Solution set for 'Queues: A Course in Queueing Theory' by Moshe Haviv

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I received much help in composing this solution set from Yoav Kerner, Binyamin Oz, and Liron Ravner. Credit is given when due next to the appropriate questions. I am indebted to all three of them.

Contents

1 Chapter 1

1.1 Question 1

We use induction. First, $\mathrm{E}(X^0) = \mathrm{E}(1) = 1$, which proves the formula for the case where $n = 0$. Assume the induction hypothesis that $\mathrm{E}(X^{n-1}) = (n-1)!/\lambda^{n-1}$. Next,

$$\mathrm{E}(X^n) = \int_{x=0}^{\infty} x^n \lambda e^{-\lambda x}\,dx = -x^n e^{-\lambda x}|_{x=0}^{\infty} + \int_{x=0}^{\infty} n x^{n-1} e^{-\lambda x}\,dx$$

$$= 0 + \frac{n}{\lambda}\int_{x=0}^{\infty} x^{n-1}\lambda e^{-\lambda x}\,dx = \frac{n}{\lambda}\mathrm{E}(X^{n-1}) = \frac{n}{\lambda}\frac{(n-1)!}{\lambda^{n-1}} = \frac{n!}{\lambda^n}.$$

where the one before last equality is due to the indiction hypothesis

1.2 Question 2

$$h(t) = \frac{f(t)}{\overline{F}(t)} = -\frac{d}{dt}\log_e(\overline{F}(t).$$

Hence, for some constant C

$$-\int_{t=0}^{x} h(t)\,dt = \log_e \overline{F}(x) + C$$

and for some constant K

$$\overline{F}(x) = K e^{-\int_{t=0}^{x} h(t)\,dt}.$$

Since $\overline{F}(\infty) = 1$ and since $\int_{t=0}^{x} h(t)\,dt = \infty$ by assumption, we conclude that $K = 1$ and complete the proof.

1.3 Question 3

a) $\mathrm{E}(X) = 1/\lambda$, $\mathrm{E}(Y) = 1/\mu$ and since $\min\{X,Y\}$ follows an exponential distribution with parameter $\lambda+\mu$, $\mathrm{E}(\min\{X,Y\}) = 1/(\lambda+\mu)$. Hence, with the help of the hint, $\mathrm{E}(\max\{X,Y\}) = 1/\lambda + 1/\mu - 1/(\lambda+\mu)$.

b)

$$F_W(t) = \mathrm{P}(W \le t) = \mathrm{P}(\max\{X,Y\} \le t) = \mathrm{P}(X \le t)P(Y \le t)$$

$$= (1-e^{-\lambda t})(1-e^{-\mu t}) = 1 - e^{-\lambda t} - e^{-\mu t} + e^{-(\lambda+\mu)t}.$$

c) It was already proved that $\mathrm{E}(\min_{i=1}^{n}\{X_i\}) = 1/(n\lambda)$. Once, the minimum among the n is realized, then, by the memoryless property, all starts from stretch but now with $n-1$ random variables. Again, we look at the minimum among them and its expected value equals $1/((n-1)\lambda)$. This is then repeated but now with $n-2$. These mean values need to be added and we get the required result. The fact that the ratio between this summation (called the harmonic series) and $\log_e n$ goes to 1 is well known. The moral here is that if you are waiting for a group to be formed where each individual's time of arrival follows an exponential distribution, you will wait quite a lot.

1.4 Question 4

We need to show that

$$\int_{x_1=0}^{y}\int_{x_2=x_1}^{y}\cdots\int_{x_{i-2}=x_{i-1}}^{y}\int_{x_{i+1}=y}^{x}\int_{x_{i+2}=x_{i+1}}^{x}\cdots\int_{x_n=x_{n-1}}^{x}\mathbf{1}\,dx_1\ldots dx_{i-1}\,dx_{i+1}\ldots dx_n$$

equals

$$\frac{y^{i-1}}{(i-1)!}\frac{(x-y)^{n-i}}{(n-i)!},\ 1\leq i\leq n.$$

Indeed taking the most inner integral with respect to x_n, we get $(x - x_{n-1})$. Repeating that, now with respect to x_{n-1}, we get $(x-x_{n-2})^2/2$. Next with respect to x_{n-2} and the integral equals $(x-x_{n-3})^3/3!$. Doing that until (and inclusive) x_{i+1}, results in $(x-y)^{n-i}/(n-i)!$. This is now a constant with respect to the rest of the integration, so we have

$$\frac{(x-y)^{n-i}}{(n-i)!}\int_{x_1=0}^{y}\int_{x_2=x_1}^{y}\cdots\int_{x_{i-2}=x_{i-1}}^{y}\mathbf{1}\,dx_1\ldots dx_{i-1}.$$

Again doing the integration one by one, leads first to $(y-x_{i-2})$, then to $(y-x_{i-3})^2/2$ and eventually to $(y-0)^{i-1}/(i-1)!$, concluding the proof.

An alternative way (but somewhat heuristic is as follows). Suppose you fix who is the observation who comes as the i-th observation. There are n option here. Also, you fix the indices of those to proceed it. There are $n-1$ choose $i-1$ options here, namely there are $n!/((i-1)!(n-i)!)$ options. The probability (in fact, density) that the one selected to have exactly a value of y is $1/x$. The probability that the selected $i-1$ will have a smaller value than y is $(y/x)^{i-1}$ and for the rest to have higher values is $((x-y)/x)^{n-i}$. All needed now is to multiply all these probabilities.

1.5 Question 5

$$A(t) = \sum_{i=1}^{\infty} p(1-p)^{i-1}t^i = tp\sum_{i=0}^{\infty}[(1-p)t]^i =$$
$$= tp\frac{1}{1-(1-p)t}.$$

1.6 Question 6

$$A(t) = \sum_{i=0}^{\infty} e^{-\lambda}\frac{\lambda^i}{i!}t^i = e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i t^i}{i!}$$
$$= e^{-\lambda}e^{\lambda t} = e^{-\lambda(1-t)}.$$

1.7 Question 7

Assume X and Y are two independent nonnegative random variables. Then,

$$A_{X+Y}(t) = \sum_{i=0}^{\infty} \mathrm{P}(X+Y=i)t^i = \sum_{i=0}^{\infty}\sum_{k=0}^{i}\mathrm{P}(X=k)\mathrm{P}(Y=i-k)t^i$$

which by the change of the order to summation equals

$$= \sum_{k=0}^{\infty}\mathrm{P}(K=k)t^k\sum_{i=k}^{\infty}\mathrm{P}(Y=i-k)t^{i-k}$$

$$= \sum_{k=0}^{\infty}\mathrm{P}(K=k)t^k\sum_{i=0}^{\infty}\mathrm{P}(Y=i)t^i$$

$$= \sum_{k=0}^{\infty}\mathrm{P}(K=k)t^k A_Y(t) = A_Y(t)\sum_{k=0}^{\infty}\mathrm{P}(K=k)t^k = A_Y(t)A_X(t).$$

The proof can now be conclude by induction: $\Sigma_{i=1}^{n}X_i = (\Sigma_{i=1}^{n-1}X_i) + X_n$.

