Iterative Methods for Approximating the Subdominant Modulus of an Eigenvalue of a Nonnegative Matrix

Moshe Haviv
Department of Statistics
The Hebrew University
Jerusalem 91905, Israel

Yakov Ritov
Department of Statistics
The Hebrew University
Jerusalem 91905, Israel

Uriel G. Rothblum*
Faculty of Industrial Engineering and Management
Technion – Israel Institute of Technology
Haifa 32000, Israel

Submitted by Hans Schneider

ABSTRACT

Two easily computable sequences of bounds on the subdominant modulus of an eigenvalue of a square nonnegative matrix are obtained. In particular it is shown that the sequences converge to the subdominant modulus. A sequence of bounds generated by a method of Brauer (1971) turns out to be a subsequence of one of our sequences. Thus, our results imply the convergence of Brauer’s sequence.

1. INTRODUCTION

Hoffman [3], Brauer [2], and Rothblum and Tan [6] use the following method to obtain bounds on the subdominant modulus of an eigenvalue of an

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We continue by introducing some notational conventions. The section

\[ 2. \text{ NOTATION AND CONVENTIONS} \]

We are interested in examples in Section 4, and prove them in Section 5. The methods we use

\[ 3. \text{ APPROXIMATING THE SUBDOMINANT MODULI OF AN ENVELOPE} \]

in method 2 in Section 2 and prove them in Section 3. The methods we use

\[ 4. \text{ ENVELOPE} \]

will be denoted \( M \) and \( \mathfrak{m} \) respectively, and their spectral radius will be denoted \( r(M) \) and \( r(\mathfrak{m}) \) respectively. The section

\[ 5. \text{ RESULTS} \]

on \( S \times S \) matrices will be denoted \( S \) and \( S \) respectively, and their spectral radius will be denoted \( r(S) \) and \( r(S) \) respectively. The section

\[ 6. \text{ APPLICATIONS} \]

are used in Section 4, and prove them in Section 5. The methods we use

\[ 7. \text{ ACKNOWLEDGMENTS} \]

are very similar to the methods used in Section 2. Here, we expect the corresponding matrix to be a
to combine elements from numerical

\[ 8. \text{ REFERENCES} \]

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\[ 9. \text{ APPENDIX} \]

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\[ 13. \text{ APPENDIX} \]

are used in Section 4, and prove them in Section 5. The methods we use
Theorem 1. For $q = 1, 2, \ldots$

Similarly, the right-hand side of (2) is obtained by selecting $q = P_T'$. Theorem 2.1 applies when $q = m$. We next state our main result. It asserts that by taking roots of the above

\[ \left( \frac{1}{n^2} \right)^n \leq \left( \frac{a}{g} \right)^{\frac{1}{n}} \]

Thus (1) is proportional to $n$ and $q$. Theorem 2.1 applies when $q = m$. We next state our main result. It asserts that by taking roots of the above

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\[ \left( \frac{1}{n^2} \right)^n \leq \left( \frac{a}{g} \right)^{\frac{1}{n}} \]
\[
\left\{ (2^\lambda - 1) - \bar{2}, (2^\lambda + 1) - \bar{2}, \bar{2} \right\} = (\lambda - 1) \left( b \cdot g \right) a - \bar{g}
\]

Of course, in order to use this recursive formula one has to argue that for
\[
\forall \lambda (\lambda \geq 0), (b)_{\bar{g}}^\lambda \in [1, 0]
\]

... and for \( g = 1 \) and \( \lambda \geq 0 \) we denote \( \bar{g} \) by \((b)_{\bar{g}}\) to signify the sequence of \( S \) matrices by \((b)_{\bar{g}}\) is defined on \( g = 1 \).

It follows that \( \bar{g} = \left( b \cdot \bar{g} \right) a - \bar{g} \). In particular, the sequence \( \left( \bar{g} \right) \) is decaying and

\[
\forall \lambda (\lambda \geq 0), (b)_{\bar{g}}^\lambda \in [1, 0]
\]

Theorem 2. Then for all \( \lambda \), the sequence \( (b)_{\bar{g}}^\lambda \) is decaying and

\[
(b)_{\bar{g}}^\lambda \leq \left( b \cdot (1 + b) \right)_{\bar{g}}^\lambda = \left( b \cdot (1 + b) \right)_{\bar{g}}^\lambda
\]

Hence, the sequence is a subsequence of \((b)_{\bar{g}}^\lambda\) and is a convergent sequence of \( (b)_{\bar{g}}^\lambda \).

Furthermore, the subsequence \((b)_{\bar{g}}^\lambda\) is not monotonically decreasing. Therefore, Theorem 1 asserts that a

... and for all \( \lambda \), the sequence \((b)_{\bar{g}}^\lambda\) is decaying and

\[
\forall \lambda (\lambda \geq 0), (b)_{\bar{g}}^\lambda \in [1, 0]
\]

Moreover, the limit of the sequence \((b)_{\bar{g}}^\lambda\) exists and is equal to \( b \).
\[ 0 \leq [\lambda(q)m - q] = [\lambda(q)m - q]^{-1} \]

\[ \text{as } \lambda(q)m \leq q \]

Next, let \( q \leq m \), then

\[ n(q, \lambda(q)m - q) = n(q, \lambda(q)m - q) \]

Establishing (20).

