (recall that $|D| = n$ and $|S| = m$):

The constants in the above algorithm may depend exponentially on k_D and k_S. We can get a linear time algorithm where the constants depend subexponentially on k_D

Function	*Input* $\|D\|$	*Input* $\|S\|$	# *iterations*	*Sample from*	*Sample size*
DLP	n	m	k_d	D	$\sqrt{n}$
M	$\sqrt{n}$	m	k_s	S	$\sqrt{m}$
I	$\sqrt{n}$	$\sqrt{m}$	$\log m$	S	const
Demand	$\sqrt{n}$	const	$\sqrt{n}$	D	1

and k_S when (D, S, ω) is basis regular. The idea is to call the modified algorithm of Sharir and Welzl only after the sizes of both the d-element set and the s-element set are reduced to constants and use the fact that this algorithm runs in linear time where the constants depend subexponentially on the dimension of the problem, when the problem is basis regular. Recall that Function I is a modified version of Function x_i^* in [8], applied on the s-element set. Instead of calling Function Demand in lines 2 and 3(b), we change it to call a new and similar Function I', which is a modified version of Function x_i^* in [8], applied on the d-element set. Function I' will call Function Demand in the lines corresponding to lines 2 and 3(b) in Function I. Thus Function Demand is called with both d-element and s-element sets of constant size. Recall that Function Demand is a modified version of Function lptype in [31], applied on the d-elements set. Instead of calling Function Basis in lines 1 and 2(c), we change it to call a new and similar Function Supply, which is a modified version of Function lptype in [31], applied on the s-element set. Function Supply will call Function Basis in the lines corresponding to lines 1 and 2(c) in Function Demand. Using similar arguments to the ones mentioned earlier in this section, we get that the resulting 6-layer algorithm proves the following theorem.

THEOREM 5.5. *Let (D, S, ω) be a (k_D, k_S)-dimensional DLP-type problem which meets the VC. (D, S, ω) is solved in $O((|S| + |D|)t_v + \sqrt{|D||S|} \log |D| \log |S| t_b)$ randomized time, where t_v (t_b) is the time needed for the violation test (basis calculation of a set consisting of at most $k_D + 1$ d-elements and $k_S + 1$ s-elements) primitive. If (D, S, ω) is basis regular, then the constants depend subexponentially on k_D and k_S.*

6. Continuous lexicographic Helly-type theorems and their relations to the LP-type model. Amenta [2] concludes her paper with "The major open problem is to characterize the Helly systems (X, H) for which there is an objective function ω that gives a fixed-dimensional LP-type[1] problem (H, ω)." We give a partial answer for her question in this section, by showing that *every* lexicographic Helly system (to be defined below) admits an objective function ω that gives a fixed-dimensional LP-type problem (H, ω).

Let $(X \times \Lambda, \bar{H})$ be a parameterized Helly system with Helly number k and ω be a natural objective function. If ω meets the UMC, then, by Theorem 2.17, $(\bar{H}, \omega)$ is an LP-type problem of combinatorial dimension k. If ω does not satisfy the UMC, in order to get a fixed-dimensional LP-type problem, one normally uses the following two "tricks." If possible, assume that the input is in such a general position that ω satisfies the UMC. Alternatively, explicitly change ω to be a lexicographic function ν whose first parameter is ω. The resulting LP-type problem $(\bar{H}, \nu)$ has usually combinatorial dimension greater than k (see [2, 32]).

Consider, for instance, LP. As noted in section 3 in [2], the parameterized Helly system corresponding to LP does not generally satisfy the UMC, but by using a lexicographic objective function, it does. As an additional example, consider the smallest

[1] Amenta uses the term GLP rather than LP-type.

enclosing ball problem defined in section 2.3. This problem does not necessarily satisfy the UMC. When we assume that the points in H are in general positions, such that no two different congruent balls are realized by points of H, this problem does satisfy the UMC.

Our approach is different. We provide a machinery which converts any parameterized lexicographic Helly system (to be defined below) into an LP-type problem. In this way, instead of extending the objective function, using (standard) Helly theorems, assuming UMC, and applying Theorem 2.17, we use lexicographic objective functions, lexicographic Helly theorems, and our framework. Unlike Theorem 2.17, this machinery does not require that the natural objective function meets the UMC.

We give some definitions first. For every totally ordered set Λ and $d \in \mathbb{N}$ we impose a lexicographic order on Λ^d such that for any $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \Lambda^d$ we say that $x <_L y$ (x is lexicographically smaller than y (*lsmaller*, in short)) if $x_1 < y_1$ or there exists $d \geq k > 1$ such that $x_i = y_i$ for $i = 1, 2, \ldots, k-1$, and $x_k < y_k$. We say that $x \geq_L y$ if $x <_L y$ does not hold. For every $X \subseteq \Lambda^d$ and $x \in \Lambda^d$ we let $X_x = \{x' \in X \mid x' \leq_L x\}$ and let $X^x = \{x' \in X \mid x' \geq_L x\}$. We note that if X is a convex set, then for every $x \in X$, X_x and X^x are convex sets as well.

Definition 6.1. *Let* Λ *be a totally ordered set. A* Helly system with lexicographic Helly number l *is a set system* (X, H)*, where* $X \subseteq \Lambda^d$ *for some positive integer* d*, such that, for any* $x \in X$*,* $(X, \{h \cap X_x \mid h \in H\})$ *is a Helly system with Helly number* l*.*

This means that for any $x \in X$, whenever every l or less elements of H have a common point which is not lgreater than x, we get that all elements of H have a common point which is not lgreater than x.

In order to get LP-type problems from lexicographic Helly theorems, we impose a lexicographic order on the ground set X and parameterize the Helly system (X, H) with lexicographic Helly number l.

Definition 6.2. *A set system* $(X \times X, \bar{H})$ *is a* parameterized Helly system with lexicographic Helly number l *if there exists a Helly system with lexicographic Helly number* l*,* (X, H)*, such that, for all* $h \in H$*,* $\bar{h} = \{(y, x) \mid x \in X, y \in h \cap X_x\}$ *and* $\bar{H} = \{\bar{h} \mid h \in H\}$*.*

From the definitions it is easy to verify the following.

Observation 6.3. Let $(X \times X, \bar{H})$ be a parameterized Helly system with lexicographic Helly number l. For every $x, y \in X$ and $\bar{h} \in \bar{H}$ the following attributes hold:

1. $\{h_x \mid x \in X\}$ is a nested family for all $\bar{h} \in \bar{H}$.
2. (X, H_x) is a Helly system with lexicographic Helly number l.
3. $(X \times X, \bar{H})$ is a parameterized Helly system with Helly number l.
4. $(y, x) \in \bar{h} \rightarrow (y, y) \in \bar{h}$.
5. $(y, x) \in \bar{h} \rightarrow y \leq_L x$.

The importance of lexicographic Helly theorems follows partly from the following two results.

Theorem 6.4. *Let* $(X \times X, \bar{H})$ *be a parameterized Helly system with lexicographic Helly number* l*. If* ω *is its natural objective function, then* $(\bar{H}, \omega)$ *is an LP-type problem of combinatorial dimension* l*.*

Proof. We show that all of the conditions of Theorem 2.17 are satisfied. Due to attribute 3 in Observation 6.3, $(X \times X, \bar{H})$ is a parameterized Helly system with Helly number at most l. It remains to show that the natural objective function ω meets the UMC. Suppose on the contrary that there is $\bar{G} \subseteq \bar{H}$, with $\omega(\bar{G}) = x$, such that there are two different points $x', x'' \in X$ such that both $(x', x), (x'', x) \in \bigcap \bar{G}$ realize $\omega(\bar{G})$. Due to attribute 5 in Observation 6.3, $x', x'' \leq_L x$. Without loss of generality $x'' <_L x'$. Hence $x'' <_L x$, and from attribute 4 in Observation 6.3 we get that $(x'', x'') \in \bigcap \bar{G}$, so $\omega(\bar{G}) \leq_L x'' <_L x$ in contradiction. □

THEOREM 6.5. *Let $(X \times \Lambda, \bar{H})$ be a parameterized Helly system with Helly number k and natural objective function ω. If, for every $\lambda \in \Lambda$, (X, H_λ) is a Helly system with lexicographic Helly number l, then there is a function $\nu : 2^{\bar{H}} \to \Lambda \times X$ such that for all $\bar{G} \subseteq \bar{H}$ the first part of $\nu(\bar{G})$ is $\omega(\bar{G})$ and $(\bar{H}, \nu)$ is an LP-type problem of combinatorial dimension $\leq k + l$.*

Proof. For every $\lambda \in \Lambda$ we parameterize the Helly system (X, H_λ) such that $(X \times X, \bar{H}_\lambda)$ is a parameterized Helly system with lexicographic Helly number l. If its natural objective function ν_λ is not well-defined, we symbolically compactify the space X by representing points at infinity. Due to Theorem 6.4 the resultant abstract problem (H_λ, ν_λ) is an LP-type problem of combinatorial dimension l. We conclude our proof by using Theorem 2.18. □

This theorem is useful when we want to omit general position assumptions. For instance, we reconsider the smallest enclosing ball problem. In the beginning of this section we represented this problem on the set H of points in $\mathbb{R}^d$ as a parameterized Helly system with Helly number $d+1$, $(X \times \Lambda, \bar{H})$, where $\Lambda = \mathbb{R}^+$ is the set of radii, and each $h_\lambda \in H_\lambda$ is the set of centers at which a ball of radius at most λ contains a particular point $h \in H$. The natural objective function ω is just the minimal radius of a ball which encloses all of the points in H. By assuming that the input points are in general positions, we caused the natural ground set function ω' to meet the UMC. In this way all of the conditions of Theorem 2.17 are met, and $(\bar{H}, \omega)$ is a d-dimensional LP-type problem.

Using the lexicographic version of Helly's theorem, Theorem 1.2, we note that the Helly system (X, H) representing the radius theorem (i.e., the ground set $X = \mathbb{R}^d$ is the set of centers of unit balls in E^d, and H is a family of unit balls) has lexicographic Helly number $d + 1$. In this way we get that, for every $\lambda \in \Lambda = \mathbb{R}^+$, (X, H_λ) (i.e., $X = \mathbb{R}^d$ and H_λ is a family of balls of radius at most λ) is a Helly system with lexicographic Helly number $d + 1$. Applying Theorem 6.5, we get that $(\bar{H}, \nu)$ is an LP-type problem of combinatorial dimension $\leq 2(d+1)$, where the first parameter of the objective function ν is the radius of the smallest enclosing ball of H.

It is possible to bound the combinatorial dimension of the resulting LP-type problem even further. We give some more definitions first. In the Helly system (X, H) representing the radius theorem, every $h = h(p) \in H$ is a unit ball centered at p. We call such p a *reference point*. For every positive *scaling factor* $\lambda \in \mathbb{R}^+$ we let $\lambda h = \lambda h(p)$ be the λ-units ball centered at p and $\lambda H = \{\lambda h \mid h \in H\}$ be the set of λ-units balls with the same centers as the balls in H.

THEOREM 6.6. *Let $d \in \mathbb{N}$, and let H be a finite family of compact subsets in $\mathbb{R}^d$ with a reference point for each one of them. If, for every scaling factor $\lambda_0 \in \mathbb{R}^+$, $(\mathbb{R}^d, \lambda_0 H)$ is a Helly system with lexicographic Helly number l, and $(\mathbb{R}^d, \lambda_0\, Int(H))$ is a Helly system with Helly number k, where $Int(H) = \{Int(h) \mid h \in H\}$ is the family of the interiors of the sets in H, then $(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}^d, \bar{H})$ is a parameterized Helly system with Helly number $m = \max\{k, l\}$, where, for all $h \in H$ and for all $\lambda = (\lambda_0, x) \in \mathbb{R}^+ \times \mathbb{R}^d$, $h_\lambda = (\lambda_0 h \cap X_x) \cup (\lambda_0\, Int(h))$. Moreover, if ω is its natural objective function, then $(\bar{H}, \omega)$ is an LP-type problem of combinatorial dimension $m = \max\{k, l\}$.*

In Figure 6.1 below, $d = 2$, h is a rectangle of length 2 and width 1 centered at the origin, and $\lambda = (1, 0, 0)$. $h_{(1,0,0)}$ is a rectangle whose closure is h itself. The dashed line and the open circles do not belong to $h_{(1,0,0)} = (h \cap \mathbb{R}^2_{(0,0)}) \cup Int(h)$, while the solid line and the black point do.

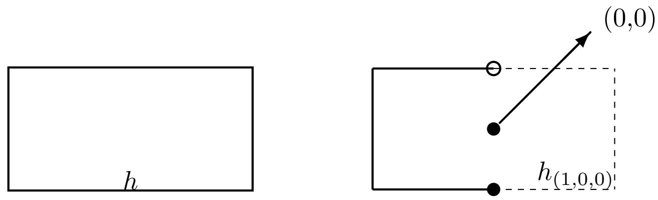

FIG. 6.1. *h and $h_{(1,0,0)}$.*

Proof. In order to prove that $(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}^d, \bar{H})$ is a parameterized Helly system with Helly number $m = \max\{k, l\}$, we need to show that $\{h_\lambda \mid \lambda \in \mathbb{R}^+ \times \mathbb{R}^d\}$ is a nested family and, for every $\lambda \in \mathbb{R}^+ \times \mathbb{R}^d$, $(\mathbb{R}^d, H_\lambda)$ is a Helly system with Helly number m.

Let $\alpha = (\lambda_0, x)$, $\beta = (\lambda'_0, x') \in \mathbb{R}^+ \times \mathbb{R}^d$ be such that $\alpha <_L \beta$. If $\lambda_0 = \lambda'_0$, then $x <_L x'$, so $X_x \subset X_{x'}$, and, from the definition of h_λ, $h_\alpha \subseteq h_\beta$. Otherwise $(\lambda_0 < \lambda'_0)$, $\lambda_0 h \subset \lambda'_0\, Int(h)$, so again $h_\alpha \subseteq h_\beta$. Hence $\{h_\lambda \mid \lambda \in \mathbb{R}^+ \times \mathbb{R}^d\}$ is a nested family.

We show now that, for every $\lambda = (\lambda_0, x) \in \mathbb{R}^+ \times \mathbb{R}^d$, $(\mathbb{R}^d, H_\lambda)$ is a Helly system with Helly number $m = \max\{k, l\}$. If every m elements in H_λ intersect in X_x, since $(\mathbb{R}^d, \lambda_0 H)$ is a Helly system with lexicographic Helly number $l \leq m$, then there is a point $x' \in X_x$ common to all of the sets in λH. Hence x' is common to all $h_\lambda \in H_\lambda$. If every m elements in H_λ intersect in $\mathbb{R}^d \setminus X_x$, then from the definition of h_λ every m elements in $\lambda_0\, Int(H)$ intersect. Since $(\mathbb{R}^d, \lambda_0\, Int(H))$ is a Helly system with Helly number $k \leq m$, all of the sets in $\lambda\, Int(H)$ have a point in common. Hence there is a point common to all $h_\lambda \in H_\lambda$. In this way we get that $(\mathbb{R}^d, H_\lambda)$ is a Helly system with Helly number m and $(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}^d, \bar{H})$ is a parameterized Helly system with Helly number m.

We will now apply Theorem 2.17 on the parameterized Helly system with Helly number m, $(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}^d, \bar{H})$. For this we need to show that the natural ground set objective function ω' meets the UMC. We observe that, due to the definition of h_λ, for every $\lambda_0 \in \mathbb{R}^+$, $\bar{h} \in \bar{H}$, and $x, y \in \mathbb{R}^d$

$$(y, \lambda_0, x) \in \bar{h} \rightarrow (y, \lambda_0, y) \in \bar{h} \tag{6.1}$$

holds. Second, we note that if $\lambda^* = (\lambda_0^*, x^*)$ is the value of the optimal solution over $\bar{G} \subseteq \bar{H}$, that is, $\omega(\bar{G}) = \omega'(x, \lambda^*) = \lambda^*$, then, for every point $(y, \lambda^*) \in \mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}^d$ realizing this value, there exists $h' \in H$ such that y lies on the boundary of $\lambda_0^* h' \cap X_x^*$. (Otherwise, y must be in $\cap_{h \in H} \lambda_0^*\, Int(h)$, and we can decrease λ_0^* slightly, say, to λ'_0, and still have a nonempty intersection $(y, \lambda'_0, x^*) \in \bigcap \bar{G}$, so $\omega(\bar{G}) \leq_L \omega'(y, \lambda'_0, x^*) = (\lambda'_0, x^*) <_L \lambda^*$ in contradiction to the optimality of λ^*.) Thus we have for every $y \in \mathbb{R}^d$

$$(y, \lambda_0^*, x^*) \in \bigcap \bar{G} \rightarrow y \leq_L x^*. \tag{6.2}$$

Suppose on the contrary that there exists $\bar{G} \subseteq \bar{H}$, with $\omega(\bar{G}) = \lambda^* = (\lambda_0^*, x^*)$, and there are two different points $y', y'' \in \mathbb{R}^d$ such that both $(y', \lambda^*), (y'', \lambda^*) \in \bigcap \bar{G}$ realize $\omega(\bar{G})$. Due to (6.2), $y', y'' \leq_L x^*$. Without loss of generality $y'' <_L y'$. Hence $y'' <_L x^*$, and (6.1) implies that $(y'', \lambda_0^*, y'') \in \bigcap \bar{G}$, so $\omega(\bar{G}) \leq_L (\lambda_0^*, y'') <_L (\lambda_0^*, x) = \lambda^*$ in contradiction to the optimality of λ^*. Hence ω' satisfies the UMC on $(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}^d, \bar{H})$, and, by Theorem 2.17, $(\bar{H}, \omega)$ is an LP-type problem of combinatorial dimension m. $\square$

Considering once again the smallest enclosing ball problem, we note that, for every $\lambda_0 \in \mathbb{R}^+$, $(X, \lambda_0 H)$ ($X = \mathbb{R}^d$ and $\lambda_0 H$ is a family of balls of radius λ_0) is a Helly system with lexicographic Helly number $d+1$. Since Helly's theorem is valid for finite families of open convex sets, $(X, \lambda_0\, Int(H))$ is a Helly system with Helly number $d+1$. Applying the theorem above, we get that $(\bar{H}, \omega)$ is an LP-type problem of combinatorial dimension $d+1$, where the first coordinate (parameter) of ω is the radius of the smallest enclosing ball of the points in H, and the remaining parameters are its center location. We demonstrate the usage of Theorem 6.6 in the next section.

7. Solving the planar lexicographic rectilinear p-center problem. In the lexicographic rectilinear p-piercing *decision* problem (p-*lpiercing* decision problem, in short) we are given a finite set B of closed axis-parallel boxes in $\mathbb{R}^d$ and a p-tuple $A = (a_1, \ldots, a_p)$ of p points in $\mathbb{R}^d$, with $a_i \leq_L a_j$ for all $i < j$. We need to decide whether there exists a p-tuple $A' = (a'_1, a'_2, \ldots, a'_p)$ such that $\{a'_1, a'_2, \ldots, a'_p\}$ p-pierces B and $A' \leq_L A$. If such a p-tuple A' exists, we say that B is A-p-*pierceable* and call A' a p-*piercing vector of* B.

In the lexicographic rectilinear p-piercing *optimization* problem (p-*lpiercing* optimization problem, in short) we are given a finite set B of closed boxes in $\mathbb{R}^d$ with edges parallel to the coordinate axes and need to find the lexicographically least p-tuple A such that A p-pierces B. If no such p-tuple exists, we return a special symbol ∞.

The Helly-type theorem related to these problems is about the least $h_L(p)$ such that, for all A, B is A-p-pierceable if each $B' \subseteq B$, with $|B'| \leq h_L$, is A-p-pierceable.

Theorem 7.1 (Theorem 2.7 in [20]). *Let B be a finite set of axis-parallel closed rectangles in the plane and $A = (a_1, \ldots, a_p)$ be a p-tuple of p points in $\mathbb{R}^d$, with $a_i \leq_L a_j$ for all $i < j$. For $p = 1, 2, 3$ the rectangles in B are A-p-pierceable if every subfamily $G \subset B$ of size at most $h_L(p)$ is A-p-pierceable, where $h_L(1) = 2$, $h_L(2) = 6$, and $16 \leq h_L(3) \leq 34$.*

Its corresponding nonlexicographic Helly-type theorem is the following.

Theorem 7.2 (see [9]). *Let B be a finite set of axis-parallel rectangles in the plane such that all of the rectangles are either closed or open. For $p = 1, 2, 3$ the rectangles in B are p-pierceable if every subfamily $G \subset H$ of size at most $h(p)$ is p-pierceable, where $h(1) = 2$, $h(2) = 5$, and $h(3) = 16$.*

In this section we solve the planar lexicographic weighted p-center problem for $p = 1, 2, 3$ in randomized linear time by applying Theorem 7.2 on open rectangles, using its corresponding lexicographic version Theorem 7.1, and Theorem 6.6.

We start by defining the parameterized Helly system corresponding to our problem. Let the ground set of all possible p-centers be

$$X_p = \{(x_1, y_1, \ldots, x_p, y_p) \mid x_1, y_1, \ldots, x_p, y_p \in \mathbb{R}\} = \mathbb{R}^{2p},$$

where $(x_1, y_1), \ldots, (x_p, y_p)$ are the p centers. Let the range of the objective function be the radius, so $\Lambda = \mathbb{R}^+$.

We consider the $2p$-dimensional space X_p. For each reference point $h = h_j = (x_0, y_0) \in H$ we define $h_p = \cup_{i=1}^{p} h_p^i$, where

$$h_p^i = \left\{ (x_1, y_1, \ldots, x_p, y_p) \;\middle|\; |x_i - x_0| \leq \frac{1}{w_j};\ |y_i - y_0| \leq \frac{1}{w_j} \right\} \tag{7.1}$$

is the set of all points in X_p such that the ith center is at weighted distance at most 1 from h. We let $H_p = \{h_p \mid h \in H\}$. For every $\lambda_1 \in \mathbb{R}^+$ and $h = h_j \in H$ we define

$\lambda_1 h_p$ as h_p scaled by λ_1, that is, $\lambda_1 h_p = \cup_{i=1}^{p} \lambda_1 h_p^i$, where $\lambda_1 h_p^i = \{(x_1, y_1, \ldots, x_p, y_p) \mid |x_i - x_0| \leq \frac{\lambda_1}{w_j}; |y_i - y_0| \leq \frac{\lambda_1}{w_j}\}$.

Due to Theorem 7.1, for every scaling factor $\lambda_0 \in \mathbb{R}^+$, the set system $(X_p, \lambda_0 H_p)$ is a Helly system with lexicographic Helly number 2 (6, a constant bounded by 34) for $p = 1$ ($p = 2, 3$). Due to Theorem 7.2 (applied on open rectangles) $(X_p, \lambda_0\, Int(H_p))$ is a Helly system with Helly number 2 (5, 16) for $p = 1$ ($p = 2, 3$). Theorem 6.6 implies that $(X_p \times \mathbb{R}^+ \times X_p, \bar{H}_p)$, where for all $h_p \in H_p$ and for all $\lambda = (\lambda_0, x) \in \mathbb{R}^{2p+1}$, $h_{p\lambda} = (\lambda_0 h_p \cap X_x) \cup (\lambda_0\, Int(h_p))$ is a parameterized Helly system with Helly number 2 (6, a constant bounded by 34) for $p = 1$ ($p = 2, 3$). Moreover, if ω_p is its natural objective function, the theorem says that $(\bar{H}_p, \omega)$ is an LP-type problem of combinatorial dimension 2 (6, 34) for $p = 1$ ($p = 2, 3$).

THEOREM 7.3. *The lexicographic weighted planar p-center problem with an l_∞ norm is solvable in (randomized) linear time for $p = 1, 2, 3$.*

Proof. Until now we have shown that the lexicographic planar p-center problem with an l_∞ norm is an LP-type problem of dimension at most 2 (6, 34) for $p = 1$ ($p = 2, 3$). We solve this problem by using the LP-type randomized algorithms, such as the one of Sharir and Welzl (see section 2.2.2). In order to obtain a linear running time it remains to show how to implement the violation test and basis calculation primitives such that they run in constant time. We slightly change the structure of these two primitives: We implement the basis calculation primitives such that when called with input (B, h) it returns, in addition to a basis $B(B \cup \{h\})$ for $B \cup \{h\}$, also the value $\omega(B \cup \{h\})$ of the objective function on $B \cup \{h\}$ and the point $x(B \cup \{h\})$ which realizes this value (there is only such a point since the objective function is lexicographic). The input for the violation tests consists of $x(B)$ in addition to B (i.e., we call Violation$(B, h, x(B))$). The violation test primitive checks whether $x(B) \in \bar{h}_p(h)$. This is done in constant time since $\bar{h}_p(h)$ is of constant complexity. We implement the basis calculation primitive Basis(B, h) in constant time as follows. For any proper subset $B' \subset B \cup \{h\}$ we calculate explicitly $\omega(B')$ and the point $x(B')$ realizing this value. Then for every $h \in B \cup \{h\} \setminus B'$ we call Violation$(B', h, x(B'))$. B' is a basis for $B \cup \{h\}$ if and only if all of these calls return "false." $\square$

We note that, since the optimal solution for the lexicographic planar p-center problem is an optimal solution for the nonlexicographic problem, we get an alternative solution to the one of [32]. We summarize as follows.

COROLLARY 7.4. *The planar p-center problem with an l_∞ norm is solvable in (randomized) linear time for $p = 1, 2, 3$.*

We note that the combinatorial dimension of the lexicographic problem is smaller than the combinatorial dimension given by [32] for the corresponding nonlexicographic problems (6 instead of 13 for the case $p = 2$ and 34 instead of 43 for the case $p = 3$).

8. Discrete Helly-type theorems and their relations to the DLP-type model.

8.1. DLP-type problems specialized to mathematical programming.

In the DLP-type framework both D and S are sets of abstract objects, and the objective function applies to elements of $2^D \times 2^S$. We consider an extended version of mathematical programming which is a quadruple (X, D, S, ω'), where X is a *ground set* (usually $\mathbb{R}^d$), D is a set of d-elements, S is a set of s-elements (both of which are subsets of the ground set), and ω' is an objective function from X to some totally ordered set Λ. We call the elements of X points. For $G = (D', S') \in 2^D \times 2^S$ we write $\bigcap G$ for $(\bigcap D) \cap (\bigcup S)$. The points in $\bigcap (D, S)$ are called *feasible*. The goal is to minimize ω' over the set of feasible points.

One can think of a discrete mathematical programming problem (X, D, S, ω') as a mathematical programming problem on a grid made by $\bigcup S$, that is, the mathematical programming problem $(X \cap (\bigcup S), D \cap (\bigcup S), \omega')$. However, our definition of a discrete mathematical programming problem enables us to solve fixed-dimensional DLP-type problems efficiently, as explained later.

To simplify our proofs later, we will make a few observations about the DLP-type framework specialized to mathematical programming.

Definition 8.1. *Let (X, D, S, ω') be a discrete mathematical programming problem. For $G = (D', S') \in 2^D \times 2^S$, let $\omega(G) = \infty$ when $\bigcap G = \emptyset$ and $\omega(G) = \min\{\omega'(m) \mid m \in \bigcap G\}$ elsewhere. We call $\omega : 2^D \times 2^S \to \Lambda$ the* induced *subfamily objective function of (X, D, S, ω') and call the triple (D, S, ω) the* induced discrete abstract problem.

For instance, in the discrete 1-center problem on the real line we are given two finite sets of real numbers H_1 and H_2. We need to find a point $h \in H_2$ which minimizes the maximum distance between points in H_1 and h. We call this point a *center* and call the distance it realizes the *radius*. We formulate this problem as a discrete mathematical programming problem (X, D, S, ω'), where $X = \mathbb{R}^2$, D is the set of $\frac{\pi}{4}$ radians cones whose origins are the points of H_1, S is a set of vertical rays whose origins are the points of H_2, and, for all $(x, y) \in \mathbb{R}^2$, $\omega'(x, y) = y$. In Figure 8.1 we have an instance of the problem where $H_1 = \{5, 9\}$ (the black points) and $H_2 = \{4, 8\}$ (the white points). In the solution of this problem the center is 8, and the radius is 3. If the center is not restricted to be a point of H_2, the radius realized by choosing a center at 7 will be 2. In the next section we will discuss in detail other p-center problems such as the 1-center problem in $\mathbb{R}^d$ with either l_1 or l_∞ norm.

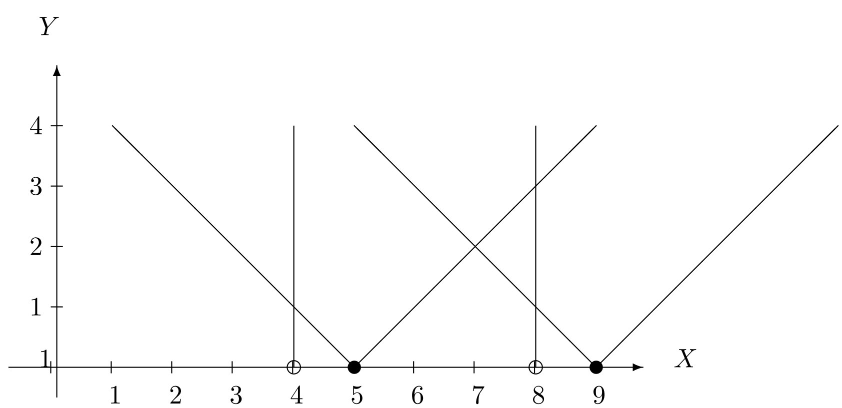

Fig. 8.1. *An instance of the general 1-center problem.*

Observation 8.2. Let (X, D, S, ω') be a discrete mathematical programming problem. The induced discrete abstract problem (D, S, ω) satisfies both monotonicity conditions of the DLP-type framework.

This follows from the fact that adding a d-element (i.e., a constraint) eliminates only feasible points, so the value of the minimum on the remaining feasible points cannot decrease. Adding an s-element (i.e., a relaxation) increases the set of feasible points, so the value of the minimum on the new enlarged set of feasible points cannot increase.

Observation 8.3. Let (X, D, S, ω') be a discrete mathematical programming problem. Its induced discrete abstract problem (D, S, ω) is a 1-supply problem which satisfies both monotonicity conditions and the locality of supply condition.

Proof. In order to show that (D, S, ω) is a 1-supply problem, it is sufficient to show that for every feasible $G = (D', S') \in 2^D \times 2^S$ there exists $S'' \subseteq S$, with $|S''| = 1$ such that $\omega(G) = \omega(D', S'')$. Since (D, S, ω) is an induced discrete abstract problem, there exists $x \in \bigcap G$ such that $\omega(G) = \omega'(x)$. From the definition of $\bigcap G$, there is $h \in S'$ such that $x \in h$ so $x \in \bigcap(D', \{h\})$ and $\omega(D', \{h\}) = \omega(G)$. Hence we choose $S'' = \{h\}$.

By Observation 8.2, (D, S, ω) obeys both monotonicity conditions.

We now show that (D, S, ω) satisfies the locality of supply condition. Let $G = (D', S') \in 2^D \times 2^S$ be feasible, and let $S'' \subseteq S'$ such that $\omega(D', S') = \omega(D', S'')$. We need to show that, for every $h \in S$, $\omega(D', S' \cup \{h\}) < \omega(G)$ implies $\omega(D', S'' \cup \{h\}) < \omega(G)$. Since (D, S, ω) is a 1-supply problem, $\omega(D', S' \cup \{h\}) < \omega(G)$ only if $\omega(D', \{h\}) < \omega(G)$. From the monotonicity of supply condition we conclude that $\omega(D', S'' \cup \{h\}) \leq \omega(D', \{h\}) < \omega(G)$. ∎

Definition 8.4. *Let (X, D, S, ω') be a discrete mathematical programming problem, and let (D, S, ω) be a discrete abstract problem, where ω is the objective function induced by ω'. If, for all $G = (D', S') \in 2^D \times 2^S$, $|\{x \in \bigcap G \mid \omega'(x) = \omega(G)\}| = 1$, we say that ω' satisfies the* UMC.

This definition says that every subfamily not only has a minimum but that this minimum is achieved by a unique point. There is one simple sufficient condition to satisfy the UMC.

Observation 8.5. If $\omega'(x) \neq \omega'(y)$ for any two distinct points $x, y \in X$, then ω' satisfies the UMC.

Lemma 8.6. *Let (X, D, S, ω') be a discrete mathematical programming problem. If its ground set function ω' meets the UMC on (X, D, S), then its induced abstract problem (D, S, ω) is a 1 s-dimensional DLP-type problem.*

Proof. By Observation 8.3, (X, D, S) is a 1-supply problem which satisfies both monotonicity conditions as well as the locality of supply condition.

We prove now that the locality of demand condition is satisfied. Let $G = (D', S') \in 2^D \times 2^S$ be bounded, and let $D'' \subseteq D'$ such that $\omega(G) = \omega(D'', S')$. We need to show that for all $h \in D$, $\omega(D' \cup \{h\}, S') > \omega(G) \to \omega(D'' \cup \{h\}, S') > \omega(G)$. Due to the UMC, the value $\omega(D', S') = \omega(D'', S')$ is achieved at a single point $x \in X$. This means that $\omega(D' \cup \{h\}, S') > \omega(G)$ only if $x \notin h$, in which case $\omega(D'' \cup \{h\}, S') > \omega(G)$, so the locality of demand condition is satisfied, and (D, S, ω) is a DLP-type problem. By Lemma 4.14 it is 1 s-dimensional. ∎

We concentrate for a moment on lexicographic integer programming (lex IP, for short) in $\mathbb{Z}^d$. The corresponding discrete mathematical programming formulation is $(\mathbb{Z}^d, D, S, \omega')$, where D is a finite set of half-spaces in $\mathbb{Z}^d$, S is the (exponentially large) set of the integer lattice points inside a "bounding box" around the problem, and ω' is defined for every $x \in \mathbb{Z}^d$ as $\omega'(x) = x$. Since S is finite and ω' satisfies the UMC, we get from Lemma 8.6 that (D, S, ω) is a 1 s-dimensional DLP-type problem. It remains to consider its d-dimension. Alternatively, we consider the combinatorial dimension of its induced LP-type problem (D, α). Suppose its combinatorial dimension is k and that the optimal value is $\alpha(D) = x^*$. We will first show that $k \leq 2^d$. Suppose on the contrary that $k > 2^d$. This means that if B is a basis for D, then every proper subset of B, $B' \subset B$ has $\alpha(B') <_L x^*$. Let $x^{\max} = \max\{\alpha(B') \mid B' \subset B\}$ be the maximal value of α on proper subsets of B. Since B is finite, $x^{\max}$ is well-defined, and

$x^{\max} <_L x^*$. Since the lexicographic version of Theorem 1.3 has Helly number 2^d (see Theorem 2.5 in [20]), applying it on B and $x^{\max}$ implies that the half-spaces in B have a common point which is not lexicographically greater than $x^{\max}$, in contradiction.

We now give a lower bound of $2^d - 1$ on the combinatorial dimension k of lex IP. Theorem 8.7 below applied on lex IP tells us that the special case of the lexicographic version of Theorem 1.3 over half-spaces has a Helly number of at most $k+1$. Since it is known (see section 2 in [20]) that this special case has Helly number 2^d, we get that $2^d \leq k+1$ as needed.

8.2. (Nonlexicographic) discrete case. We first show that there is a discrete Helly theorem corresponding to the constraint set of every fixed-dimensional DLP-type problem.

THEOREM 8.7. *Let (D, S, ω) be a (k_D, k_S)-dimensional DLP-type problem. For every $\lambda \in \Lambda$, $H = (D, S)$ has the property $\omega(H) \leq \lambda$ if and only if every $B_D \subseteq D$, with $|B_D| \leq k_D + 1$, has the property $\omega(B_D, S) \leq \lambda$. Moreover, H has the property $\omega(H) \geq \lambda$ if and only if every $B_S \subseteq S$, with $|B_S| \leq k_S + 1$, has the property $\omega(D, B_S) \geq \lambda$.*

Proof. We prove the first part of the theorem. The proof of the second part of the theorem is analogous. Let $\omega(H) \leq \lambda$. By the monotonicity of demand condition, $\omega(B_D, S) \leq \omega(H) \leq \lambda$. Going in the other direction, H must contain a basis $B = (B_D, B_S)$, with $|B_D| \leq k_D + 1$, and $\omega(B_D, S) = \omega(H)$ (if H is feasible, $|B_D| \leq k_D$; otherwise, every subset of B_D is feasible, so $|B_D| \leq k_D + 1$). So if every subfamily (B_D, S), with $|B_D| \leq k_D + 1$, has $\omega(B_D, S) \leq \lambda$, then $\omega(H) = \omega(B_D, S) \leq \lambda$. □

We next show how to get fixed-dimensional DLP-type problems from discrete Helly-type problems.

We first "discretize" set systems and Helly systems. A *discrete set system* is a triple (X, D, S), where X is a set and D, S are families of subsets of X. A discrete set system (X, D, S) is a *discrete Helly system* if there exists a finite integer k such that the intersection of every k or less d-elements of D has a common element in $\bigcup S$ implies that $\bigcap D \cap (\bigcup S) \neq \emptyset$. Let $(X \times \Lambda, \bar{D}, \bar{S})$ be a discrete set system, where Λ is a totally ordered set which contains a maximal element ∞. For every $\lambda \in \Lambda$ and $\bar{h} \in \bar{D} \cup \bar{S}$, we write $h_\lambda = \{x \in X \mid \exists \nu \leq \lambda \text{ s.t. } (x, \nu) \in \bar{h}\}$ for the projection into X of the part of $\bar{h}$ with Λ-coordinate no greater than λ. Also, for $G \in 2^D \times 2^S$, we write G_λ as a shorthand for $\{h_\lambda \mid \bar{h} \in \bar{G}\}$.

We next discretize parameterized Helly systems.

DEFINITION 8.8. *A discrete set system $(X \times \Lambda, \bar{D}, \bar{S})$ is a* discrete parameterized Helly system *with Helly number k, when*

1. *$\{h_\lambda \mid \lambda \in \Lambda\}$ is a nested family for all $\bar{h} \in \bar{D} \cup \bar{S}$, and*
2. *$(X, D_\lambda, S_\lambda)$ is a discrete Helly system, with Helly number k for all λ.*

We say that $\bar{G} = (\bar{D}', \bar{S}') \in 2^{\bar{D}} \times 2^{\bar{S}}$ *intersects at* λ if $\bigcap D'_\lambda \cap (\bigcup S'_\lambda) \neq \emptyset$. $\omega(\bar{D}', \bar{S}')$ is then the least value in Λ at which $\bar{G} = (\bar{D}', \bar{S}')$ intersects, i.e., $\omega(\bar{D}', \bar{S}') = \lambda^* = \inf\{\lambda \mid \bigcap D'_\lambda \cap (\bigcup S'_\lambda) \neq \emptyset\}$, and $\omega(\bar{D}', \bar{S}') = \infty$ if $\bar{G}$ fails to intersect at all $\lambda \in \Lambda$.

Figure 8.2 is a schematic diagram of a discrete parameterized Helly system. The whole stack represents $X \times \Lambda$, each of the pyramids represents a set $\bar{h} \in \bar{D}$, and each of the vertical lines represents a set $\bar{h} \in \bar{S}$. Each $\bar{h}$ is a subset of $X \times \Lambda$. Since all of the $\bar{h}$ are indexed with respect to Λ, the cross section at λ (represented by one of the planes) is equivalent to the discrete Helly system $(X, D_\lambda, S_\lambda)$ corresponding to Theorem 1.4. The discrete parameterized Helly system drawn in this figure is related to the discrete weighted 1-center problem with an l_∞ norm, which we solve in the next section. $\omega(\bar{D}, \bar{S})$ is the smallest value in Λ at which the intersection of the pyramids

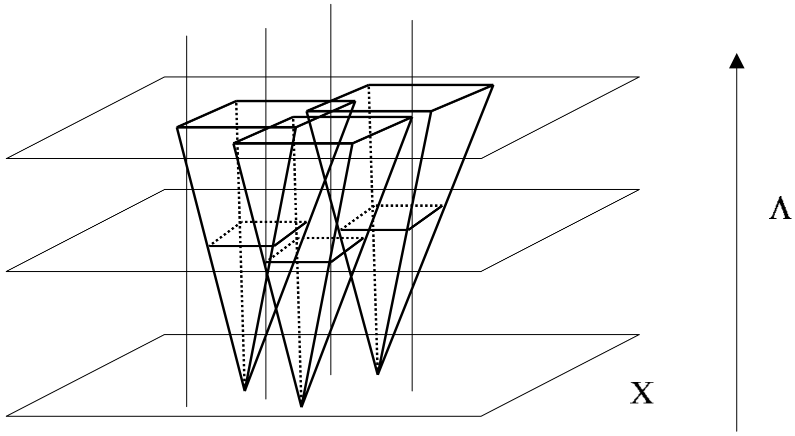

FIG. 8.2. *A discrete parameterized Helly system.*

in $\bar{D}$ "touches" a vertical line from $\bar{S}$.

We extend the main theorem in [2] to the discrete case and get the following.

THEOREM 8.9. *Let $(X \times \Lambda, \bar{D}, \bar{S})$ be a parameterized discrete Helly system with Helly number k, a natural ground set function ω', and a natural objective function ω. If ω' meets the UMC, then $(\bar{D}, \bar{S}, \omega)$ is a DLP-type problem of combinatorial dimension $(k, 1)$.*

Proof. Since ω is induced by the natural ground set objective function ω' on the space $X \times \Lambda$, and since ω' meets the UMC on $(X \times \Lambda, \bar{D}, \bar{S}, \omega')$ (Definition 8.4), by Lemma 8.6 $(\bar{D}, \bar{S}, \omega)$ is a 1 s-dimensional DLP-type problem.

It remains to prove that $(\bar{D}, \bar{S}, \omega)$ is k-d-dimensional. Consider any feasible $\bar{G} = (\bar{D}', \bar{S}') \in 2^{\bar{D}} \times 2^{\bar{S}}$ and a basis $\bar{B} = (\bar{B}_D, \bar{B}_S)$ for $\bar{G}$. Let $(\bar{D}', \alpha_{S'})$ be the induced LP-type problem of $(\bar{D}', \bar{S}', \omega)$. $\bar{B}_D$ is then a basis for $\bar{D}'$. We need to prove that $|\bar{B}_D| \leq k$. From the minimality of a basis we get that, for any $\bar{h} \in \bar{B}_D$, $\alpha_{S'}(\bar{B}_D \backslash \{\bar{h}\}) < \alpha_{S'}(\bar{B}_D)$. Let $\lambda^{\max} = \max\{\alpha_{S'}(\bar{B}_D \setminus \{\bar{h}\}) \mid \bar{h} \in \bar{B}_D\}$. Since $\bar{D}'$ is finite, so is $\bar{B}_D$, and this maximum is guaranteed to exist.

The basis $\bar{B}$ does not intersect at $\lambda^{\max}$, but for any $\bar{h} \in \bar{B}_D$, $\alpha_{S'}(\bar{B}_D \backslash \{\bar{h}\}) \leq \lambda^{\max}$, which means that $(\bar{B}_D \setminus \{\bar{h}\}, \bar{S}')$ intersects at $\lambda^{\max}$. Since $(X, D_{\lambda^{\max}}, S_{\lambda^{\max}})$ is a discrete Helly system with Helly number k, $\bar{B}_D$ must contain some subfamily $\bar{A}$, with $|\bar{A}| \leq k$, such that $(\bar{A}, \bar{S}')$ does not intersect at $\lambda^{\max}$. Every $\bar{h} \in \bar{B}_D$ must be in $\bar{A}$, since otherwise it would be the case that $\bar{A} \in (\bar{B}_D \setminus \{\bar{h}\})$ for some $\bar{h}$. This cannot be, because $(\bar{A}, \bar{S}')$ does not intersect at $\lambda^{\max}$, while every $(\bar{B}_D \setminus \{\bar{h}\}, \bar{S}')$ does. Therefore $\bar{B}_D = \bar{A}$ and $|\bar{B}_D| \leq k$. ∎

8.3. Lexicographic-discrete case. In this rather technical section we discretize the results in section 6. We start by "lexifying" discrete Helly systems.

DEFINITION 8.10. A discrete Helly system with lexicographic Helly number ld *is a discrete set system (X, D, S) such that, for every $x \in X$, $(X, \{d \cap X_x \mid d \in D\}$, $\{s \cap X_x \mid s \in S\})$ is a discrete Helly system with Helly number ld.*

This means that for every $x \in X$, whenever every ld or less elements of D have a common point in S which is not lgreater than x, we get that all elements of D have a common point in S which is not lgreater than x.

We next discretize Theorem 2.18. Let $(X \times \Lambda, \bar{D}, \bar{S})$ be a parameterized discrete Helly system with Helly number k and natural objective function ω. For all $\lambda \in \Lambda$, we assume a function $\nu_\lambda : 2^{D_\lambda} \times 2^{D_\lambda} \to \Lambda'$, where Λ' is a totally ordered set containing a maximal element ∞ such that $(D_\lambda, S_\lambda, \nu_\lambda)$ is a DLP-type problem of d-dimension at most d and s-dimension 1. The functions ν_λ may themselves be lexicographic. Similarly to [2], we impose a lexicographic order on $\Lambda \times \Lambda'$ with $(\lambda, \kappa) > (\lambda', \kappa')$ if $\lambda > \lambda'$ or if $\lambda = \lambda'$ and $\kappa > \kappa'$. We define a lexicographic objective function $\nu : 2^{\bar{D}} \times 2^{\bar{S}} \to \Lambda \times \Lambda'$ in terms of ω and the functions ν_λ as seen in the following.

THEOREM 8.11. *Let Λ' be a totally ordered set. Let $(X \times \Lambda, \bar{D}, \bar{S})$ be a parameterized discrete Helly system with Helly number k and natural objective function ω. If, for all λ, $(D_\lambda, S_\lambda, \nu_\lambda)$ is a DLP-type problem of combinatorial dimension $(d, 1)$, where $\nu_\lambda : 2^{D_\lambda} \times 2^{S_\lambda} \to \Lambda'$, then $(\bar{D}, \bar{S}, \nu)$ is a DLP-type problem of d-dimension $\leq k+d$ and s-dimension 1, where $\nu : 2^{\bar{D}} \times 2^{\bar{S}} \to \Lambda \times \Lambda'$ is defined as $\nu(\bar{G}) = (\omega(\bar{G}), \nu_{\omega(\bar{G})}(G_{\omega(\bar{G})}))$ for all $\bar{G} \subseteq \bar{D} \times \bar{S}$.*

Proof. Due to Observation 8.2, $(\bar{D}, \bar{S}, \omega)$ obeys both monotonicity conditions. For every λ, $(D_\lambda, S_\lambda, \nu_\lambda)$ obeys both monotonicity conditions. Hence, since ν is a composition of monotone functions, we get that $(\bar{D}, \bar{S}, \nu)$ obeys both monotonicity conditions as well. We next show that $(\bar{D}, \bar{S}, \nu)$ obeys both locality conditions.

Consider $\bar{D''} \subseteq \bar{D'} \subseteq \bar{D}$ and $\bar{S'} \subseteq \bar{S}$, with $\nu(\bar{D''}, \bar{S'}) = \nu(\bar{D'}, \bar{S'}) = (\lambda^*, \kappa^*)$ and $\nu(\bar{D'} \cup \{\bar{h}\}, \bar{S'}) = (\lambda, \kappa) > \nu(\bar{D'}, \bar{S'})$. We must have either $\lambda > \lambda^*$ or $\kappa > \kappa^*$. If $\lambda > \lambda^*$, by the definition of ω, $\nu_{\lambda^*}(D'_{\lambda^*} \cup \{h_{\lambda^*}\}, S'_{\lambda^*}) = \infty$, so $\nu_{\lambda^*}(D'_{\lambda^*} \cup \{h_{\lambda^*}\}, S'_{\lambda^*}) > \nu_{\lambda^*}(D'_{\lambda^*}, S'_{\lambda^*})$. Otherwise, $\kappa > \kappa^*$, that is, $\nu_{\lambda^*}(D'_{\lambda^*} \cup \{h_{\lambda^*}\}, S'_{\lambda^*}) > \nu_{\lambda^*}(D'_{\lambda^*}, S'_{\lambda^*})$. In either case, by the locality of demand condition on ν_{λ^*}, $\nu_{\lambda^*}(D''_{\lambda^*} \cup \{h_{\lambda^*}\}, S'_{\lambda^*}) > \nu_{\lambda^*}(D''_{\lambda^*}, S'_{\lambda^*})$ and $\nu(\bar{D''} \cup \{\bar{h}\}, \bar{S'}) > \nu(\bar{D''}, \bar{S'})$. So the lexicographic function ν also satisfies the locality of demand condition.

We now consider the locality of supply condition. We note that, due to Observation 8.3, $(\bar{D}, \bar{S}, \omega)$ is a 1-supply problem which meets the locality of supply condition. For every λ, $(D_\lambda, S_\lambda, \nu_\lambda)$ obeys both locality conditions. We will show that, since ν is a composition of functions satisfying the locality of supply condition, $(\bar{D}, \bar{S}, \nu)$ meets the locality of supply condition. Let $\bar{S''} \subseteq \bar{S'} \subseteq \bar{S}$ and $\bar{D'} \subseteq \bar{D}$, with $\nu(\bar{D'}, \bar{S''}) = \nu(\bar{D'}, \bar{S'}) = (\lambda^*, \kappa^*)$ and $\nu(\bar{D'}, \bar{S'} \cup \{\bar{h}\}) = (\lambda, \kappa) < \nu(\bar{D'}, \bar{S'})$. We need to show that $\nu(\bar{D'}, \bar{S''} \cup \{\bar{h}\}) < (\lambda^*, \kappa^*)$. Clearly, we must have either $\lambda < \lambda^*$ or $\kappa < \kappa^*$. If $\lambda < \lambda^*$, since $(\bar{D}, \bar{S}, \omega)$ obeys the locality condition of supply, $\omega(\bar{D'}, \bar{S''} \cup \{\bar{h}\}) < \lambda^*$, so $\nu(\bar{D'}, \bar{S''} \cup \{\bar{h}\}) < (\lambda^*, \kappa^*)$, as needed. Otherwise, $\kappa < \kappa^*$, that is, $\nu_{\lambda^*}(D'_{\lambda^*}, S'_{\lambda^*} \cup \{h_{\lambda^*}\}) < \nu_{\lambda^*}(D'_{\lambda^*}, S'_{\lambda^*})$. In this case, by the locality of supply condition on ν_{λ^*}, $\nu_{\lambda^*}(D'_{\lambda^*}, S''_{\lambda^*} \cup \{h_{\lambda^*}\}) < \nu_{\lambda^*}(D'_{\lambda^*}, S''_{\lambda^*})$ and again $\nu(\bar{D'}, \bar{S''} \cup \{\bar{h}\}) < \nu(\bar{D'}, \bar{S''})$, as needed.

We now consider the combinatorial s-dimension. It is sufficient to show that, for every feasible $(\bar{D'}, \bar{S'}) \in 2^{\bar{D}} \times 2^{\bar{S}}$ and every basis $\bar{B} = (\bar{BD}, \bar{BS})$ for $(\bar{D'}, \bar{S'})$, $|\bar{BS}| = 1$. Since $\bar{B}$ is a basis for $(\bar{D'}, \bar{S'})$, we have $\nu(\bar{B}) = \nu(\bar{D'}, \bar{S'}) = (\lambda^*, \kappa^*)$. Let $B_{\lambda^*} = (BD'_{\lambda^*}, BS'_{\lambda^*})$ be a basis for $(D'_{\lambda^*}, S'_{\lambda^*})$ in the DLP-type problem $(D'_{\lambda^*}, S'_{\lambda^*}, \nu_{\lambda^*})$. Since the s-dimension of $(D'_{\lambda^*}, S'_{\lambda^*}, \nu_{\lambda^*})$ is 1, there is $\bar{h'} \in \bar{S'}$ such that $BS'_{\lambda^*} = \{h'_{\lambda^*}\}$. Let $\bar{S''} \subseteq \bar{S'}$ be the set of all such $\bar{h'}$. Since $(\bar{D'}, \bar{S'}, \omega)$ is an induced discrete abstract problem, there is a feasible point $x \in \bigcap(\bar{D'}, \bar{S'})$ such that $\omega(\bar{D'}, \bar{S'}) = \omega'(x) = \lambda^*$, $x \in h'_{\lambda^*}$ (so $x \in \bar{h'}$), and

$$\nu(\bar{D'}, \bar{S'}) = \nu(\bar{D'}, \{\bar{h'}\}). \tag{8.1}$$

Let BD''_{λ^*} be a basis for D_{λ^*} in the induced LP-type problem of $(D_{\lambda^*}, S_{\lambda^*}, \nu_{\lambda^*})$ such that $BD''_{\lambda^*} \subseteq BD_{\lambda^*}$. It is possible to choose such a basis since $\nu_{\lambda^*}(BD_{\lambda^*}, S'_{\lambda^*}) = \kappa^*$. Similarly, let BS''_{λ^*} be a basis for S_{λ^*} in the induced dual LP-type problem of $(D_{\lambda^*}, S_{\lambda^*}, \nu_{\lambda^*})$ such that $BS''_{\lambda^*} \subseteq BS_{\lambda^*}$. It is possible to choose such a basis since $\nu_{\lambda^*}(D'_{\lambda^*}, BS_{\lambda^*}) = \kappa^*$. Due to the definition of a basis (Definition 4.4), $(BD''_{\lambda^*}, BS''_{\lambda^*})$

is a basis for $(D'_{\lambda^*}, S'_{\lambda^*})$. Hence $\bar{S''} \cap \bar{BS''} \neq \emptyset$, and consequently $\bar{S''} \cap \bar{BS} \neq \emptyset$, so there exists

$$\bar{h'} \in \bar{S''} \cap \bar{BS}. \tag{8.2}$$

We claim that $\bar{BS} = \{\bar{h'}\}$. Since $(\bar{BD}, \bar{BS})$ is a basis for $(\bar{D'}, \bar{S'})$ we get

$$\nu(\bar{D'}, \bar{S'}) = \nu(\bar{D'}, \bar{BS}). \tag{8.3}$$

Combining (8.1) and (8.3) together implies that $\nu(\bar{D'}, \bar{BS}) = \nu(\bar{D'}, \{\bar{h'}\})$. Since $\bar{BS}$ is a basis for the induced dual LP-type problem $(\bar{S'}, \beta)$, the last equality implies that $\bar{BS} = \{\bar{h'}\}$, so $(\bar{D}, \bar{S}, \nu)$ has s-dimension 1.

Finally, we consider the combinatorial d-dimension. We note that, since $\bar{B}$ is a basis, $\nu(\bar{BD} \setminus \{h\}, \bar{S'}) = (\lambda, \kappa) < \nu(\bar{B}) = (\lambda^*, \kappa^*)$ for any $\bar{h} \in \bar{BD}$. Let the subset $\bar{B_1} = \{\bar{h} \in \bar{BD} \mid \nu(\bar{BD} \setminus \{\bar{h}\}, \bar{S'}) = (\lambda, \kappa)$ and $\lambda < \lambda^*\}$. Since the d-dimension of $(B_{\lambda^*}, \nu_{\lambda^*})$ is d, $\bar{BD} \setminus \bar{B_1}$ contains at most d constraints. If $\bar{B_1} = \emptyset$, we are done. Otherwise, we let

$$\lambda^{\max} = \max\{\lambda \mid \nu(\bar{BD} \setminus \{\bar{h}\}, \bar{S'}) = (\lambda, \kappa),\, \bar{h} \in \bar{B_1}\}.$$

Since $\lambda^{\max} < \lambda^*$, $\bar{BD}$ fails to intersect with S' at $\lambda^{\max}$ and hence must contain a set $\bar{A}$ of size $\leq k$ that also fails to intersect. Every $\bar{h} \in \bar{B_1}$ must also be in $\bar{A}$, since $\bar{BD} \setminus \{\bar{h}\}$ intersects with S' at $\lambda^{\max}$ and $\bar{A}$ does not, so $\bar{A} \not\subseteq \bar{BD} \setminus \{\bar{h}\}$. So $|\bar{B_1}| \leq |\bar{A}| \leq k$ and $|\bar{BD}| \leq k + d$. ∎

In order to get DLP-type problems from lexicographic-discrete Helly theorems, we impose lexicographic order on the ground set X and parameterize the discrete Helly system (X, D, S) with lexicographic Helly number ld (see Definition 8.10) in the following way (recall that $X_x = \{x' \in X \mid x' \leq_L x\}$).

DEFINITION 8.12. *A discrete set system* $(X \times X, \bar{D}, \bar{S})$ *is a* parameterized discrete Helly system with lexicographic Helly number ld *if there exists a discrete Helly system with lexicographic Helly number ld, (X, D, S), such that for all $h \in D$, $\bar{h} = \{(y, x) \mid x \in X,\, y \in h \cap X_x\}$, $\bar{D} = \{\bar{h} \mid h \in D\}$, and for all $h \in S$, $\bar{h} = \{(y, x) \mid x \in X,\, y \in h\}$, $\bar{S} = \{\bar{h} \mid h \in S\}$.*

From the definitions it is easy to verify the following.

Observation 8.13. Let $(X \times X, \bar{D}, \bar{S})$ be a parameterized discrete Helly system with lexicographic Helly number ld. For every $x, y \in X$ and $\bar{h} \in \bar{D} \cup \bar{S}$ the following attributes hold:

1. $\{h_x \mid x \in X\}$ is a nested family for all $\bar{h} \in \bar{D} \cup \bar{S}$.
2. (X, D_x, S_x) is a discrete Helly system with lexicographic Helly number ld.
3. $(X \times X, \bar{D}, \bar{S})$ is a parameterized discrete Helly system with Helly number ld.
4. $(y, x) \in \bar{h} \rightarrow (y, y) \in \bar{h}$.
5. $(y, x) \in \bar{h} \rightarrow y \leq_L x$.

We give the discrete versions of Theorems 6.4 and 6.5 and prove them similarly to the way we proved the continuous versions.

THEOREM 8.14. *Let $(X \times X, \bar{D}, \bar{S})$ be a parameterized discrete Helly system with lexicographic Helly number ld and ω be its natural objective function. Then $(\bar{D}, \bar{S}, \omega)$ is a DLP-type problem of combinatorial dimension $(ld, 1)$.*

Proof. We show that all of the conditions of Theorem 8.9 are satisfied. Due to attribute 3 in Observation 8.13, $(X \times X, \bar{D}, \bar{S})$ is a parameterized discrete Helly system with Helly number at most ld. It remains to show that the natural objective function ω meets the UMC. Suppose on the contrary that there is $\bar{G} = (\bar{D'}, \bar{S'}) \in 2^{\bar{D}} \times 2^{\bar{S}}$, with $\omega(\bar{G}) = x$, such that there are two different points $x', x'' \in X \cap (\bigcup S')$ so that both $(x', x), (x'', x) \in \bigcap \bar{G}$ realize $\omega(\bar{G})$. Due to attribute 5 in Observation 8.13, $x', x'' \leq_L x$. Without loss of generality $x'' <_L x'$. Hence $x'' <_L x$, and from attribute 4 in Observation 8.13 we get that $(x'', x'') \in \bigcap \bar{G}$, so $\omega(\bar{G}) \leq_L x'' <_L x$ in contradiction. □

THEOREM 8.15. *Let $(X \times \Lambda, \bar{D}, \bar{S})$ be a parameterized discrete Helly system with Helly number k and natural objective function ω. If, for every $\lambda \in \Lambda$, $(X, D_\lambda, S_\lambda)$ is a discrete Helly system with lexicographic Helly number ld, then there is a function $\nu : 2^{\bar{D}} \times 2^{\bar{S}} \to \Lambda \times X$ such that for all $\bar{G} \in 2^{\bar{D}} \times 2^{\bar{S}}$ the first part of $\nu(\bar{G})$ is $\omega(\bar{G})$ and $(\bar{D}, \bar{S}, \nu)$ is a DLP-type problem of d-dimension $\leq k + l$ and s-dimension 1.*

Proof. For every $\lambda \in \Lambda$ we parameterize the discrete Helly system $(X, D_\lambda, S_\lambda)$ such that $(X \times X, \bar{D_\lambda}, \bar{S_\lambda})$ is a parameterized discrete Helly system with lexicographic Helly number ld. If its natural objective function ν_λ is not well-defined, we symbolically compactify the space X by representing points at infinity. Due to Theorem 8.14 the resulted discrete abstract problem $(D_\lambda, S_\lambda, \nu_\lambda)$ is a DLP-type problem of combinatorial dimension $(ld, 1)$. We conclude our proof by using Theorem 8.11. □

It is possible to bound the combinatorial dimension of the resulting LP-type problem further by using the following discrete version of Theorem 6.6 (whose proof is similar to the one of Theorem 6.6).

THEOREM 8.16. *Let $d \in \mathbb{N}$, D be a finite family of compact subsets in $\mathbb{R}^d$ and S be a finite family of closed subsets in $\mathbb{R}^d$. If, for every scaling factor $\lambda_0 \in \mathbb{R}^+$, $(\mathbb{R}^d, \lambda_0 D, S)$ is a discrete Helly system with lexicographic Helly number l, and $(\mathbb{R}^d, \lambda_0\, Int(D), S)$ is a discrete Helly system with Helly number k, where $Int(D) = \{Int(h) \mid h \in D\}$ is the family of the interiors of the sets in D, then $(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}^d, \bar{D}, \bar{S})$ is a parameterized discrete Helly system with Helly number $m = \max\{k, l\}$, where, for all $h \in D$ and for all $\lambda = (\lambda_0, x) \in \mathbb{R}^+ \times \mathbb{R}^d$, $h_\lambda = (\lambda_0 h \cap X_x) \cup (\lambda_0\, Int(h))$, and for all $h \in S$ and for all $\lambda = (\lambda_0, x) \in \mathbb{R}^{d+1}$, $h_\lambda = h$. Moreover, if ω is its natural objective function, then $(\bar{D}, \bar{S}, \omega)$ is a DLP-type problem of combinatorial dimension $(m, 1)$.*

9. Solving the discrete weighted 1-center problem in $\mathbb{R}^d$ with either l_1 or l_∞ norm. In this section we show how to solve the discrete weighted 1-center problem in $\mathbb{R}^d$ with an l_∞ norm (1-center problem, in short) in linear time by formulating it as a fixed-dimensional DLP-type problem which satisfies the VC.

Given an instance D, S, W of the 1-center problem, for every $G = (D', S') \in 2^D \times 2^S$ let $r(D', S')$ be the optimal radius of the 1-center problem on D', S', W, realized by making $s^*(D', S') \in S'$ the center.

Considering the set of boxes $r(D', S')D' = \{r(D', S')d_i \mid d_i \in D'\}$, where $r(D', S')d_i$ is the box with center at d_i and radius $\frac{r(D', S')}{w_i}$, we note that $s^*(D', S')$ intersects all of the boxes of $r(D', S')D'$. The proof of Theorem 1.5 applied on the set of boxes $r(D', S')D'$ and the set of points S' tells us that the following $2d$ boxes "define" the optimal solution:

For every $i = 1, \ldots, d$, let $L_i(D', S') \in D'$ be a d-element $d_{j(i)} \in D'$ such that the projection of box $r(D', S')d_{j(i)}$ on the ith coordinate results in an interval $[l_i, r_i]$ with the smallest r_i. Let $l_i(D', S') \in \mathbb{R}$ be the right end point of the projection of $r(D', S')d_{j(i)}$ on the ith coordinate.

For every $i = 1, \ldots, d$, let $G_i(D', S') \in D'$ be a d-element $d_{j(i)} \in D'$ such that the projection of box $r(D', S')d_{j(i)}$ on the ith coordinate results in an interval $[l'_i, r'_i]$ with the greatest l'_i. Let $g_i(D', S') \in \mathbb{R}$ be the left end point of the projection of $r(D', S')d_{j(i)}$ on the ith coordinate.

We also define $C(D', S') \in S'$ to be the lexicographically smallest optimal center. We let the range of the objective function be $\mathbb{R}^+ \times \mathbb{R}^{3d}$ and define the objective function to be

$$\omega(D', S') = (r(D', S'), C(D', S'), l_1(D', S'), -g_1(D', S'), \ldots, l_d(D', S'), -g_d(D', S')).$$

Clearly (D, S, ω) is a discrete abstract problem.

For every $(D', S') \in 2^D \times 2^S$ we let $\mathit{Feasible}(D', S')$ denote the set of points that intersect all of the boxes $r(D', S')D'$. From the definitions of the variables and the optimality of the solution we get the following.

Observation 9.1. Let D, S, W be an instance of the discrete weighted 1-center problem in $\mathbb{R}^d$ with an l_∞ norm. For every $(D', S') \in 2^D \times 2^S$, $\mathit{Feasible}(D', S')$ is the minimal axis-parallel box containing the 2 points $(g_1(D', S'), \ldots, g_d(D', S'))$ and $(l_1(D', S'), \ldots, l_d(D', S'))$. Furthermore, $C(D', S')$ lies on its boundary.

We show now that (D, S, ω) is a $(2d, 1)$-dimensional DLP-type problem. (D, S, ω) obeys the monotonicity of demand condition since adding a new element h to D' cannot lexicographically decrease the value, i.e., $\omega(D', S') \leq_L \omega(D' \cup \{h\}, S)$. Similarly, (D, S, ω) obeys the monotonicity of supply condition since adding a new point h cannot lexicographically increase the objective function value.

We now show that (D, S, ω) obeys both locality conditions and that it obeys the VC. Let $G = (D', S') \in 2^D \times 2^S$ and $F = (D'', S'') \in 2^{D'} \times 2^{S'}$ be such that $\omega(G) =_L \omega(F)$ (so due to Observation 9.1 $\mathit{Feasible}(G) = \mathit{Feasible}(F)$). It suffices to show that the following 3 properties hold:

1. For all $h \in S$, $\omega(D', S' \cup \{h\}) <_L \omega(G)$ if and only if $\omega(D'', S'' \cup \{h\}) <_L \omega(F)$.
2. For all $h \in D$, $\omega(D' \cup \{h\}, S') >_L \omega(G) \rightarrow \omega(D'' \cup \{h\}, S') >_L \omega(F)$.
3. For all $h \in D$, $\omega(D'' \cup \{h\}, S'') >_L \omega(G) \rightarrow \omega(D' \cup \{h\}, S') >_L \omega(F)$.

Regarding the first property we note (for every set $X \in \mathbb{R}^d$, let $\mathit{Int}(X)$ denote its interior, and let $\partial(X)$ denote its boundary) that

$$\begin{aligned}
\omega(D'', S'' \cup \{h\}) <_L \omega(F) &\iff C(D'', S'' \cup \{h\}) = h \\
&\iff (h \in \mathit{Int}(\mathit{Feasible}(F))) \vee ((h \in \partial(\mathit{Feasible}(F))) \\
&\qquad\qquad \wedge (h <_L C(F))) \\
&\iff (h \in \mathit{Int}(\mathit{Feasible}(G))) \vee ((h \in \partial(\mathit{Feasible}(G))) \\
&\qquad\qquad \wedge (h <_L C(G))) \\
&\iff C(D', S' \cup \{h\}) = h \\
&\iff \omega(D', S' \cup \{h\}) <_L \omega(G).
\end{aligned}$$

We now consider the remaining two properties. Let $d_i \in D$. $\omega(D'' \cup \{d_i\}, S'') >_L \omega(F)$ if and only if one of the following cases occurs:

1. $r(D'' \cup \{d_i\}, S'') > r(F)$, or
2. $r(D'' \cup \{d_i\}, S'') = r(F)$ and $C(D'' \cup \{d_i\}, S'') >_L C(F)$, or
3. $r(D'' \cup \{d_i\}, S'') = r(F)$, $C(D'' \cup \{d_i\}, S'') =_L C(F)$, and $(l_1(D'' \cup \{d_i\}, S''), -g_1(D'' \cup \{d_i\}, S''), \ldots, l_d(D'' \cup \{d_i\}, S''), -g_d(D'' \cup \{d_i\}, S'')) >_L (l_1(F), -g_1(F), \ldots, l_d(F), -g_d(F))$.

Regarding case 1, we have

$$\begin{aligned}
&r(D'' \cup \{d_i\}, S'') > r(F) \iff r(F)d_i \cap \mathit{Feasible}(F) \cap S'' = \emptyset \\
\iff\ & (r(G)d_i \cap \mathit{Feasible}(G) \cap S' = \emptyset) \vee ((r(G)d_i \cap \partial(\mathit{Feasible}(G) \cap (S' \setminus S'')) \neq \emptyset) \\
&\qquad\qquad\qquad\qquad \wedge C(G) \notin r(G)d_i) \\
\iff\ & (r(D' \cup \{d_i\}, S') > r(G)) \vee ((r(D' \cup \{d_i\}, S') = r(G)) \\
&\qquad\qquad\qquad\qquad \wedge (C(D' \cup \{d_i\}, S') >_L C(G))).
\end{aligned}$$

We now consider case 2.

$$\begin{aligned}
&(r(D'' \cup \{d_i\}, S'') = r(F)) \wedge (C(D'' \cup \{d_i\}, S'') >_L C(F)) \\
&\iff (C(F) \notin r(F)d_i) \wedge (r(F)d_i \cap \partial(\mathit{Feasible}(F)) \cap S'' \neq \emptyset) \\
&\Rightarrow (C(G) \notin r(G)d_i) \wedge (r(G)d_i \cap \partial(\mathit{Feasible}(G)) \cap S' \neq \emptyset) \\
&\iff (r(D' \cup \{d_i\}, S') = r(D', S')) \wedge (C(D' \cup \{d_i\}, S') >_L C(D', S')).
\end{aligned}$$

When $S' = S''$, the other direction of implications is also correct.

Case 3 occurs if and only $C(F) \in r(F)d_i$ and there exists j such that, among the projections of the boxes in $r(F)D'' \cup \{r(F)d_i\}$ on the jth coordinate, the projection of $r(F)d_i$ results in an interval $[l, r]$ with either the smallest r or the greatest l. This happens if and only if $C(G) \in r(G)d_i$, and among the projections of the boxes in $r(G)D' \cup \{r(G)d_i\}$ on the jth coordinate, the projection of $r(G)d_i$ results in an interval $[l, r]$ with either the smallest r or the greatest l. This occurs if and only if $r(D' \cup \{d_i\}, S') = r(G)$, $C(D' \cup \{d_i\}, S') =_L C(G)$, and $(l_1(D' \cup \{d_i\}, S'), -g_1(D' \cup \{d_i\}, S'), \ldots, l_d(D' \cup \{d_i\}, S'), -g_d(D' \cup \{d_i\}, S')) >_L (l_1(G), -g_1(G), \ldots, l_d(G), -g_d(G))$.

From the above analysis we get that the last two properties are indeed satisfied. Hence (D, S, ω) is a DLP-type problem which satisfies the VC.

It is easy to verify that $B(D, S) = (\{L_1(D, S), G_1(D, S), \ldots, L_d(D, S), G_d(D, S)\}, \{C(D, S)\})$ is a basis of a feasible and bounded (D, S) and that the problem is of d-dimension $2d$ and s-dimension 1.

The violation test can easily be implemented in constant time. For a basis $B = (B_D, B_S)$ and a d-element d_i, $\omega(B_D \cup \{d_i\}, B_S) > \omega(B)$ if and only if either $r(B)d_i$ does not contain $C(B)$ or there exists j such that, among the projections of the boxes in $r(B)D'' \cup \{r(B)d_i\}$ on the jth coordinate, the projection of $r(B)d_i$ results in an interval $[l, r]$ with either the smallest r or the greatest l. For an s-element h, $\omega(B_D, B_S \cup \{h\}) < \omega(B)$ if and only if either h lies in the interior of $\mathit{Feasible}(B)$ or h lies on the boundary of $\mathit{Feasible}(B)$ and $h \leq_L C(B)$. The basis calculation can be implemented in constant time by calling the violation test a constant number of times. Using a DLP algorithm such as the one stated in section 5, we conclude as follows.

Theorem 9.2. *The discrete weighted* 1*-center problem in* $\mathbb{R}^d$ *with an* l_∞ *norm is solvable in (randomized) linear time for every fixed* d.

The rectilinear 1-center problem in $\mathbb{R}^d$ (i.e., with an l_1 norm) is solved similarly by using the rectilinear Helly-type versions of Theorems 1.4 and 1.5 (i.e., with rectilinear "balls" instead of axis-parallel boxes), which have Helly number 2^d instead of $2d$. We get the following theorem.

Theorem 9.3. *The discrete weighted rectilinear* 1*-center problem in* $\mathbb{R}^d$ *is solvable in (randomized) linear time for every fixed* d.

We note that, while the Euclidean 1-center problem in $\mathbb{R}^d$ can be formulated as a $(d+1)$-dimensional LP-type problem and thus is solved in randomized linear time [31], the corresponding discrete problem admits an $\Omega(n \log n)$ lower bound under the algebraic computation tree model and is solved in the same time bound [24]. This demonstrates that sometimes the complexity of a discrete optimization version of a continuous optimization problem is strictly harder, as discussed also in the introduction.

10. Solving problems related to line transversals in the plane.

10.1. Continuous case. We first consider the *lexicographic (continuous) line transversal of axis-parallel rectangles problem.* The input is a family $D = \{d_1, \ldots, d_n\}$ of axis-parallel (closed) rectangles in the plane, together with a set of their reference points $C = \{c_1, \ldots, c_n\}$ such that c_i lies in the interior of d_i for every $i = 1, \ldots, n$. For a particular rectangle $d_i \in D$, let λd_i be the homothet of d_i that results from scaling d_i by a factor of λ, relatively to c_i (i.e., while keeping the point c_i fixed in the plane). Let $\lambda D = \{\lambda d \mid d \in D\}$. In the lexicographic (continuous) line transversal of axis-parallel rectangles *optimization* problem, we are interested in the smallest scaling factor λ^* and lexicographically smallest vector (a'', b'') such that the line $y = a''x + b''$ intersects each of the scaled rectangles in $\lambda^* D$. In the corresponding (nonlexicographic) *decision* problem (i.e., no scaling is allowed), we ask whether there exists a line transversal which intersects all of the rectangles in D. We note that this decision problem is solved in linear time via LP-type algorithms or by reducing it to linear programming [1, 2]. We are unaware of any linear time algorithms for the (nonlexicographic) optimization problem. We solve this problem by solving the (more general) lexicographic problem in linear time and noting that the optimal scaling factors of the lexicographic and nonlexicographic problems are equal. We solve the lexicographic problem by using the LP-type framework and the following two Helly-type theorems.

THEOREM 10.1 (see [29]). *Let D be a family of parallel open rectangles in the plane. If every subset of at most 6 rectangles admits a line transversal, then H does as well.*

THEOREM 10.2 (Theorem 2.12 in [20]). *Let D be a family of axis-parallel (closed) rectangles in the plane. For every pair of reals a' and b', if every subfamily of at most 6 rectangles admits a line transversal $y = ax + b$, with $(a, b) \leq_L (a', b')$, then D does as well.*

We note that, for every line direction (e.g., vertical to the x-axis), the restricted problem of finding the smallest scaling factor λ^*, such that there exists a line transversal for $\lambda^* D$ in this direction, is solvable in linear time by projecting the problem on the vertical direction (e.g., on the x-axis) and formulating it as a 2-dimensional LP problem. Hence it is enough to solve in linear time the problem where the line transversal must not be vertical to the x-axis.

We show that this problem is a 6-dimensional LP-type problem by formulating it as a parameterized Helly system with lexicographic Helly number 6 and using Theorem 6.6. Let the ground set $X = \mathbb{R}^2$ be the set of lines in the plane which are not vertical to the x-axis (i.e., $(a, b) \in X$ is the line $Y = aX + b$), and let $\Lambda = \mathbb{R}^+$. For every $d \in D$, let $t(d)$ be the set of lines intersecting d, let $\bar{d} = \{t(\lambda d) \mid \lambda \in \mathbb{R}^+\}$, and let $\bar{D} = \{\bar{d} \mid d \in D\}$.

Every line that intersects the homothet $\lambda_1 d$ also intersects $\lambda_2 d$ for any $\lambda_2 > \lambda_1$, so each $\bar{d}$ is a nested family of lines. Due to Theorem 10.2 every $(X, \lambda D)$ is a Helly system with lexicographic Helly number 6. Due to Theorem 10.1 every $(X, \lambda\, Int(D))$ is a Helly system with Helly number 6. Hence, Theorem 6.6 implies that $(\mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^2, \bar{D})$ is a parameterized Helly system with Helly number 6 and that $(\bar{D}, \omega)$ is a 6-dimensional LP-type problem. The natural objective function $\omega(\bar{D})$ is the lsmallest vector $\lambda = (\lambda_0, a, b)$ such that D_λ intersects at the point (a, b) (i.e., the line $aX + b$ intersects each of the scaled rectangles in $\lambda_0 D$). Recall that the algorithm in [28] runs in $O(t_v n + t_b \log n)$ time, where t_v is the time needed for a violation test and t_b is the time required for a basis calculation. Since both violation test and basis calculation primitives can easily be implemented in constant time, we have just proved the following theorem.

THEOREM 10.3. *The line transversal of axis-parallel rectangles optimization problem is solvable in (randomized) linear time.*

We conclude this section by considering several variants of the line transversal of a totally separable set of convex planar objects problem (problem 3 in the introduction). The input for the *lexicographic* version of this problem is a totally separable family $D = \{d_1, \ldots, d_n\}$ of simple convex objects, a family $C = \{c_1, \ldots, c_n\}$ of reference points such that c_i lies in the interior of d_i for every $i = 1, \ldots, n$, and a vector (a', b'). In the *decision problem* we want to decide whether there exists a line $Y = aX + b$, with $(a, b) \leq_L (a', b')$, which intersects all of the objects in D. In the *optimization problem* we are interested in the smallest scaling factor λ^* and lexicographically smallest vector (a, b) such that the line $y = ax + b$ intersects each of the scaled objects in $\lambda^* D$. Clearly, the answer for the decision problem is positive if and only if the solution of the optimization problem is at most $(1, a', b')$. We solve this problem in linear time using the LP-type framework and the following two Helly-type theorems.

THEOREM 10.4 (see [23]). *Let D be a totally separable finite family of open convex sets. If every subset of at most 3 sets admits a line transversal, then D does as well.*

THEOREM 10.5 (Corollary 2.20 in [20]). *Let D be a totally separable family of (closed) convex sets. For every pair of reals a' and b', if every subfamily of at most 3 sets admits a line transversal $y = ax + b$, with $(a, b) \leq_L (a', b')$, then H does as well.*

It is easy to show, using similar arguments to the ones mentioned earlier in this section, that the optimization problem is indeed a 3-dimensional LP-type problem and that it is solved in linear time. We thus have just proved the following theorem.

THEOREM 10.6. *The line transversal of totally separable set of convex planar objects decision problem is solvable in (randomized) linear time.*

10.2. Discrete case I—A finite number of permissible directions of line transversals. In this section we define a discrete version for the line transversal of axis-parallel rectangles optimization problem. We solve it in randomized linear time by using the DLP-type framework.

Problem 10.7. Given are a set $D = \{d_1, \ldots, d_n\}$ of (not necessarily pairwise disjoint) axis-parallel compact rectangles in the plane, together with a set of their reference points $C = \{c_1, \ldots, c_n\}$, such that c_i lies in the interior of d_i, for every $i = 1, \ldots, n$, and a set $S = \{s_1, \ldots, s_m\}$ of permissible line directions. Find the minimal scaling factor $\lambda_1^* = \lambda_1(D, S) \in \mathbb{R}^+$ such that $\lambda_1^* D$ admits a line transversal whose direction is in S.

If we choose S to be the (infinite) set of all possible directions, this problem coincides with the continuous one. We can assume that S does not contain the vertical direction and that the directions in S are such that the permissible lines are $\{y = ax + b \mid a \in S\}$. (If S does contain a vertical direction, we will take the minimal solution (i.e., scaling factor) among the ones of Problem 10.7 on S without the vertical direction and on the vertical direction alone. As already mentioned in the previous section, the latter problem is solved in linear time by formulating it as an LP problem.)

A special case of Problem 10.7 is when the line transversal must be nondescending.

Problem 10.8. Given are a set $D = \{d_1, \ldots, d_n\}$ of (not necessarily pairwise disjoint) axis-parallel compact rectangles, together with a set of their reference points $C = \{c_1, \ldots, c_n\}$, such that c_i lies in the interior of d_i for every $i = 1, \ldots, n$, and a set $S = \{s_1, \ldots, s_m\}$ of permissible line directions. Find the minimal scaling factor $\lambda_1^* = \lambda_1(D, S) \in \mathbb{R}^+$ such that $\lambda_1^* D$ admits a nondescending line transversal whose direction is in S.

The solution of Problem 10.7 is the minimum scaling factor between the solution of Problem 10.8 and the analog problem where the line transversal must be nonascending.

10.2.1. A formulation as a discrete LP-type problem. In this section we formulate Problem 10.8 as a fixed-dimensional DLP-type problem by using Theorem 8.16 and the following Helly-type theorems.

THEOREM 10.9 (Theorem 2.13 in [20]). *Let D be a family of open rectangles in the plane with edges parallel to the axes, and let S be a set of nonnegative reals (line directions). If every subfamily of at most 4 rectangles admits a line transversal with a slope from S, then D does as well.*

THEOREM 10.10 (Theorem 5.8 in [20]). *Let D be a family of rectangles in the plane with edges parallel to the axes, and let S be a set of nonnegative reals (line directions). For every pair of reals $a' \geq 0$ and b', and a pair of nonnegative reals $sl_{\min} \leq sl_{\max}$, if every subfamily of at most 5 rectangles admits a line transversal $y = ax + b$, with $a \in S$, $sl_{\min} \leq a \leq sl_{\max}$, and $(a, b) \leq_L (a', b')$, then D does as well.*

Let $G = (D', S') \in 2^D \times 2^S$ be an arbitrary set such that $G \neq (\emptyset, \emptyset)$. We first look closely at an optimal solution for Problem 10.8 on G. Let λ_1^* be the optimal scaling factor. Due to Theorem 10.9 there is a direction $s^* \in S'$ and a set $D'' \subseteq D'$ of at most 4 rectangles such that the solution of Problem 10.8 on $(D'', \{s^*\})$ is λ_1^*. For this solution we define the following variables:

- $\lambda_1(D', S') \in \mathbb{R}^+$ is λ_1^*, the optimal scaling factor.
- $DIR(D', S') \in S$ is s^*, the minimal direction in S' in which there exists a nondecreasing line transversal for $\lambda_1(D', S')D'$.
- $LINE(D', S') = (DIR(D', S'), b(D', S'))$ is the line $y = DIR(D', S')x + b(D', S')$, which intersects every $\lambda_1(D', S')d \in \lambda_1(D', S')D$.

We note that, due to the optimality of λ_1^*, there exists only one line transversal to D' with direction s^*, and this line is tangent to at least 2 rectangles in $\lambda_1^* D'$.

In order to solve Problem 10.8, we first define and solve a lexicographic version of it, containing 4 more parameters. Let us consider the dual space $\mathbb{R}^2$ of all possible line transversals for $\lambda_1^* D'$. In this dual space, each nonvertical line $y = ax + b$ is represented by the point (a, b). We will use the following observation.

Observation 10.11 (Observation 5.7 in [20]). Let D be a family of axis-parallel rectangles in the plane, and, for every $d \in D$, let $\mathcal{L}(d)$ be the set of line transversals which d admits in the dual space of line transversals. Let $P(D) = \cap_{d \in D} \mathcal{L}(d)$ be the set of line transversals which D admits. The intersection of $P(D)$ with either the x nonnegative or x nonpositive half-planes is a convex polygon. The slopes of line transversals with nonnegative (nonpositive) slopes for D generate a slope range interval $[sl^{\min}, sl^{\max}]$ ($[sl_{\min}, sl_{\max}]$) resulted by the projection of $P(D)$ on the nonnegative (nonpositive) part of the x-axis, respectively. Each of the 4 end points of these two intervals is determined by two rectangles.

Due to this observation, the range of slopes of the possible nondecreasing line transversals is a closed interval contained in $\mathbb{R}^+$. We call this interval the *slope range* corresponding to $\lambda_1^* D'$ and denote it by $[SL^{\min}(D', S'), SL^{\max}(D', S')]$, where

- $SL^{\min}(D', S')$ is the slope of the line transversal for $\lambda_1^* D'$ with a minimal nonnegative slope, and
- $SL^{\max}(D', S')$ is the slope of the line transversal for $\lambda_1^* D'$ with a maximal nonnegative slope.

(Due to the optimality of the scaling factor, the slope range does not contain any direction from S' in its interior.) We are ready to define a lexicographic version for Problem 10.8.

Problem 10.12. Given are a set $D = \{d_1, \dots, d_n\}$ of (not necessarily pairwise disjoint) axis-parallel rectangles, together with a set of their reference points $C = \{c_1, \dots, c_n\}$, such that c_i lies in the interior of d_i for every $i = 1, \dots, n$, and a set $S = \{s_1, \dots, s_m\}$ of permissible line directions. Find the lexicographically minimal vector $\lambda = (\lambda_1, s, b, sl^{\min}, -sl^{\max})$ such that the line $y = sx + b$ ($s \in S$) is nondescending, intersects all of the rectangles in $\lambda_1 D$, and the slope range corresponding to $\lambda_1 D$ is $[sl^{\min}, sl^{\max}]$.

Clearly, the optimal solution of Problem 10.12 is an optimal solution for Problem 10.8.

We now apply Theorem 10.10 in order to construct a parameterized discrete Helly system. Let the ground set be $X = \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^-$, the space of all nondecreasing lines and slope ranges. In this space, each point represents a line by the geometric duality transformation mentioned above and a slope range, as shown below. Let the range of the objective function be $\Lambda = \mathbb{R}^+$. For every $h \in D$ and $\lambda \in \mathbb{R}^+$ we define

(10.1)
$$h_\lambda = \left\{ x = (a, b, sl^{\min}, -sl^{\max}) \in X \;\middle|\; \begin{array}{l} y = ax + b \text{ is a line transversal for } \lambda h \text{ and} \\ a \in [sl^{\min}, sl^{\max}]. \end{array} \right\}.$$

As usual, we let $\bar{h} = \{h_\lambda \mid \lambda \in \Lambda\}$.

LEMMA 10.13. *For all $h \in D$, $\bar{h}$ is a nested family.*

Proof. We need to show that for all $\alpha, \beta \in \Lambda$, with $\alpha < \beta$, $h_\alpha \subseteq h_\beta$, i.e., for all $x \in h_\alpha$, x is also in h_β. This is true by monotonicity: A line transversal for αh is also a line transversal for βh. ∎

For every $h \in S$ and $\lambda \in \Lambda$ we let

$$h_\lambda = \{x \mid x \text{ is a line with direction } s\}.$$

Obviously, for every $h \in S$, $\bar{h} = \{h_\lambda \mid \lambda \in \Lambda\}$ is a nested family as well, and h_λ does not depend on λ.

From the definitions and Theorem 10.9 we get that, for all $\lambda \in \Lambda$, $(X, D_\lambda, S_\lambda)$ is a discrete Helly system with Helly number 4, and thus $(X \times \Lambda, \bar{D}, \bar{S})$ is a parameterized discrete Helly system with Helly number 4 (see Definition 8.8). From Theorem 10.10, we get that, for all $\lambda \in \Lambda$, $(X, D_\lambda, S_\lambda)$ is a discrete Helly system with lexicographic Helly number 5. Thus all of the conditions in Theorem 8.16 are fulfilled, and (D, S, ν) is a DLP-type problem of d-dimension at most 5 and s-dimension 1, where for all $G = (D', S') \in 2^D \times 2^S$

$$\nu(G) = (\lambda_1(G), DIR(G), b(G), SL^{\min}(G), -SL^{\max}(G)). \tag{10.2}$$

We have just proved the following lemma.

LEMMA 10.14. *(D, S, ν) is a $(5, 1)$-dimensional DLP-type problem.*

Before we continue, we explain the values that the decision and optimization problems return. The decision problem returns "yes" if and only if $\lambda_1^* = \lambda_1(D, S) \leq 1$. The optimization problem returns the minimal scaling factor $\lambda_1(D, S)$, the minimal nonnegative direction from S such that $\lambda_1(D, S)$ admits a line transversal with direction $DIR(D, S)$ and intersection point $b(D, S)$ with the y-axis, the slope range defined by $SL^{\min}(D, S), SL^{\max}(D, S)$, and a basis $B = (B_D, B_S)$ for (D, S). We note that there exist line transversals for $\lambda_1^* D$ with each of the slopes $SL^{\min}(D, S)$ and $SL^{\max}(D, S)$. We can view B and $LINE(D, S)$ as witnesses for the optimality of the scaling factor $\lambda_1(D, S)$. We need only to check that $\lambda_1(B_D, S) = \lambda_1^*$ (the monotonicity of demand condition implies $\lambda_1^*(D, S) \geq \lambda_1^*$) and that the line transversal $LINE(D, S)$ intersects each one of the rectangles $\lambda_1^* h$, $h \in D$ (the monotonicity of supply condition implies $\lambda_1^*(D, S) \leq \lambda_1^*$). The first test can be executed in $|S|$ time and the second in $|D|$ time.

10.2.2. A linear time algorithm. In this section we apply the linear time algorithm stated in section 5. We need to show that the conditions stated in Theorem 5.4 are satisfied and implement each of the violation test and basis calculation primitives in constant time. In the last section we formulated Problem 10.8 as a $(5, 1)$-dimensional DLP-type problem. Thus, it remains to show the following.

LEMMA 10.15. *(D, S, ν) meets the VC (Definition 4.9).*

Proof. We need to show that for every $(D', S') \in 2^D \times 2^S$ and $(D'', S'') \in 2^{D'} \times 2^{S'}$ with $\nu(D', S') = \nu(D'', S'')$ the following properties hold:

1. For every $h \in D$, if $\nu(D'' \cup \{h\}, S'') > \nu(D'', S'')$, then $\nu(D' \cup \{h\}, S') > \nu(D', S')$.
2. For every $h \in S$, if $\nu(D'', S'' \cup \{h\}) < \nu(D'', S'')$, then $\nu(D', S' \cup \{h\}) < \nu(D', S')$.

Let $\lambda_1 = \lambda_1(D', S') = \lambda_1(D'', S'')$. We define the following functions related to $P(\lambda_1 D')$, the set of line transversal which $\lambda_1 D'$ admits (see Observation 10.11 for the definition and structure of $P(\lambda_1 D')$ in the dual space of line transversals). Let $l^{\min}(D', S')$ be the unique line transversal with direction $SL^{\min}(D', S')$ that $\lambda_1(D', S')D'$ admits. Let $b^{\min}(D', S')$ be its intersection point with the y-axis. In this way $(SL^{\min}(D', S'), b^{\min}(D', S'))$ is the leftmost point in $P(\lambda_1 D')$. We define $l^{\max}(D', S')$ and $b^{\max}(D', S')$ similarly, so $(SL^{\max}(D', S'), b^{\max}(D', S'))$ is the rightmost point in $P(\lambda_1 D')$.

The proof of both properties relies on the following observation which is true due to (10.2):

(10.3)

$$\nu(D', S') = \nu(D'', S'') \rightarrow \quad \begin{array}{l} \text{the functions } SL^{\min}, b^{\min}, l^{\min}, SL^{\max}, b^{\max}, \text{ and } l^{\max} \\ \text{have the same values on } (D', S') \text{ and on } (D'', S''). \end{array}$$

We first show that the first property holds. $h \in D$ violates (D'', S'') if and only if the set of line transversals which $\lambda_1 h$ admits does not contain both lines $l^{\min}(D'', S'')$ and $l^{\max}(D'', S'')$ (i.e., $\{(SL^{\min}(D'', S''), b^{\min}(D'', S''));\ (SL^{\max}(D'', S''), b^{\max}(D'', S''))\} \not\subset P(\lambda_1 h)$). Using (10.3), the latter condition occurs if and only if the set of line transversals which $\lambda_1 h$ admits does not contain both lines $l^{\min}(D', S')$ and $l^{\max}(D', S')$, which in turn occurs if and only if $h \in D$ violates (D', S').

Regarding the second property, we observe that $h \in S$ violates (D'', S'') if and only if either the slope h lies in the interior of the slope range corresponding to $\lambda_1 D''$ (so the scaling factor decreases) or h is the left end point of the slope range with $h < DIR(D'', S'')$. We conclude the proof by using (10.3) again. $\square$

We are ready to make the complexity calculations. Given a basis $B = (B_D, B_S)$ and its optimal scaling factor λ_1, we compute in constant time the functions $SL^{\min}(B)$, $b^{\min}(B), l^{\min}(B), SL^{\max}(B), b^{\max}(B), l^{\max}(B), DIR(B)$, and the set of all line transversals for $\lambda_1 B$, $P(\lambda_1 B_D) = \cap_{h\in B_D} \mathcal{D}(\lambda_1 h)$ (see Observation 10.11 for the notations). The following violation tests are implemented in constant time as follows:

t_{vS}: A new s-element h violates B if and only if either h lies in the interior of the slope domain $(SL^{\min}(B), SL^{\max}(B))$ or $h = SL^{\min}(B)$ and $DIR(B) = SL^{\max}(B)$.

t_{vD}: A new d-element h violates B if and only if $\mathcal{D}(\lambda_1 h)$ does not contain $\{(SL^{\min}(D'', S''), b^{\min}(D'', S'')); (SL^{\max}(D'', S''), b^{\max}(D'', S''))\}$.

Using the violation tests it is easy to see that the basis calculation for (D', S') where both $|D'|, |S'|$ are constants can be implemented in constant time. We have proved the following.

THEOREM 10.16. *Problem* 10.7 *is solvable in (randomized) linear time.*

COROLLARY 10.17. *The lexicographic discrete line transversal of axis-parallel rectangles problem is solvable in (randomized) linear time.*

10.3. Discrete case II—A finite number of permissible line transversals. In this section we show that the problem below has a lower bound of $\Omega(n \log n)$.

Problem 10.18. Given are a set $D = \{d_1, \ldots, d_n\}$ of (not necessarily pairwise disjoint) axis-parallel compact rectangles, together with a set of their reference points $C = \{c_1, \ldots, c_n\}$, such that c_i lies in the interior of d_i for every $i = 1, \ldots, n$, and a set $S = \{s_1, \ldots, s_m\}$ of permissible lines. Find the minimal scaling factor $\lambda_1^* = \lambda_1(D, S) \in \mathbb{R}^+$ such that $\lambda_1^* D$ admits a line transversal from S.

Clearly it is sufficient to show that the corresponding decision problem has that lower bound.

THEOREM 10.19. *Given a set D of axis-parallel rectangles and a set S of lines, deciding whether D admits a line transversal from S requires $\Omega(n \log n)$ time under the algebraic computation tree model (when $m = n$).*

Proof. We reduce in linear time the set equality problem (see definition in section 4) to this decision problem. Given an instance of the set equality problem, i.e., two sets A, B of n real numbers each, we act as follows. We find $\min_A$ and $\max_A$ ($\min_B$ and $\max_B$) the minimal and maximal elements in A (B), respectively. We define two new sets $A' = \{\frac{2(a-\min_A)}{\max_A - \min_A} - 1 \mid a \in A\}$ and $B' = \{\frac{2(b-\min_B)}{\max_B - \min_B} - 1 \mid b \in B\}$. All of the elements in A' and B' are numbers between -1 to 1, and $A = B$ if and only if $A' = B'$. For any $-1 \le r \le 1$ let $p(r)$ be the intersection point of the unit circle and the ray originating at the origin and having an angle of r radians with the positive part of the x-axis. We define two instances for the problem. The first instance, instance I, has a set of lines $S_{\mathrm{I}} = S(A')$ and a set of rectangles (intervals) $D_{\mathrm{I}} = D(B')$ defined as follows. We let $S(A') = \{s(a') \mid a' \in A'\}$, where $s(a')$ is the line tangent to the unit circle at point $p(a')$. We let $D(B') = \{i(b') \mid b' \in B'\}$, where $i(b')$ is a horizontal interval of length M, where M is a large number (e.g., 100), whose left end is slightly to the right of $p(b')$ (from a computation point of view we build the left end of the interval at exactly $p(b)$ but symbolically do not include this point in the interval). From the above construction we get that $D(B')$ admits a line transversal from $S(A')$ if and only if $A \not\subset B$. The second instance, instance II, has the set of lines $S_{\mathrm{II}} = S(B')$ and the set of rectangles $D_{\mathrm{II}} = D(A')$. We get that $D(A')$ admits a line transversal from $S(B')$ if and only if $B \not\subset A$. We conclude the proof by observing that $A = B$ if and only if both instances of the problem return negative responses. $\square$

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