1.8 Question 8

$f_X(t) = \Sigma_n p_n \lambda_n e^{-\lambda_n t}$. Hence, $\overline{F}_X(t) = \Sigma_n p_n e^{-\lambda_n t}$ and finally $F_X(t) = 1 - \Sigma_n p_n e^{-\lambda_n t}$

1.9 Question 9

The proof is given in fact in the footnote.

1.10 Question 10

(Benny) Let $X|Y$ be Erlang with parameters Y and λ where Y is geometric with parameter p. Then

$$F_X^*(s) = E(e^{-sX}) = E(E(e^{-sX}|Y)).$$

From (1.24) we learn that

$$E(e^{-sX}|Y) = (\frac{\lambda}{\lambda+s})^Y$$

and therefore, using (1.19) we get

$$F_X^*(s) = E((\frac{\lambda}{\lambda+s})^Y) = A_Y(\frac{\lambda}{\lambda+s}) = \frac{p\frac{\lambda}{\lambda+s}}{1-(1-p)\frac{\lambda}{\lambda+s}} = \frac{p\lambda}{\lambda+s-(1-p)\lambda}$$
$$= \frac{p\lambda}{s+p\lambda}$$

which is the LST of an exponential random variable with parameter $p\lambda$.

1.11 Question 11

$$F_X^*(s) = \int_{x=0}^{\infty} e^{-sx}\lambda e^{-\lambda x}\,dx = \frac{\lambda}{\lambda+s}\int_{x=0}^{\infty}(\lambda+s)e^{-(\lambda+s)x}\,dx$$
$$= \frac{\lambda}{\lambda+s}.$$

1.12 Question 12

It is sufficient to show the result for the case where $n=2$ as the rest will fellow with an inductive argument. This case be done as follows:

$$f_{X_1+X_2}^*(s) = \int_{x=0}^{\infty} e^{-sx} f_{X_1+X_2}(x)\,dx = \int_{x=0}^{\infty} e^{-sx}\int_{y=0}^{x} f_{X_1}(y)f_{X_2}(x-y)\,dy\,dx$$
$$= \int_{x=0}^{\infty}\int_{y=0}^{x} f_{X_1}(y)e^{-sy}f_{X_2}(x-y)e^{-s(x-y)},dy\,dx$$
$$= \int_{y=0}^{\infty} e^{-sy}f_{X_1}(y)\int_{x=y}^{\infty} e^{-s(x-y)}f_{X_2}(x-y)\,dx\,dy$$
$$= \int_{y=0}^{\infty} e^{-sy}f_{X_1}(y)\int_{x=0}^{\infty} e^{-sx}f_{X_2}(x)\,dx\,dy$$
$$F_{X_1}^*(s)F_{X_2}^*(s).$$

1.13 Question 13

We prove the result by induction and the use of the memoryless property. The case where $n = 2$ was established in Lemma 2.1(2).

$$\mathrm{P}(X_1 \geq \sum_{i=2}^{n} X_i) = \mathrm{P}(X_1 \geq \sum_{i=2}^{n} X_i | X_1 \geq X_2)\mathrm{P}(X_1 \geq X_2).$$

which by the memoryless equals

$$\mathrm{P}(X_1 \geq \sum_{i=3}^{n} X_i)\mathrm{P}(X_1 \geq X_2).$$

The left term equals, by the induction hypothesis to $\Pi_{i=3}^{n}\lambda_i/(\lambda_i + \lambda_1)$ while the second, due to Lemma 2.1(2) equals $\lambda_2/(\lambda_2 + \lambda_1)$. This completes the proof.

1.14 Question 14

In the case where all parameters in the previous question coincide, we get $1/2^{n-1}$ in the righthand side at the previous question. If we ask what is the probability that X_2 is larger than the sum of all others, we of course will get the same answer. Likewise for any X_i, $1 \leq i \leq n$ being greater than or equal to the sum of all others. The union of all these events which are clearly disjoint, is that one of them will be greater than or equal to the sum of the other. This one is the fact the maximum among them (and if the maximum is grater than the sum of the others, then clearly there exists such one of them). Summing up the probabilities we get $n/2^{n-1}$.

1.15 Question 15

Proof 1. We next show that the reciprocal function of $h_X(t)$ is monotone decreasing in t. Divide (1.12) by (1.10) and get, up to a multiplicative constant,

$$\frac{\sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}}{t^{n-1}} = \sum_{k=0}^{n-1} \frac{(\lambda t)^{k+1-n}}{k!}$$

which is clearly decreasing with t since $k + 1 - n \leq 0$ for all $0 \leq k \leq n$.

Proof 2. Recall that an Erlang random variable with parameters n and λ is the sum of n independent exponential random variables with parameter

λ. Thus, while one waits for a realization of it, it can be looked as being in one of n possible stages which progress as a Poisson process. Also, in order to see its termination in the next instant, in needs to be in its final stage. Moreover, given that, the hazard rate is then λ (as the time until conclusion is now exponential with parameter λ). Note that the hazard rate given any other stage is zero as one needs to conclude more than one stage in no time. Using the notation on Poisson process what we look for is

$$\mathrm{P}(N(t) = n | N(t) \leq n).$$

By (1.14) this probability equals

$$\frac{e^{-\lambda t}\frac{(\lambda t)^n}{n!}}{\sum_{k=0}^{n} e^{-\lambda t}\frac{(\lambda t)^k}{k!}}.$$

As in the previous proof it can easily seen that the reciprocal of this value is monotone decreasing with t.

1.16 Question 16

a)

$$f_X(t) = \sum_{i=1}^{n} \alpha_i \lambda_i e^{-\lambda_i t}.$$

b) $\mathrm{E}(X|I = i) = 1/\lambda_i$, $1 \leq i \leq n$ and since $\mathrm{E}(X) = \mathrm{E}(\mathrm{E}(X|I))$, $\mathrm{E}(X) = \Sigma_{i=1}^{n} \alpha_i/\lambda_i$. As for the variance, first $\mathrm{Var}(X|I = i) = 1/\lambda_i^2$, and then $\mathrm{E}(\mathrm{Var}(X)) = \Sigma_{i=1}^{n} \alpha_i/\lambda_i^2$. Second, $\mathrm{Var}(\mathrm{E}(X|I)) = \mathrm{E}(\mathrm{E}(X|I)^2) - \mathrm{E}^2(\mathrm{E}(X|I))$ which equals $\Sigma_{i=1}^{n} \alpha_i/\lambda_i^2 - (\Sigma_{i=1}^{n} \alpha_i/\lambda_i)^2$. Finally, use the fact that $\mathrm{Var}(X) = \mathrm{E}(\mathrm{Var}(X|I)) + \mathrm{E}(\mathrm{Var}(X|I))$ and sup up these two values to get

$$\Sigma_{i=1}^{n} 2\alpha_i/\lambda_i^2 - (\Sigma_{i=1}^{n} \alpha_i/\lambda_i)^2.$$

c) Bayes' formula leads to

$$\mathrm{P}(I = i | X \geq t) = \frac{\mathrm{P}(I = i)\mathrm{P}(X \geq t | I = i)}{\mathrm{P}(X \geq t)}$$

$$\frac{a_i e^{-\lambda_i t}}{\sum_{j=1}^{n} \alpha_j e^{-\lambda_i t}}, \quad 1 \leq i \leq n.$$

d)
$$h_X(t) = \frac{f_X(t)}{\bar{F}_X(t)} = \frac{\sum_{i=1}^n \alpha_i \lambda_i e^{-\lambda_i t}}{\sum_{i=1}^n \alpha_i e^{-\lambda_i t}}. \tag{1}$$

We need to show that this ratio is decreasing in t. We differentiate it with respect to t and need to observe that the derivative is negative for any value of t. Taking derivative in a standard way for a quotient, we next look only in the numerator of the derivative (since the denumerator is always positive). Specifically, this numerator equals of

$$-\sum_{i=1}^n \alpha_i \lambda_i^2 e^{-\lambda_i t} \sum_{i=1}^n \alpha_i e^{-\lambda_i t} + (\sum_{i=1}^n \alpha_i \lambda_i e^{-\lambda_i t})^2.$$

We argue that the above is negative by Cauchy-Swhartz inequality. It says that for any two positive series $\{a_i\}_{i=1}^n$, and $\{b_i\}_{i=1}^n$,

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - (\sum_{i=1}^n a_i b_i)^2 \geq 0.$$

Using this inequality with $a_i = \lambda_i \sqrt{\alpha_i e^{-\lambda_i t}}$ and $b_i = \sqrt{\alpha_i e^{-\lambda_i t}} \lambda_i$, $1 \leq i \leq n$, concludes the proof.

e) Inspect the ratio in (1). Multiply numerator and denominator by $e^{\min_{j=1}^n \{\lambda_j\} t}$ Take limit when t goes to infinity. All terms of the type $e^{-\lambda_i t + min_{j=1}^n \{\lambda_j\} t}$ will go to zero with the exception of $j^* \equiv \arg\min_{i=1}^n \{\lambda_i\}$ which is a constant. In particular, this is its limit. This leaves us with the $\min_{j=1}^n \{\lambda_j\}$. as required.

Comment: It is possible here to see that given $X \geq t$, namely the age equals t, the residual life time follows an hyper-exponential distribution with α_i being replace with $\mathrm{P}(I = i | X \geq t)$, whose explicit expression is given above. In words, as time progresses the weights on the possible exponential distributions move. In particular, the hazard function is a weighted average of the current posteriors for each of the options on I. When time is set to infinity, full weight is given to the slowest option possible. To visualize this, suppose a lightbut follows an exponential distribution conditional on quality, where quality is measured by the rate of burning, the slowest the better. The longer the lightbulb is functioning, the more likely it is to be of the best quality (with the corresponding exponential distribution) and hence the more time (stochastically) is ahead, as said by the DFR property.

1.17 Question 17

Given that server-i finishes first (a probability $\lambda_i/(\lambda_1+\lambda_2)$ event), the probability that this customer is not the last to leave is $\lambda_i/(\lambda_1+\lambda_2)$, $i=1,2$. Hence, the prior probability of this event is

$$\sum_{i=1}^{2}(\frac{\lambda_i}{\lambda_1+\lambda_2})^2.$$

1.18 Question 18

Integration by parts (note that s is considered as constant here) leads to

$$\int_{x=0}^{\infty} F_X(x)e^{-sx}\,dx = -\frac{1}{s}F_X(x)e^{-sx}|_{x=0}^{\infty} + \frac{1}{s}\int_{x=0}^{\infty} f_X(x)e^{-sx}\,dx.$$

The fact that the first term is zero concludes the proof.

1.19 Question 19

One needs to assume here that $F_X(0)=0$ as in the previous exercise.

$$\mathrm{P}(X \le Y) = \int_{y=0}^{\infty} \mathrm{P}(X \le y) f_Y(y)\,dy = \int_{y=0}^{\infty} \mathrm{P}(X \le y) s e^{-sy}\,dy$$

$$s\int_{y=0}^{\infty} F_X(y)e^{-sy}\,dy = s\frac{F_X^*(s)}{s} = F_X^*(s)$$

where the one before last equality is based on the previous exercise.

1.20 Question 20

(Was solved by Liron Ravner) Taking the derivative of the negative of this tail function, we get the density function $-e^{-(\lambda x)^\alpha}\lambda\alpha(\lambda x)^{\alpha-1}$. Dividing this density by the tail function we get that the hazard function equals

$$\alpha\lambda(\lambda x)^{\alpha-1}.$$

a) The hazard is decreasing with x when $0<\alpha<1$.

b) The hazard is increasing with x when $\alpha>1$. Note that in the case where $\alpha=1$ we get an exponential distribution which is both IHR and DHR.

1.21 Question 21

a) Straightforward differentiation leads to $f_X(t) = \alpha\beta^\alpha/(\beta+t)^{\alpha+1}$. This implies that $h_X(t) = \alpha/(\beta+t)$ which is decreasing with t.

b)
$$\mathrm{P}(X \geq x | X \geq t) = \frac{\mathrm{P}(X \geq x)}{\mathrm{P}(X \geq t)} = \frac{\beta^\alpha}{(\beta+x)^\alpha} / \frac{\beta^\alpha}{(\beta+t)^\alpha}$$
$$= \frac{(\beta+t)^\alpha}{(\beta+x)^\alpha}, \quad x \geq t.$$

Hence,
$$\mathrm{P}(X - t \geq x | X \geq t) = \frac{(\beta+t)^\alpha}{(\beta+x+t)^\alpha}, \quad x \geq 0.$$

Integrating this from zero to infinity leads to the expected value we are after (see Lemma 1.1). Specifically, the integral equals

$$\frac{(\beta+t)^\alpha}{-\alpha+1}(\beta+x+t)^{-\alpha+1}|_{x=0}^{\infty} = \frac{\beta+t}{\alpha-1}$$

which is decreasing with t.

1.22 Question 22

$$\mathrm{P}(\lfloor X \rfloor = i) = \int_{x=i}^{i+1} \lambda e^{-\lambda x} = e^{-\lambda x}|_{x=i}^{i+1} = e^{-\lambda i} - e^{-\lambda(i+1)}$$

$$(e^{-\lambda})^i(1 - e^{\lambda}), \quad i \geq 0.$$

2 Chapter 2

2.1 Question 1

$$\mathrm{E}(X) = \sum_{i=1}^{\infty} ip_i = \sum_{i=1}^{\infty}\sum_{j=1}^{i} p_i.$$

Changing the order of summation, we get

$$\mathrm{E}(X) = \sum_{j=1}^{\infty}\sum_{j=i}^{\infty} p_i = \sum_{j=1}^{\infty} q_j,$$

as required.

2.2 Question 2

$\mathrm{E}(X) = \Sigma_{i=1}^{\infty} p_n/\lambda_n$ and $\overline{F}_X(x) = \Sigma_{i=1}^{\infty} p_n e^{-\lambda_n x}$. Dividing the latter by the former gives the age density.

2.3 Question 3

Recall that $\mathrm{E}(X) = n/\lambda$. The tail density function is given in formula (1.12) (see Chapter 1). Dividing the latter by the former implies that

$$f_X(x) = \frac{1}{n}\sum_{k=0}^{n-1} \lambda e^{-\lambda x}\frac{(\lambda x)^k}{k!}.$$

This density is a mixture with equal probabilities among n Erlang densities with parameters $(1, \lambda), \ldots, (n, \lambda)$.

2.4 Question 4

(Benny) The first step is to compute the LTS of the right hand side. Then we show that it coincides with the LST of the left hand side. Before that, note the following facts:

a) The LST of a sum of N iid random variables with LTS $F_X^*(s)$ equals

$$\Pi_N(F_X^*(s))$$

where $\Pi_N(t) = E(t^N)$ is the z-transform of N.

b) The LTS of a sum of $N-1$ such random variables equals

$$\Pi_{N-1}(F_X^*(s)) = E(F_X^*(s)^{N-1}) = \frac{\Pi_N(F_X^*(s))}{F_X^*(s)}.$$

c) The z-transform of the length bias distribution of N equals

$$\begin{aligned}\Pi_{L_N}(t) &= \textstyle\sum_{i=1}^{\infty} t^i P(L_N = i) \\ &= \textstyle\sum_{i=1}^{\infty} t^i \frac{iP(N=i)}{E(N)} \\ &= \textstyle\frac{t}{E(N)} \sum_{i=1}^{\infty} t^{i-1} iP(N=i) \\ &= \textstyle\frac{t}{E(N)} \frac{d\,\Pi_N(t)}{d\,t}\end{aligned}$$

Now, recalling Lemma 2.3 and the fact that the LTS of an independent sum is their LTS product, the LST of the right hand side equals

$$F_{L_X}^*(s)\frac{\Pi_{L_N}(F_X^*(s))}{F_X^*(s)} = -\frac{d\,F_X^*(s)}{d\,s}\frac{1}{E(X)}\frac{\Pi_{L_N}(F_X^*(s))}{F_X^*(s)}$$

The LTS of the left hand side equals

$$\begin{aligned}F_{L_Y}^*(s) &= \textstyle -\frac{d\,F_Y^*(s)}{d\,s}\frac{1}{E(Y)} \\ &= \textstyle -\frac{d\,\Pi_N(F_X^*(s))}{d\,s}\frac{1}{E(N)E(X)} \\ &= \textstyle -\left.\frac{d\,\Pi_N(t)}{d\,t}\right|_{t=F_X^*(s)} \frac{d\,F_X^*(s)}{d\,s}\frac{1}{E(N)E(X)} \\ &= \textstyle -\frac{\frac{t}{E(N)}\left.\frac{d\,\Pi_N(t)}{d\,t}\right|_{t=F_X^*(s)}}{F_X^*(s)}\frac{d\,F_X^*(s)}{d\,s}\frac{1}{E(X)} \\ &= \textstyle -\frac{\Pi_{L_N}(F_X^*(s))}{F_X^*(s)}\frac{d\,F_X^*(s)}{d\,s}\frac{1}{E(X)}\end{aligned}$$

as required.
An explanation for this theorem is as follows. Sampling length biased Y is equivalent to sampling length biased X and length biased N, and then compute Y as this (biased) X plus a sum of (biased) $N-1$ of (unbiased) X-es.

2.5 Question 5

a) The event of being at stage i at time t, means that up to this time $i-1$ stages have being completed while the i-th is still in process. Its probability in the case where $i=1$ is $e^{-\lambda t}$ as this is the probability that a single exponential random variable with parameter λ is greater than or equal to t. In case of $i \geq 2$, it equals

$$\int_{y=0}^{t} \lambda e^{-\lambda y} \frac{(\lambda y)^{i-2}}{(i-2)!} e^{\lambda(t-y)}\, dy.$$

Note that we are using the complete probability theorem when the integration is with respect to where exactly stage $i-1$ ended at. Simple calculus leads to the value of

$$e^{-\lambda t} \frac{(\lambda t)^{i-1}}{(i-1)!}.$$

This needs to be divided by $\mathrm{P}(X \geq t)$ in order to get $p_i(t)$. This value can be read from formula (1.12) (see Chapter 1). Hence, we conclude that

$$p_i(t) = \frac{(\lambda t)^{i-1}/(i-1)!}{\sum_{j=0}^{n-1} (\lambda t)^j / j!}, \quad 1 \leq i \leq n.$$

b)

$$h_X(t) = \frac{f_X(t)}{\overline{F}_X(t)} = \frac{\lambda \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}}{\sum_{i=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^i}{i!}} = \lambda p_n(t).$$

The interpretation of this is as follows. The hazard rate corresponds to an immediate end. This can take place only if the current stage is stage n (immediate end in other stages requires the end of a number of stages in no time). Once the current stage is stage n, the rate of 'death' is λ. Hence the product of λ and $p_n(t)$ is the hazard rate in time t.

2.6 Question 6

(Benny & Liron) Let X be a non-negative random variable and denote the corresponding age variable by A. Suppose that X has a decreasing hazard rate, i.e. $\frac{f_X(x)}{\overline{F}_X(x)}$ is monotone decreasing w.r.t x. We prove that $\mathrm{E}(A) \geq \mathrm{E}(X)$ by showing that A stochastically dominates X: $F_X(x) \geq F_A(x),\ \forall x \geq 0$. Recall that the density of the age is $f_A(x) = \frac{\overline{F}(x)}{\mathrm{E}(X)}$. Since $\overline{F}_X(x)$ is monotone decreasing w.r.t. x, then so is $f_A(x)$. The decreasing hazard rate clearly implies that $f_X(x)$ is also monotone decreasing w.r.t. x, and at a faster rate than $\overline{F}_X(x)$ for any $x \geq 0$. Consider the equation $f_X(x) = f_A(x)$ or

$$f_X(x) = \frac{\overline{F}_X(x)}{\mathrm{E}(X)}, \tag{2}$$

It has a unique solution, denoted by x', because it is equivalent to:

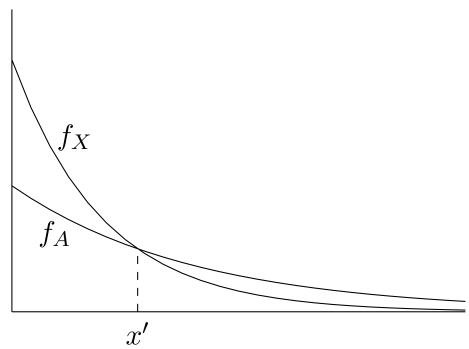
$$h_X(x) = \frac{1}{\mathrm{E}(X)}, \qquad (3)$$

where the LHS is decreasing and the RHS is constant.
All of the above lead to the following observations:

a) $f_X(x) > f_A(x)$ for any $x < x'$.

b) $f_X(x') = f_A(x')$.

c) $f_X(x) < f_A(x)$ for any $x > x'$.



The first and second imply that $F_X(x) > F_A(x)$, $\forall x \leq x'$. The third, together with the fact that $\lim_{x\to\infty} F_X(x) = lim_{x\to\infty} F_A(x) = 1$, implies that $F_X(x) = F_A(x)$ only at the bounds of the support of X. Finally, we can conclude that $F_X(x) \geq F_A(x), \forall x \geq 0$, which implies $\mathrm{E}(A) \geq \mathrm{E}(X)$ and thus completes the proof.

As for the second part, recall that the coefficient of variation of a random variable equals the ratio between its standard deviation and its mean. We next show that the square of this ratio is greater than or equal to one if and and only if $\mathrm{E}(A) \geq \mathrm{E}(X)$. Indeed,

$$\frac{\mathrm{E}(X^2) - \mathrm{E}^2(X)}{\mathrm{E}^2(X)}$$

is larger than or equal to one, if and only if

$$\mathrm{E}(X^2) \geq 2\mathrm{E}^2(X),$$

which is easily seen to be the case if and only if $\mathrm{E}(A) \geq \mathrm{E}(X)$ since $\mathrm{E}(A) = \mathrm{E}(X^2)/2\mathrm{E}(X)$.

2.7 Question 7

The approach in the first two items is to consider the kernel of a density function in the family, then to multiply it by x and get yet again a kernel in this family.

a) The kernel of the density of a gamma random variable with parameters α and β equals
$$x^{\alpha-1}e^{-\beta x}, \quad x \geq 0.$$
Multiplying it by x makes it a gamma kernel but with parameters $\alpha+1$ and β.

b) The kernel of the density of a beta random variable with parameters α and β equals
$$x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \leq x \leq 1.$$
Multiply it by x just replaces the α with $\alpha+1$.

c) This is a special case of the previous case but with the case where $\beta = 1$. In order to conclude the proof we need to show that what is claimed to be the density function of L is a proper density function, i.e., $\theta+1$ is the correct constant. Indeed,
$$\int_{x=0}^{1} x^{\theta}\, dx = \frac{1}{\theta+1}x^{\theta+1}|_{x=0}^{1} = \frac{1}{\theta+1}.$$

2.8 Question 8

a) By (2.14) for the case where X follows a binomial distribution
$$\mathrm{P}(L-1=l) = \frac{(l+1)p_{l+1}}{\mathrm{E}(X)}, \quad 0 \leq l \leq n-1.$$
Since $\mathrm{E}(X) = np$, the righthand side equals
$$\frac{(n-1)!}{l!(n-1-l)!}p^{l}(1-p)^{n-1-l}, \quad 0 \leq l \leq n-1$$
which is indeed the probability of $l-1$ for a binomial random variable with parameters $n-1$ and p.

b) The random variable X follows a negative binomial distribution with parameters r and p for some integer $r \geq 1$ and a fraction $0 < p < 1$. This means that

$$\mathrm{P}(X = k) = \left(\begin{array}{c} k-1 \\ r-1 \end{array} \right) p^r (1-p)^{k-r}, \quad k \geq r.$$

Hence,

$$\mathrm{P}(L - 1 = \ell) = \mathrm{P}(L = \ell + 1) = \frac{(\ell+1)\mathrm{P}(X = \ell+1)}{\mathrm{E}(X)}, \ \ell + 1 \geq r.$$

Since $\mathrm{E}(X) = r/p$, we get that

$$\mathrm{P}(L = \ell - 1) = \frac{(\ell+1)\frac{l!}{(r-1)!(\ell+1-r)!} p^r (1-p)^{\ell+1-r}}{\frac{r}{p}}.$$

With minimal algebra this equals

$$\left(\begin{array}{c} \ell+1 \\ r \end{array} \right) p^{r+1} (1-p)^{\ell-(r+1)},$$

which is the probability of $\ell{+}1$ for a negative binomial random variable with parameters $r+1$ and p.

c)

$$\mathrm{P}(L - 1 = l) = \frac{(l+1)p_{l+1}}{\mathrm{E}(X)} = (l+1)e^{-\lambda} \frac{\lambda^{l+1}}{(l+1)!} \frac{1}{\lambda}.$$

This easily seen to equal $\mathrm{P}(X = l)$. Next we show the converse. If $\mathrm{P}(L - 1 = l) = \mathrm{P}(X = l)$, we can conclude that

$$\mathrm{P}(L = l + 1) = (l+1)\mathrm{P}(X = l+1)/\mathrm{E}(X) = \mathrm{P}(X = l).$$

Hence, $\mathrm{P}(X = l + 1) = \mathrm{E}(X)\mathrm{P}(X = l)/(l + 1)$. By induction we get that that $\mathrm{P}(X = l) = \mathrm{P}(X = 0)\mathrm{E}^l(X)/l!$. Since the probabilities sum up to one, we get that $\mathrm{P}(X = 0) = e^{-\mathrm{E}(X)}$ and that $\mathrm{P}(X = l) = e^{-\mathrm{E}(X)}\frac{E^l(X)}{l!}$. This is the Poisson distribution when $\mathrm{E}(X)$ is usually denoted by λ.

2.9 Question 9

a) If $X \leq a + r$ the process is never on. Otherwise, it is on only when it is in the 'middle' of its life, when it crosses the age of a and before it enters the period when its residual is smaller than or equal to r. Moreover, the length of this period is $X - a - r$. Thus, the expected time of 'on' is $\mathrm{E}(\max\{X - a - r, 0\})$. Clearly,

$$\mathrm{E}(\max\{X - a - r, 0\}) = \int_{x=a+r}^{\infty} (x - a - r) f_X(x).$$

b) By the analysis of Section 2.3, we learn that

$$\mathrm{P}(\text{'}on'\text{'}) = \mathrm{P}(A \geq a, R \geq r) = \frac{\mathrm{E}(\max\{X - a - r, 0\})}{\mathrm{E}(X)}.$$

Taking derivative with respect to both a and r leads us to the joint density for (A, R) at the point (a, r). This is done in detail next but first note that

$$\int_{x=a+r}^{\infty} (x-a-r) f_X(x)\, dx = \int_{x=a+r}^{\infty} x f_X(x)\, dx - a \int_{a+r}^{\infty} f_X(x)\, dx - r \int_{x=a+r}^{\infty} f_X(x)\, dx.$$

Taking derivative with respect to a, leads to

$$-(a + r) f_X(a + r) - \int_{x=a+r}^{\infty} f_X(x) + a f_X(a + r) + r f_X(a + r).$$

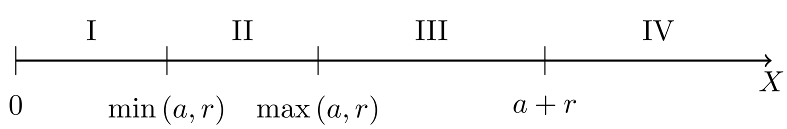
And now with respect to r,

$$-f_X(a+r) - (a+r) f'_X(a+r) + f_X(a+r) + a f'_X(a+r) + r f'_X(a+r) + f_X(a+r),$$

which is easily seen to equal $f_X(a + r)$. Hence, the joint density function of A and R equals $f_X(a, r)/\mathrm{E}(X)$.

2.10 Question 10

a) (Benny) The values of a and r splits the possible values of X in to 4 cases as described below:



In case I X is smaller than a and r, therefore the condition is true for all $0 < t < X$ so the 'on' length is X.

In case II X is between a and r which means $a < X < r$ or $r < X < a$. In the first case, the age is smaller than a only for $0 < t < a$ and the residual is smaller then r for all $0 < t < X$ so the 'on' length is a. In the second case, the age is smaller than a for all $0 < t < X$ but the residual is smaller then r only for $X - r < t < X$ so the 'on' length is r. Concluding both cases for case II, the 'on' length is $\min(a, r)$

In case III and IV X is grater than a and r, therefore the age is smaller than a only for $0 < t < a$ and the residual is smaller then r only for $X - r < t < X$ so both conditions are true for $X - r < t < a$. The 'on' length is grater then 0 only if $X - r < a$ namely $X < a + r$ that is true in case III but not in case IV. Concluding both cases (III and IV), the 'on' length is $\min(a + r - X, 0)$

Now it is easy to see that, given X, the 'on' length is equal to,

$$\min(X, a) + \min(X, r) - \min(X, a + r)$$

by simply checking the 4 cases above.

Thus,

$$\mathrm{P}(A \leq a, R \leq r) = \frac{\mathrm{E}(\min\{a, X\} + \min\{r, X\} - \min\{a + r, X\})}{\mathrm{E}(X)}.$$

b) We next deal with the numerator above. We need to take its derivative first with respect a and then with respect to r (or with the reversed order). The first two terms will of course end with zero. So look only at the third:

$$-\mathrm{E}(\min\{a + r, X\}) = -\int_{x=0}^{a+r} x f_X(x)\, dx - (a + r)\overline{F}_X(a + r).$$

Taking derivative with respect to a, we get

$$-(a + r)f_X(a + r) - \overline{F}_X(a + r) + (a + r)f_X(a + r) = -\overline{F}_x(a + r).$$

Taking now the derivative with respect to r, we get $f_X(a + r)$ as was required.

3 Chapter 3

3.1 Question 1

$$\begin{aligned}&\mathrm{P}(X_{n+2}=i_{n+2},X_{n+1}=i_{n+1}|X_0=i_0,\ldots,X_n=i_n)\\&=\mathrm{P}(X_{n+2}=i_{n+2}|X_{n+1}=i_{n+1},X_0=i_0,\ldots,X_n=i_n)\mathrm{P}(X_{n+1}=i_{n+1}|X_0=i_0,\ldots,X_n=i_n),\end{aligned}$$

which by the definition of a Markov process equals

$$\begin{aligned}&\mathrm{P}(X_{n+2}=i_{n+2}|X_{n+1}=i_{n+1},X_n=i_n)\mathrm{P}(X_{n+1}=i_{n+1}|X_n=i_n)\\&=\mathrm{P}(X_{n+2}=i_{n+2},X_{n+1}=i_{n+1}|X_n=i_n),\end{aligned}$$

as required.

3.2 Question 2

$$\begin{aligned}&\mathrm{P}(X_{n-1}=i_{n-1},X_{n+1}=i_{n+1}|X_n=i_n)\\&=\frac{\mathrm{P}(X_{n-1}=i_{n-1},X_{n+1}=i_{n+1},X_n=i_n)}{\mathrm{P}(X_n=i_n)}\\&=\frac{\mathrm{P}(X_{n+1}=i_{n+1}|X_n=i_n,X_{n-1})\mathrm{P}(X_n=i_n,X_{n-1}=i_{n-1})}{\mathrm{P}(X_n=i_n)},\end{aligned}$$

which, due to the fact that we have a Markov process, equals

$$\begin{aligned}&\mathrm{P}(X_{n+1}=i_{n+1}|X_n=i_n)\frac{\mathrm{P}(X_n=i_n,X_{n-1}=i_{n-1})}{\mathrm{P}(X_n=i_n)}\\&=\mathrm{P}(X_{n+1}=i_{n+1}|X_n=i_n)\mathrm{P}(X_{n-1}=i_{n-1}|X_n=i_n),\end{aligned}$$

as required.

3.3 Question 3

$$\begin{aligned}&\mathrm{P}(X_n=i_n|X_{n+2}=i_{n+2},X_{n+1}=i_{n+1})\\&=\frac{\mathrm{P}(X_{n+2}=i_{n+2},X_{n+1}=i_{n+1},X_n=i_n)}{\mathrm{P}(X_{n+2}=i_{n+2},X_{n+1}=i_{n+1})}\\&=\frac{\mathrm{P}(X_{n+2}=i_{n+2}|X_{n+1}=i_{n+1},X_n=i_n)\mathrm{P}(X_{n+1}=i_{n+1},X_n=i_n)}{\mathrm{P}(X_{n+2}=i_{n+2}|X_{n+1}=i_{n+1})\mathrm{P}(X_{n+1}=i_{n+1})}.\end{aligned}$$

Since the process is a Markov process, the first terms in the numerator and the denominator, coincide. Hence, we got

$$\frac{\mathrm{P}(X_{n+1}=i_{n+1},X_n=i_n)}{\mathrm{P}(X_{n+1}=i_{n+1})}=\mathrm{P}(X_n=i_n|X_{n+1}=i_{n+1}),$$

as required.

3.4 Question 4

The result can be established by the help of an induction argument. The cases where $n = 1$ and $n = 2$ are stated in page 39. Assuming, it holds for n. Then, by the complete probability theorem,

$$\mathrm{P}(X_{n+1} = j) = \sum_i \mathrm{P}(X_n = i) P_{ij}(n).$$

Inserting above the value of $P(X_n = i)$ as stated by the induction hypothesis, concludes the proof.

3.5 Question 5

a) We next prove that if two matrices (with the same dimension) are stochastic, then the same is the case with their product. The theorem is now established by the use of induction. Specifically, suppose A and B are two stochastic matrices in $R^{n \times n}$. Then, for any i, $1 \le i \le n$,

$$\sum_{j=1}^n (AB)_{ij} = \sum_{j=1}^n \sum_{k=1}^n A_{ik} B_{kj} = \sum_{k=1}^n A_{ik} \sum_{j=1}^n B_{kj} = 1 \times 1 = 1.$$

b) This item is also shown with the help of an induction argument. The result is correct for the case where $n = 1$ by definition. Then,

$$\mathrm{P}(X_{n+1} = j | X_0 = i)$$

$$= \sum_k \mathrm{P}(X_{n+1} = j | P(X_n = k, X_0 = i) \mathrm{P}(X_n = k | X_0 = i) \sum_k \mathrm{P}(X_{n+1} = j | X_n = k) \mathrm{P}(X_n = k | X_0 = i)$$

since the process is a Markov chain. This equals, by the time-homogeneity of the process and the induction hypothesis, to

$$\sum_k P_{kj} P_{ik}^n = P_{ij}^{n+1},$$

as required.

c) Y_{n+1} and Y_n are two observations from the original Markov process, which are k time epochs from each others. Hence, given Y_n, Y_{n+1} does not depend on $Y_0, \ldots, Y_{n-1}$ which makes in a Markov chain itself. Also, $Y_{n+1} | Y_n$ is distributed as $Y_1 | Y_0$. The transition matrix of this process is P^k as required.

3.6 Question 6

Let v be a probability vector for which is it given that $v_j = \Sigma_i v_i P_{ij}$ for all states but some state j_0. We next show that the same equation holds for j_0 too. This is done by some algebra as follows:

$$v_{j_0} = 1 - \sum_{j \neq j_0} v_j = 1 - \sum_{j \neq j_0} \sum_i v_i P_{ij} = 1 - \sum_i v_i \sum_{j \neq j_0} P_{ij} = 1 - \sum_i v_i (1 - P_{ij_0}),$$

the last equality being true do to the fact that P is stochastic. This then equals to

$$1 - \sum_i v_i + \sum_i v_i P_{ij_0} = 1 - 1 + \sum_i v_i P_{ij_0} = \sum_i v_i P_{ij_0},$$

as required.

3.7 Question 7

(Liron) Let P be the probability matrix of a time-reversible discrete-time Markov process with some finite state space S. According to Theorem (3.4), the limit probability vector by u satisfies

$$u_j = \sum_{i \in S} u_i P_{ij}, \ j \in S \tag{4}$$

and

$$\sum_{j \in S} u_j = 1. \tag{5}$$

Now let us divide the state space into two disjoint partitions J and J' such that $J \cup J' = S$. We will prove that the normalized vector $\hat{u}_J \in \mathbb{R}^{|J|}$:

$$\hat{u}_j = \frac{u_j}{\sum_{j \in J} u_j}, \ j \in J \tag{6}$$

is the limit probability vector of the Markov process with probability matrix:

$$\hat{P}_{ij} = \begin{cases} P_{ij}, & i \neq j, \ i, j \in J \\ P_{ii} + \sum_{k \in J'} P_{ik}, & i = j, \ i \in J \end{cases} . \tag{7}$$

Recall the definition of time-reversibility: $u_i P_{ij} = u_j P_{ji}, \ \forall i, j \in S$. We use this property to show that

$$u_j = \sum_{i \in J} u_i \hat{P}_{ij}. \tag{8}$$

We compute the RHS:

$$\begin{aligned}
\sum_{i\in J} u_i \hat{P}_{ij} &= \sum_{i\neq j,\ i\in J} u_i P_{ij} + u_j \left(P_j j + \sum_{k in J'} P_{jk} \right) \\
&= \sum_{i\in J} u_i P_{ij} + \sum_{k\in J'} u_j P_{jk} \\
&\stackrel{TR}{=} \sum_{i\in J} u_i P_{ij} + \sum_{k\in J'} u_k P_{kj} \\
&= \sum_{i\in S} u_i P_{ij} = u_j.
\end{aligned}$$

We have shown that u_J solves the balance equations of the process defined by $\hat{P}$. Therefore, $\hat{u}_J$ clearly solves the equations too, along with the probability vector condition.

3.8 Question 8

We start with the necessity part: Assume the Markov chain is time-reversible and hence that the detailed balanced equations (see (3.7)) hold. Hence, for any given path $(i_1, i_2, \ldots, i_k)$ one gets that

$$\begin{aligned}
u_{i_1} P_{i_1 i_2} &= u_{i_2} P_{i_2 i_1}, \\
u_{i_2} P_{i_2 i_3} &= u_{i_3} P_{i_3 i_2}, \\
&\vdots \\
u_{i_k} P_{i_k i_1} &= u_{i_1} P_{i_1 i_k}.
\end{aligned}$$

Multiplying all left hand sides and then (separately) all right hand sides and noting that the products of the u components have the same contribution in both sides, leads to (3.8). For the converse, fix a pair of states i and j. Then for any k intermediate states $i_1, \ldots, i_k$, condition (3.8) implies that

$$P_{ii_1} P_{i_1 i_2} \cdots P_{i_{k-1} i_k} P_{i_k j} P_{ji} = P_{ij} P_{j i_k} P_{i_k i_{k-1}} \cdots P_{i_2 i_1} P_{i_1 i}.$$

Summing up the above with respect to all k-length paths, we get that

$$P_{ij}^{k+1} P_{ji} = P_{ij} P_{ji}^{k+1}.$$

Taking limit with respect to k and recalling that $\lim_{k\to\infty} P_{ij}^{k+1} = u_j$ and that $\lim_{k\to\infty} P_{ji}^{k+1} = u_i$, we conclude that

$$u_j P_{ij} = u_i P_{ij},$$

as required.