Proof.

The definition of \( \eta(q, n(q, m - q)) \) directly implies the equivalence of the

Lemma 3. For \( q \in \mathbb{R}^+ \),

\[ n(q, \lambda(q)m - q) \leq n(q, \lambda(q)m - q) \]

Finally, for \( q \in \mathbb{R}^+ \),

\[ n(q, \lambda(q)m - q) \leq n(q, \lambda(q)m - q) \]

and

\[ n(q, \lambda(q)m - q) \leq n(q, \lambda(q)m - q) \]

with equality holding if and only if \( q = \lambda(q)m \).

Also,

\[ n(q, \lambda(q)m - q) \leq n(q, \lambda(q)m - q) \]

In particular for \( q \) satisfying the equivalent conditions of (17),

\[ n(q, \lambda(q)m - q) \leq n(q, \lambda(q)m - q) \]

Lemma 3. For \( q \in \mathbb{R}^+ \).

We will now establish the parts of Theorem 1 that concern \( \lambda(q) \).

Unfortunately, the first summation of (9) and \( q \in \mathbb{R}^+ \).
Proof of Theorem 1. Applying (24) to $B$, we have that

\[
[(n', n, g', g)][(n', n, g', g)] = (g - d)(g' - d) = g' - d + g' = (s + g,d) 
\]

(46)

Subordinant Modulus of an Eigenvalue

We are now ready for the proof of Theorem 1.

Lemma 1. For $g \in H$, the following equation summarizes some properties of $g$ and

\[
(n', n, g)[(n', n, g)] = a(n,n')g - (g - d)
\]

(47)

Under this selection of $g$, (see Lemma 2) will be derivable $(n', n, g)[(n', n, g)] = a(n,n')g - (g - d)$.

(48)

\[
\begin{align*}
\int_1^0 \frac{d}{dx} \left[ \frac{1}{n^2} \int_1^0 \left( \frac{g}{n} - g \right) dx \right] &= \left( \frac{g}{n} - g \right) \\
\text{where } q &= \text{proportion of } q \text{ is determined by selecting } \frac{\left( \frac{g}{n} - g \right)}{n} = q
\end{align*}
\]

(49)

This is a special case of the theorem of Lemma 1. Therefore, we will use them to establish the desired conclusion.

We next consider bounds on $(g')$. By applying Lemma 2 to vectors $g'$, the definition of $\|g'\|$ directly implies the equivalence of the corresponding propostion to Lemma 1. (This is possible because $1 = n, n'$.)

Finally, for $t = 1, 2, \ldots, n$

(50)

\[
[(n', n, g', g)] = a[(n', n, g) - g] 
\]

(51)

\[
(n', n, g') \supset (n', n, g) \supset (g') 
\]

Also
6. COMPUTATION

We first establish Theorem 2. (2)'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$, and the well-known finite integral of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$, and the well-known finite integral of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$.

It follows directly from (17) and the result of Theorem 1.

Theorem 1. (17) is the well-known finite integral of $\mathbf{y}$, and (17) is the well-known finite integral of $\mathbf{y}$.

We note that the essential convergence of the sequence $\{a^i\}$ follows.

Thus, we conclude from (32) that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$, and the well-known finite integral of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$, and the well-known finite integral of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$.

Next, let $\mathbf{y}$ be the well-known finite integral of $\mathbf{y}$, and the well-known finite integral of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$, and the well-known finite integral of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$.

Therefore, the essential convergence of the sequence $\{a^i\}$ follows.

and the definition of $\mathbf{y}$ and the definition of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$, and the well-known finite integral of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$.

Next, let $\mathbf{y}$ be the well-known finite integral of $\mathbf{y}$, and the well-known finite integral of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$, and the well-known finite integral of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$.

Thus, with $\mathbf{y}$, we conclude that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$, and the well-known finite integral of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$, and the well-known finite integral of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$.

This result is trivial for $\mathbf{y}$, and the well-known finite integral of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$, and the well-known finite integral of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$.

and consider $\mathbf{y} + b$ By the well-known finite integral of $\mathbf{y}$, and the well-known finite integral of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$, and the well-known finite integral of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$.

Next, assume that it holds for $\mathbf{y}$, then $\mathbf{y}$, and the well-known finite integral of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$, and the well-known finite integral of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$.

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Next, we observe that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$, and the well-known finite integral of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$, and the well-known finite integral of $\mathbf{y}$'s, p. 117, we have that $\mathbf{y}$ is the well-known finite integral of $\mathbf{y}$.

Therefore, the essential convergence of the sequence $\{a^i\}$ follows.
In the theory of non-negative matrices and non-negative cones, a fundamental result by Gajewski and Neumann states that if \( C \subseteq \mathbb{R}^{n \times n} \) is an ordered cone and \( A, B \in C \), then the least upper bound of \( A \) and \( B \) in \( C \) is given by:

\[
\min(A, B) = \left( \min_{i,j} (a_{ij}, b_{ij}) \right)
\]

for matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \). This result is crucial in understanding the structure of non-negative matrices and their applications in various fields, including economics and computer science.

In the context of non-negative matrices, the document highlights the importance of understanding the least upper bound in the ordered cone, which is essential for solving problems related to matrix inequalities and optimization.

Reference: