Equilibrium strategies in queues based on time or index of arrival

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Abstract

In most decision models dealing with unobservable stochastic congested environments, one looks for a (Nash) equilibrium behavior among customers. This is a strategy that if adopted by all, then under the resulting steady-state conditions, the best response for an individual is to adopt this strategy too. The purpose of this paper is to look for a simple decision problem but where the assumption of steady-state conditions is removed. Specifically, we consider an M/M/N/N loss model in which one pays for trying to get service but is rewarded only if one finds an available server. The initial conditions at time zero are common-knowledge and each customer possesses his arrival time as his private information. The equilibrium profile tells each arrival if to try or not (randomization allowed) given his time of arrival. We show that all join up to some point of time. At this point, there

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is a quantum drop in the joining probability from one to some fraction. From then on their joining probability continuously converges to the equilibrium joining probability under the model which assumes steady-state.

1 Introduction

By now there is a large literature on customers' behavior in queueing models. See [1] for a review. In these models selfish customers are decision makers trying to maximize their utility functions. Typically, there is an infinite number of customers who are generated in accordance with a Poisson process. Usually customers' whereabouts interact and hence the solution concept adopted is that of a Nash equilibrium. Next we exemplify such a decision problem, state what is the traditional approach for defining equilibrium which is based on the concept of steady-state. Then, we give a possible different definition for an equilibrium behavior where the assumption of steady-state is removed.

The following model originated in [2] and is dealt with also in [1], pp. 60-61. This is an $M/M/1/1$ model with trial costs and service rewards. Specifically, to a single server station there is a *potential* demand of λ per unit of time. This potential demand is generated by a Poisson process. Service times follow exponential distribution with parameter μ. No waiting room exists and a customer who finds a busy server upon trial, leaves for good. Customers value receiving service by R. However, a trial costs C. To avoid trivialities, assume that $C < R$. Customers have to decide if to try or not (randomization allowed) while seeking to maximize their individual expected net gain. Note that they do not know the status of the server upon generation (as otherwise this would have been a trivial decision problem). Thus, in case one tries, one's utility is R times the probability that the server is idle minus C. Not trying comes with a zero utility.[1]

The analysis given in [1], pp. 60-61, is typical for such decision making in queueing systems. The approach is as follows: for any symmetric strategy profile (ie, the same strategy is selected by all), which here is being characterized by the trying probability of p, look for the corresponding steady-state conditions. These conditions are reflected by the probabilities of an idle

[1]The value of zero is without loss of generality and it reflects the opportunities elsewhere.

and of a busy server. They are $\mu/(p\lambda + \mu)$ and $p\lambda/(p\lambda + \mu)$, respectively. Under these conditions, one's utility in case that one selects strategy p' is $p'(R\mu/(p\lambda+\mu)-C)$. An equilibrium strategy profile is then a strategy which is also one's best response under steady-state conditions resulting when all select this strategy too. In our case, $0 \leq p \leq 1$, defines an equilibrium profile if

$$p \in \arg\max_{0\leq p'\leq 1} p'(R\mu/(p\lambda + \mu) - C). \tag{1}$$

We denote this (unique) probability by p_e. It is clear that if $C \leq R\mu/(p\lambda+\mu)$, for all $0 \leq p \leq 1$ (which of course holds if and only if $C \leq R\mu/(\lambda + \mu)$), then $p_e = 1$.[2] Otherwise, p_e, $0 < p_e < 1$, is such that $R\mu/(p_e\lambda + \mu) - C = 0$. Indeed, if all try with probability p_e, then for an individual customer, under the resulting steady-state conditions, trying himself with probability p_e is a best response.[3] This is the case as one is indifferent between the two pure options of trying or not, and hence one may randomize as well. In summary, trying with a probability of p_e defines a Nash equilibrium strategy.[4] The issue that this definition of Nash equilibrium avoids is how steady-state conditions have been reached. Or put differently, a question which still exists here is what makes non-asymptotic customers behave in accordance with p_e?[5]

The purpose of this paper, apparently for the first time, is to remove the assumption that steady-state conditions under some strategy (in fact, the

[2]In fact, it is a dominant strategy, ie, no matter what others do then, under steady-state conditions, it is best for an individual to try.

[3]In fact, any probability is a best response against p_e but this is immaterial.

[4]Note that if $p > p_e$ ($p < p_e$, resp.) one's best response is not to try (to try, resp.), making this an *avoid the crowd* case as one's optimal response (described by the trying probability), is monotone non-increasing by the common trying probabilities of the others. Also, it is possible to see that p_e is an *evolutionarily stable equilibrium*. By that we mean that for any $p \neq p_e$ which is a best response against p_e, p_e is a better response for an individual against all playing p. For further details see [1]. Note that for the case where $0 < p_e < 1$ any p, $0 \leq p \leq 1$ is one's best response against p_e.

[5]This does not make the traditional analysis incorrect. In particular, a strategy profile is defined as a distribution over the trial probabilities which indicates percentage of customer trying with the corresponding probabilities. For any strategy profile used by all, there is a well defined objective function for an individual customer (based on the resulting steady-state probabilities) as exemplified in (1). Moreover, for any such profile there exists individual return for any action one may used. Finally, one looks for a strategy which is the best response against itself which in particular implies that the strategy profile is with the same prescription for all, ie, a symmetric strategy.

equilibrium strategy) have been reached. Instead, we assume that customers possess some private information as their time of generation (Section 2) or their serial number of generation (Section 3).

As opposed to decision models which concentrate on steady-state analysis, for the abovementioned non-stationary type of private information, the description of the models calls for assuming some initial conditions which of course will be part of the customers' common knowledge. In the M/M/1/1 model dealt with here it is natural to assume that at time zero the server is idle. Yet, some other initial condition can be assumed without much effect on the qualitative results we report below.

Our main findings for the decision models under consideration are as follows. In Section 2 we deal with the case where at time $t = 0$ the system is empty and where the time of generation is known to the customer involved and only to him (ie, this is his private information). In Subsection 2.1 we show that in the M/M/1/1 model the equilibrium strategy is characterized by two values, time t_e and trial probability p'_e: Up to time t_e all try with probability one, and from time t_e and on all try with probability p'_e. As it turns out, the equilibrium $p'_e = p_e$, the latter being the trial probability under the model which assumes steady-state. In Subsection 2.2 we look at the same question but now for the M/M/2/2 model. Here our findings are even more surprising: The equilibrium profile is such that all who are generated from time zero to some time t_e try with probability one. Then, as in the M/M/1/1 model, there is a gap in the trial probability. Specifically, denote by $p_e(t)$ the equilibrium trial probability at time t. Then, although $p_e(t_e) = 1$, $p_e(t_e+) < 1$. Also, $\lim_{t\to\infty} p_e(t) = p_e$ where p_e is the corresponding steady-state equilibrium trial probability. Finally, all our numerical runs indicate that for $t > t_e$, $p_e(t)$ is strictly monotone decreasing. Subsection 2.3 generalizes these findings to the general $M/M/N/N$ model, $N \geq 1$.

In Section 3 we deal with the case where the serial number of one's generation is one's private information (while the same initial conditions are assumed). We consider only the relatively simple case of M/M/1/1. We show that for this model, the equilibrium strategy is periodical and is characterized by an integer $m \geq 1$. Specifically, the first arrival tries, the next $m - 1$ customers do not try, customer $m + 1$ tries, then the next $m - 1$ do not, etc. We also compare this decision model with that where the time of generation is private information. In particular, we show that no general order between p_e and $1/m_e$ exists.

2 Time dependent strategies

This section contains three subsections. They deal, in this order, with the M/M/1/1, M/M/2/2 and M/M/N/N (where $N \geq 3$) models.

2.1 The M/M/1/1 case

This section is devoted to the M/M/1/1 case where time of generation is customers' private information. Subsection 2.1.1 deals the equilibrium profile while Subsection 2.1.2 looks at a different criterion, that of maximizing social gain.

2.1.1 Equilibrium strategies

As said above, assume the initial condition that at time zero the server is idle. Suppose a customer who is generated at time t, tries with probability $p(t)$, $t \geq 0$, independent of anything else. The latter implies that the resulting trial process is a non-homogeneous Poisson process with rate $\lambda p(t)$, $t \geq 0$. Denote this behavior with P. Also, let $I_P(t)$ be the probability that the server is idle at time t when all behave in accordance with the strategy profile P. We define a Nash-equilibrium as a strategy profile P such that for any t, $t \geq 0$,

$$p(t) \in \arg\max_{0 \leq p \leq 1} p(I_P(t)R - C), \quad 0 \leq t < \infty.$$

We denote such a profile (which below is shown to be unique) by P_e and the corresponding trial probability at time t by $p_e(t)$.

Theorem 2.1 *If* $C < \frac{R\mu}{\mu+\lambda}$ *then* P_e *is with* $p_e(t) = 1$ *for all* $t \geq 0$.[6] *Otherwise, the unique* P_e *is such that for some* $t_e < \infty$, $p_e(t) = 1$ *for* $t \leq t_e$ *and* $p_e(t) = p_e$ *for some* p_e, $0 < p_e < 1$ *when* $t > t_e$. *Moreover,*

$$p_e = \frac{\mu}{\lambda}\frac{R - C}{C} \tag{2}$$

and

$$t_e = -\frac{1}{\lambda + \mu}\log_e \frac{1 - p_e}{\lambda + \mu p_e}. \tag{3}$$

In particular, when $t > t_e$, *the equilibrium behavior coincides with that of the model which assumes steady-state.*

[6] In fact, it is a dominant strategy.

Proof. It is well known (see, e.g. [4], p. 150), that if at time zero the server is idle and if all try during the time interval $[0, t]$, then the probability that the server is idle at time t, equals

$$\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}, \quad t \geq 0 \; .$$

Note that this probability is monotone decreasing with t. Denote it by $I_{\mathbf{1}}(t)$ and note that

$$\lim_{t\to\infty} I_{\mathbf{1}}(t) = \mu/(\mu+\lambda). \tag{4}$$

If $C < R\frac{\mu}{\lambda+\mu}$, then even if all try, $C < RI_{\mathbf{1}}(t)$ for any t, and one would better try oneself at any time. Thus, $p_e(t) = 1$ for all $t \geq 0$.

We first observe that $I_P(t) \geq I_{\mathbf{1}}(t)$, $t \geq 0$. To see this we can couple two queueing processes, the first where all try and the second with strategy P based on two independent Poisson processes, one with rate λ (potential arrivals) and the other with rate μ (potential departures) and an independent sequence $\{U_n |\; n \geq 1\}$ of independent uniform (on $(0,1)$) random variables as follows. At an instant τ_n of the nth potential arrival the first process jumps up by one if it is at zero. The second jumps up by one if it is at zero *and* $U_n \leq p(\tau_n)$, otherwise it remains at zero. Both processes remain at one if they are at one at a time of a potential arrival. At an instant of a potential departure both processes jump down by one if their value at this instant is one and remain at zero if it is zero. It is clear that with this coupling the first process (all try) is always greater than or equal to the second process (strategy P) for any $t \geq 0$, which in turn implies that $I_P(t) \geq I_{\mathbf{1}}(t)$.[7]

Thus, the individual's best response against any policy P is to try up to time t_e. In the case where $C > R\mu/(\mu+\lambda)$, it is easy to see that p_e (as defined in (2)) is with $0 < p_e < 1$. Also, $I_{\mathbf{1}}(t_e)$ is such that $RI_{\mathbf{1}}(t_e) = C$ (where t_e is as defined in (3)). It is also easy to see that that $I_{\mathbf{1}}(t_e) = \frac{\mu}{\lambda p_e+\mu}$. Extending (4) to any constant joining probability, it is possible to see the following stationarity point: If at some point in time the probability of idleness is $\mu/(\mu+\lambda p_e)$ and from then and on all try with probability $p(t) = p_e$, then $\mu/(\mu+\lambda p_e)$ is the idleness probability from that point in time and on. Hence, if all try in case of generation prior to time t_e and afterwards try

[7] It is interesting to note, that this coupling approach also implies that if $p_1(t) \leq p_2(t)$ for all $t \geq 0$ then $I_{P_1}(t) \geq I_{P_2}(t)$ for all $t \geq 0$, since $\{U_n \leq p_1(\tau_n)\} \subseteq \{U_n \leq p_2(\tau_n)\}$.

with probability p_e, then all those who arrive after t_e are indifferent between trying or not. In particular, one might as well try with probability p_e. In other words, this is an equilibrium behavior. $\Box$

Remark 2.1 It is interesting to observe that the equilibrium trying probability is a step function with (at most) one drop. It is no surprise that $p_e(t) = 1$ for all $0 \leq t \leq t_e$ for some t_e, and that $p_e(t) < 1$ otherwise, and that $\lim_{t\rightarrow} p_e(t) = p_e$. What we find somewhat not intuitive is that $p_e(t)$ does not decrease to p_e gradually but rather in a single jump at t_e.

Remark 2.2 As was pointed out in the introduction, p_e has a steady-state meaning. Specifically, in case that all use this trying probability (regardless of time of generation and under any initial conditions), then an asymptotic arrival is indifferent between trying or not, and hence trying with this probability is also a best response for him. Likewise, if at time zero the server is idle with probability $\mu/(\lambda p_e + \mu)$ (which is the steady-state idleness probability when all join with probability p_e), then joining with probability p_e for *any* time t, is a Nash equilibrium.

Remark 2.3 A possible question here is how the equilibrium strategy will vary with the initial conditions. Specifically, let π be the probability that at time zero the server is idle. It is evident, with a similar approach as in the proof of Theorem 2.1, that if $1 \geq \pi \geq p_e$ ($0 \leq \pi \leq p_e$, respectively) then all will try (not try, respectively) from time zero until some time $t(\pi)$ and from then on all will try with probability p_e. Moreover, $t(\pi)$ is monotone decreasing (increasing, respectively) in π.

Remark 2.4 It is possible to think of different information structures leading to equilibria which are easy to characterize. For example, suppose one knows upon one's generation when all previous customer generations took place. Inductively, it is possible to learn which among them tried (given that all did their best for themselves). Hence, one can deduce when the last trial took place. Suppose it was s unit of times before his own arrival. Of course, it is immaterial if this trail was successful or not. Then one should try if and only if $C < R(1 - e^{-\lambda s})$.

2.1.2 The social optimal strategy

The following is taken from [1], pp. 60-61, and is repeated here for completeness and comparison. See also [3] for a related problem where the decision maker possesses more information (how much time elapsed from the last trial). Suppose now that the cost of all trials and service rewards are being borne and gained, respectively, by a single entity, call it *society*. Social optimization behavior and selfish (ie, equilibrium) behavior disagree since under the latter case, externalities imposed by one who tries (in the guise of increasing the probability that the next to try finds a busy server) are ignored. Under social optimization they are being taken into consideration, leading to a reduced trial rate.
If all try with probability p, the net gain per unit of time (during an infinitely long horizon) is

$$\lambda p \left(R \frac{\mu}{\mu + p\lambda} - C \right) . \tag{5}$$

Note that this gain is maintained regardless of any initial conditions and likewise, if during some finite time, some other strategy is used. Hence, we look for p, $0 \leq p \leq 1$, which maximizes (5). Denote the maximizer p by p^*. It is an easy exercise to check that $p^* = \min\{1, \mu(\sqrt{R/C} - 1)/\lambda\}$. Also, $p_e = p^*$ if and only if $p^* = 1$. Otherwise, $p_e > p^*$, as expected: Left to themselves customers intend to overcrowd the system more than it is socially desired. This is the case since selfish customers do not mind the effect of their moves on others. In this example, one's trials reduces the utility of another. As is many other related decision models, it is possible to find T (for toll or tax) such that when looking for p_e when R is replaced with $R - T$, the resulting value coincides with p^* (under R). Note that this toll regulates customers behavior so as their resulting equilibrium behavior coincides with the social optimal one.

In the following subsections of Section 2 we deal with models with a few servers. As from the social optimization point of view there are nothing conceptually new, we do not return to this issue again.

2.2 The $M/M/2/2$ case

First we find the equilibrium trial probability for the steady-state case, p_e. This value is either one or the unique positive solution to the equation in p:

$$\frac{\frac{1}{2}(\frac{\lambda p}{\mu})^2}{1+\frac{\lambda p}{\mu}+\frac{1}{2}(\frac{\lambda p}{\mu})^2} = 1 - \frac{C}{R}\ . \tag{6}$$

This is the case since the left hand side is the probability that the two servers are busy.
The only positive root of this quadratic equation is

$$\frac{\mu}{\lambda}\frac{R-C+\sqrt{R^2-C^2}}{C}$$

Therefore,

$$p_e = \min\left\{1,\ \ \frac{\mu}{\lambda}\frac{R-C+\sqrt{R^2-C^2}}{C}\right\}. \tag{7}$$

In other words, if the positive root is smaller than one, the equilibrium profile prescribes mixing. Otherwise, it is pure and prescribes always trying. In this latter case, trying is also a dominant strategy.

We now look at the time dependent case. Suppose the system is empty at time 0 and that customers possess the private information of their time of generation. Assume that all arrivals between time zero and time t try with probability one and denote by $P_i(t)$ the probability of having i customers in the system at time t (which depends one the adopted strategy). If all try up to some time t_e, then for all $0 \leq t \leq t_e$, the following (Kolmogorov's forward) differential equations hold,

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t)$$

$$P_1'(t) = \lambda P_0(t) - (\lambda+\mu)P_1(t) + 2\mu P_2(t)$$

$$P_2'(t) = \lambda P_1(t) - 2\mu P_2(t)$$

with the initial conditions being $P_0(0) = 1$, $P_1(0) = P_2(0) = 0$.[8] The solution to this set of differential equations leads in fact to the probability that a customer who tries does not find any idle server:

[8] Note that $\Sigma_{i=1}^3 P_i(t) = 1$ or $\Sigma_{i=1}^3 P_i'(t) = 0$ which can replace one of the above three differential equations.

$$P_2(t) = \pi_2 \frac{\dfrac{1-e^{-c_1 t}}{c_1} - \dfrac{1-e^{-c_2 t}}{c_2}}{\dfrac{1}{c_1} - \dfrac{1}{c_2}} = \frac{\lambda^2}{\theta} \int_0^t \int_{c_1}^{c_2} v e^{-uv} \mathrm{d}u \mathrm{d}v \qquad (8)$$

where $\theta = \sqrt{\mu(4\lambda + \mu)}$, $c_1 = (2\lambda + 3\mu - \theta)/2$, $c_2 = (2\lambda + 3\mu + \theta)/2$ (noting that $0 < c_1 < c_2$) and $\pi_2 = \frac{(\lambda/\mu)^2/2}{1+(\lambda/\mu)+(\lambda/\mu)^2/2}$, which is the stationary probability of two busy servers when all join for all t. Clearly the right handside of (8) is increasing in t so that it is dominant to try with probability one until the first instant t_e for which

$$R(1 - P_2(t_e)) = C. \qquad (9)$$

Note that $P_2(t_e) = 1 - R/C$ is neither a function of λ nor of μ. This of course is not the case regarding t_e itself. For the case where $t > t_e$, the following differential equations hold,

$$P_0'(t) = -\lambda_e(t)P_0(t) + \mu P_1(t)$$
$$P_1'(t) = \lambda_e(t)P_0(t) - (\lambda_e(t) + \mu)P_1(t) + 2\mu P_2(t)$$
$$0 = P_2'(t) = \lambda_e(t)P_1(t) - 2\mu P_2(t) = \lambda_e(t)P_1(t) - 2\mu(1 - \frac{C}{R})$$

where $\lambda_e(t) = \lambda p_e(t)$, which is the arrival rate under equilibrium. Note that the zero in the left handside of the last equation is due to the equilibrium condition. The same can be said on replacing $P_2(t)$ with $1 - C/R$ in the righthand side there. One of the initial conditions for the latter set is that $P_2(t_e) = 1 - C/R$, while at least one of the two initial values $P_0(t_e)$ or $P_1(t_e)$ needs to be derived from the solution of the former set (the other one can be found by using the identity $\Sigma_{i=0}^2 P_i(t) = 1$). The resulting differential equations are not linear as, for example, both $\lambda_e(t)$ (or $p_e(t)$) and $P_1(t)$ are to be determined and their product appears in the last equation. This implies that one needs to consider numerical techniques in order to solve these differential equations.

Below we have our main findings regarding the M/M/2/2 model.

Theorem 2.2 *Assuming p_e, the equilibrium joining probability in the steady-state case (see (7)), is strictly smaller than one, then $p_e(t)$, $t \geq 0$, the equilibrium joining probability, possesses the following properties:*

1. $p_e(t) = 1$, *for* $t \in [0, t_e]$ *where* t_e *solves* $P_2(t) = 1 - C/R$ *for* t.[9]

2. $p_e(t)$ *has a discontinuity in* $t = t_e$, *i.e.,* $p_e(t_e) = 1$ *but* $p_e(t_e+) < 1$.

3. $\lim_{t\to\infty} p_e(t) = p_e$.

Proof. The first item was already established in our discussion preceding the theorem. We will now prove the second item. As can be observed from (8), $P_2(t)$ is increasing. Applying this fact into the last equation among the first set, we get that $\lambda P_1(t) > 2\mu P_2(t)$ for all $t \le t_e$. On the other hand, as $P_1(t)$ is continuous, the second set of differential equations implies that for all $t > t_e$, $\lambda P_1(t) > 2\mu P_2(t) = 2\mu(1 - C/R)$. This concludes the proof of the second item.
Once the first set of differential equations are solved, the value for t_e can be determined. However, the solution can not be presented analytically since it is of the type

$$Ae^{xt} + Be^{yt} + C$$

for some A, B, x, y. The second set of differential equations yields, after some manipulations, the following differential equation:

$$p_e'(t) = \frac{p_e^2}{2} p_e(t)[p_e - p_e(t)][\mu p_e + (\mu + \lambda p_e)p_e(t)] \ . \tag{10}$$

We note that at any point t for which $p_e(t) > (<) p_e$ then $p_e'(t) < (>) 0$ so that $p_e(\cdot)$ is strictly decreasing (increasing). Therefore, $p_e(\cdot)$ never crosses p_e (otherwise it would have to either increase or decrease on the "wrong" side of p_e). Thus, $p_e(\cdot)$ is monotone and bounded and thus has a limit. Therefore, so does the right handside of (10) and hence $p_e'(t)$ as well. Since either $p_e(t) \ge p_e$ for all $t > t_e$ or $p_e(t) \le p_e$ for all $t > t_e$ and $p_e(\cdot)$ is strictly increasing when $p_e(t) < p_e$, it follows that the limit of $p_e(t)$ is not zero. Moreover, this also implies that $\int_{t_0}^{\infty} p_e'(t)dt$ converges with $p_e'(t)$ being either nonnegative or nonpositive for all $t > t_e$. Thus $p_e'(t)$ converges to zero (as otherwise this integral would diverge). Hence, the right handside of (10) converges to zero as $t \to 0$. Since the limit of $p_e(t)$ is strictly positive, this implies that it is necessarily p_e. $\square$

[9] See (8) for an expression for $1 - P_2(t)$.

2.3 Generalizations for the $M/M/N/N$ case

In this subsection, we generalize some of the previous results to a loss system with $N \geq 3$ servers.

Theorem 2.3 *Let π_N be the limit probability that all the servers are busy, given that all try. Assume that the system is empty at time $t = 0$. Then,*

1. *If $\pi_N \geq 1 - \frac{C}{R}$ then trying is a dominant strategy for any t.*
2. *If $\pi_N \leq 1 - \frac{C}{R}$ then there exists $t_e > 0$ such that*
   - *$p_e(t) = 1$ is a dominant strategy for all $t < t_e$.*
   - *$p_e(t) < 1$, $t > t_e$.*
   - *$p_e < 1$.*

Proof. The theorem will be proved by the following two lemmas:

Lemma 2.1 *$P_N(t)$ is strictly increasing with t.*

To prove this lemma we need the following result.

Lemma 2.2 *Let $X(t)$ be the number of customers in the system at time t. For any t, $t \geq 0$, the random variables $X(t)|X(0) = k$ are stochastically (strictly) increasing in k. In particular, $P[X(t) = N|X(0) = k]$ is strictly increasing in k.*

Proof for lemma 2.2. The proof will follow a coupling argument. Let $M_1, \ldots, M_N$ be N independent Poisson processes each of which with rate μ (potential service completions for each of the servers) and M_0 be a Poisson process with rate λ (potential arrivals). Let $X_k(t)$ start with $0 \leq k \leq N$ customers in the system. If $k \geq 1$ and the first event is due to an arrival from one of $M_1, \ldots, M_k$, then it is an actual service and at this epoch X_k is reduced by one. If $k \leq N - 1$ and the first event is due to an arrival of one of $M_{k+1}, \ldots, M_N$ then X_k remains unchanged. If the first event is due to an arrival of M_0 then if $k \leq N - 1$, it is an actual arrival and X_k increases by one and if $k = N$ then X_k remains unchanged. After the first event one continues in the same manner but with either $k - 1$, k or $k + 1$ depending on which event occurred, and so on. Due to the memoryless property, the resulting process is the number of customers in the system in an $M/M/N/N$ queue starting from k customers at time zero. One may perform this for any initial state k and in particular for $k+1$ whenever $k \leq N-1$. It is clear that with this construction there is some $t_k \leq \infty$ such that $X_{k+1}(t) = X_k(t) + 1$ for every $t < t_k$ and $X_{k+1}(t) = X_k(t)$ for $t \geq t_k$ (if $t_k < \infty$). In particular t_k is the first event for which either X_k is increased by one but X_{k+1} does not or that X_{k+1} is decreased by one and X_k does not. Until this instant both processes increase and decrease by one at the same instants. This implies that $X_k(t) \leq X_{k+1}(t)$ for all t and since $X_k(t)$ has the conditional distribution of $X(t)|X(0) = k$, the lemma is proved upon noting that since $P[t_k > t] > 0$ for every $0 \leq t < \infty$ then the stochastic monotonicity is strict. □

Now, for all $s < t$,

$$\begin{aligned}
&P\{X(t) = N|X(0) = 0\} \\
&= \sum_{k=0}^{N} P\{X(t-s) = k|X(0) = 0\} P\{X(s) = N|X(0) = k\} \\
&> \left(\sum_{k=0}^{N} P\{X(t-s) = k|X(0) = 0\} \right) P\{X(s) = N|X(0) = 0\} \\
&= P\{X(s) = N|X(0) = 0\}
\end{aligned}$$

which proves Lemma 2.1 □
Lemma 2.1 implies that if $\pi_N < 1 - \frac{C}{R}$ then $P_N(t) < 1 - \frac{C}{R}$ for all $t \geq 0$ and hence it is dominant to try for all t. Otherwise, there exist $t_e < \infty$ such that $P_N(t_e) = 1 - \frac{C}{R}$.
We look now at the forward equations which fit this model: The last equation is $P_N'(t) = \lambda P_{N-1}(t) - N\mu P_N(t)$ and Lemma 2.1 implies that the left handside and hence the right handside, are strictly positive. As was done for the cases where $N = 1$ and $N = 2$, $P_N(t) = 1 - \frac{C}{R}$ for $t > t_e$. Hence, we obtain that $0 = \lambda(t)P_{N-1}(t) - N\mu(1 - \frac{C}{R})$. Therefore, as anything else is continuous, $\lambda_e(t_e+) < \lambda(t_e-) = \lambda$.□

3 Index dependent equilibrium strategies

3.1 Equilibrium strategy

Suppose now that the arrivals know their order of arrival, ie, the i-th to arrive customer knows that he is indeed the i-th arrival. Again, it is assumed that at time zero the server is idle and this is part of the common knowledge.[10] Based on this, each of the arrivals has to decide if to try or not. Thus, a strategy here is to assign for any index i, an action, to try or not to try (randomization allowed). Denote by $p^{(i)}$ the trying probability of the i-th customer and by $p_e^{(i)}$ his equilibrium trying probability (that below will be shown to be unique).

Before stating our main result, we introduce the following notation. For an integer $m \geq 1$, let $I(m)$ be the probability that an arrival finds an idle server given that the last trial took place by the customer whose index is smaller by m than his. It is clear that as long as the customer's index is larger than or equal to m, the actual index of the customer involved is immaterial. Likewise, it does not matter if the trial was a successful one or not. Of course,

$$I(m) = 1 - \left(\frac{\lambda}{\lambda + \mu}\right)^m \tag{11}$$

which is monotone increasing with m. Moreover, $\lim_{m\to\infty} I(m) = 1$.

Theorem 3.1 *If $C < RI(1)$ then $p_e^{(i)} = 1$ for all $i \geq 1$. In fact, it is a dominant strategy. Otherwise, let the integer m_e be defined by*

$$m_e = \arg\min_{m\geq 2}\{C \leq RI(m)\}.$$

[11] *Then $p_e^{(i)}$, $i \geq 1$, is uniquely given by $p_e^{(i)} = 1$ if $i = 1(mod\ m_e)$ and $p_e^{(i)} = 0$ otherwise.*[12]

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[10] As it will be easy to see below, we in fact also solve the case where the initial conditions are such that at time zero the server is busy. However, other initial conditions, ie, conditions under which the server at time zero is busy with a probability strictly between zero and one, are harder to analyze.

[11] Alternatively, let

$$m_e = \lceil \frac{\log C - \log R}{\log \lambda - \log(\mu - \lambda)} \rceil.$$

[12] Note that this strategy defines a cycle of a length of m_e arrivals: in the first period

Proof. Firstly, if $C < RI(1)$, then even under the least favorable conditions, namely when all previous arrivals tried[13] and hence the probability that the server is idle when one arrives (regardless of whether the last trial was successful or not as in both cases the server is busy at this point in time) is $I(1) = \mu/(\lambda + \mu)$ and hence one should try, independently of his index. Secondly, if $C > R(1)$, customer-2 would better not try (as the first one of course tried) and $I(1)$ is the probability that customer-2 finds an idle server. Given customer-2 did not try, customer-3 finds an idle server with probability $I(2)$. In the case where $m_e > 2$ he should not try. Otherwise, he should try. In case one tries, all start from this index and on as if it was the first arrival. In case he does not try, the probability that customer-4 finds an idle server is $I(3)$, etc. $\square$

An interesting question here is how p_e and $1/m_e$ are related. In particular, does there exist an order between the two values? The answer is that the order is parameter dependent. Consider the following example. Let $\lambda = \mu = R = 1$. Therefore, for all $0.5 < C \leq 0.75$, $m_e = 2$ while $p_e = \frac{1-C}{C}$. For all $\frac{1}{2} < C < \frac{2}{3}$, $p_e > \frac{1}{2} = \frac{1}{m_e}$ and for all $\frac{2}{3} < C < \frac{3}{4}$, $p_e < \frac{1}{2} = \frac{1}{m_e}$.

3.2 The socially optimal strategy

Suppose all costs and rewards are being paid from a single pocket, as in subsection 2.1.2. Note that here we assume the same information (or, in fact lack of) is available to the customers, in particular, the states of the server are not revealed to them upon arrival. The objective function here is to find a trial strategy $p^{(i)}$, $i \geq 1$, which maximizes the long-run social gain, say strategy $p_{so}^{(i)}$, $i \geq 1$, which maximizes

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} p^{(i)}(RP_i - C)$$

where P_i is the probability that the server is idle at the point of arrival of the i-th customer, given that all customers $1, \ldots, i-1$, followed the considered strategy. Usually, the socially optimal and the equilibrium strategy do not coincide. The reason behind that, as was already pointed out in (2.1.2), is that

one tries while in the consecutive $m_e - 1$ periods nobody tries. This is then repeated. Note also that the case where $m_e = 1$ was considered separately just because the corresponding strategy is dominant.

[13] What matters of course is only if the previous arrival tried.

selfish customers ignore, in their decision making process, the externalities that their acts inflict on others. In this model, a trial by an individual increases the probability of a failed trial by future customers. In other words, trying here comes with negative externalities. Thus, the socially optimal strategy would prescribe trials less often that the equilibrium strategy does. For more on this concept, see [1].

It is clear that here too there exists a socially optimal strategy which has the same shape as the equilibrium strategy, ie, it is based on a period of some length, say $m^* \geq 1$, such that customer-i should try if and only if $i = 1(mod\ m^*)$.[14]

Theorem 3.2 *The social optimal strategy is that customer-i tries if and only if $i = 1(mod\ m^*)$ where*

$$m^* \in \arg\max_{m\geq 1} \frac{RI(m) - C}{m} \tag{12}$$

Remark. It is possible to argue that the objective function in (12) is unimodal[15] and hence m^* is the index where the marginal increment of the objective becomes negative for the first time. Also, $m^* \geq m_e$ because m_e was determined by the first m such $RI(m) \geq C$ and the determination of m^* in (12) can be presented as

$$m^* \in \arg\max_{m|RI(m)>C} \frac{RI(m) - C}{m} \tag{13}$$

Remark. Since the set of strategies is discrete, m^* is not necessarily unique.

References

[1] Hassin, R. and M. Haviv (2003), *To Queue or not to Queue: Equilibrium Behavior in Queueing System*, Kluwer Academic Publishers, Norwell, MA 02061 USA.

[14]Since the behavior during any finite time horizon is irrelevant from the social optimality, the optimal strategy is not unique. Suppose one decides on a cycle of length m, the expected net gain during this cycle is $RI(m) - C$. The following result is now immediate.

[15]It is easy to show that $f(x) := (\alpha - \beta^x)/x$ is concave and has maximum under the constrains on the model's parameters.

[2] Lin, K.Y. and S.M. Ross (2003), “Admission control with incomplete information of a queueing system,” *Operations Research*, vol. 51, pp. 645-654.

[3] Lin. K.L. and S.M. Ross (2004), “Optimal admission control for a single-server loss queue,” *Journal of Applied probability*, vol. 41, pp. 535-546.

[4] Ross, S.M (1996), *Stochastic Processes, 2nd edition*, Wiley, New York.