In the special case where all off diagonal entries of P are positive, it is claimed that the condition

$$P_{ij}P_{jk}P_{ki} = P_{ik}P_{kj}P_{ji}$$

for any three states i,j and k is sufficient for time-reversibility (the necessity is trivial since the corresponding condition is necessary for all cycles). Indeed, fix a state, call it state-0, and any positive value for u_0. Then define u_j as $u_0 P_{i_0 j}/P_{j i_0}$. It is possible to show that this choice for u_j solves the detailed balance equations. Indeed, It is easy to check that for any pair of i and j, $u_j P_{jk} = u_k P_{kj}$:

$$u_j P_{jk} = u_0 \frac{P_{i_0 j}}{P_{j i_0}} P_{jk} \quad \text{and} \quad u_k P_{kj} = u_0 \frac{P_{i_0 k}}{P_{k i_0}} P_{kj},$$

which are easily seen to be equal since $P_{i_0 j}P_{jk}P_{k i_0} = P_{i_0 k}P_{kj}P_{j i_0}$.

3.9 Question 9

Consider two states i and j, both being in J. What we look for is the probability that given that i is the initial state, for the next state visited in J is state j. Of course, P_{ij} is the probability of moving directly there but to this probability we need to add the probability of hoping from i to J' and when J' is left, the first state in J visited in state j. Minding all possible states from which J' can be entered, all possible length of time in staying there and finally all possible states to exit J' from, we get that this additional probability equals

$$\sum_{k \in J'} P_{ik} \sum_{n=0}^{\infty} \sum_{l \in J'} (P^n_{J'J'})_{kl} P_{lj}.$$

Writing this in matrix notation and recalling that $\sum_{n=0}^{\infty} P^n_{J'J'} = (I - P_{J'J'})^{-1}$ since $P_{J'J'}$ is a transient matrix (see Lemma 3.1), the last display equals

$$\sum_{k \in J'} P_{ik} (I - P_{J'J'})^{-1}_{lj}.$$

In matrix notation this leads to $P_{JJ'}(I - P_{J'J'})^{-1} P_{J'J}$.

4 Chapter 4

4.1 Question 1

Denote the mean of this type of busy period by $\bar{b}_s$. Then, in an argument similar to the one leading to (4.11) one gets that

$$\bar{b}_s = \bar{s} + \lambda\bar{s}\bar{b},$$

where $\bar{b}$ is the mean of a standard busy period. Indeed, each one who arrives during the first service time can be seen as opening a standard busy period. Moreover, $\lambda\bar{s}$ is the mean number of such arrivals. Finally, from (4.10) we learn that $\bar{b} = \bar{x}/(1-\rho)$. The rest is trivial algebra. Next, let $\bar{n}_s$ be the mean number who are served during this type of busy period. Then,

$$\bar{n}_s = 1 + \lambda\bar{s}\bar{n},$$

where $\bar{n}$ is the mean number who are served during a standard busy period. From (4.13) we learn that $\bar{n} = 1/(1-\rho)$. Hence,

$$\bar{n}_s = 1 + \lambda\bar{s}\frac{1}{1-\rho}.$$

4.2 Question 2

a) Applying Little's rule to the case where the system under consideration is the server, we get the λW product is in fact $\lambda\bar{x}$ or ρ. This equals L, the mean number in service. This should be smaller than 1. Hence, $\rho < 1$ is the condition needed for stability. Note that it does not matter for how long the vacation is or under which policy the server resumes service. All required is that once he is back to service, he does not take another vacation prior to emptying the system.

b) The above argument leads to the conclusion that ρ is the utilization level.

c) The reduction of the queue length from n to $n-1$ from the instant of a service commencement is equivalent to a busy period. Thus, this server needs to complete n busy periods prior to his next vacation. Hence, the final answer is $n\bar{b} = n\bar{x}/(1-\rho)$. Note that the reduction of the queue length from n to $n-1$ when looked from an arbitrary point in not as a standard busy period. This is due to the fact that the residual service time of the one in service is not $\bar{x}$ (which is true only upon service commencement). In fact, it has a value which is a function of n. More on that and at length in Chapter 6. Finally, note that this disclaimer does not hold when service is exponentially distributed: By the memoryless property, the residual service time is always with the same mean of $\bar{x}$.

d) With probability ρ the service is busy and hence he will be ready for the next server after time whose mean equals $\bar{x^2}/2\bar{x}$. With the complementary probability of $(1-\rho)$ is on vacation which will end after time whose

mean equals $(n-1)/(2\lambda)$. Note that once in vacation the number in queue is uniformly distributed between 0 and $n-1$, implying that the vacation will end after a uniformly distributed between 0 and $n-1$ number of arrivals, each of which takes on average $1/\lambda$ units of time. Using PASTA and the same argument used in Proof 1 of Theorem 4.2, we conclude that

$$W_q = L_q\bar{x} + \rho\frac{\bar{x^2}}{2\bar{x}} + (1-\rho)\frac{n-1}{2\lambda}. \tag{9}$$

Replacing L_q above by λW_q (Little's) leads to an equation for W_q which is solved to

$$W_q = \frac{\lambda\bar{x^2}}{1-\rho} + \frac{n-1}{2\lambda}.$$

This is once again a decomposition result.

e) The second term in (9) needs to be replaced with

$$(1-\rho)\frac{\frac{n}{\lambda}(\frac{n-1}{2\lambda}+\bar{s}) + \bar{s}\frac{\bar{s^2}}{2\bar{s}}}{\frac{n}{\lambda}+\bar{s}}.$$

The reasoning is as follows. Suppose arrival finds the server idle (we consider the setup time as part of idleness). The probability that the idleness is due to waiting for the queue to be at least of size n is $\frac{n/\lambda}{n/\lambda+\bar{s}}$ in which case the mean time until (true) service commencement is time whose mean equals $(n-1)/(2\lambda)+\bar{s}$. With the complementary probability the idleness is due to the server being engaged in setup. In this case the mean time until (true) service commencement equals the mean of its residual which equals $\bar{s^2}/(2\bar{s})$.

4.3 Question 3

We start with the case where $z \geq 0$. Then, using the convolution formula,

$$f_Z(z) = \int_{x=z}^{\infty} f_X(x) f_Y(x-z)\,dx = \int_{x=z}^{\infty} \lambda\mu e^{-\lambda x} e^{-\mu(x-z)}\,dx$$

$$= \lambda\mu e^{\mu z} \frac{-1}{\lambda+\mu} [e^{-(\lambda+\mu)}]_z^{\infty} = \frac{\lambda\mu}{\lambda+\mu} e^{-\lambda z}.$$

For the case where $z \leq 0$, note the symmetry: A negative difference is as the corresponding positive difference with the roles of λ and μ being swapped.

4.4 Question 4

This is Little's rule one more time time. Specifically, consider the set of servers as the 'system.' L now is the mean number of busy servers, λ is the same arrival rate and W is the mean time in the system which is $\bar{x}$. Thus, $L = \lambda\bar{x}$. Of course, $1/\bar{x}$ is the service rate. Hence, on the righthand side we got the ratio between the arrival rate and the service rate. Note that we have reached an insensitivity result: The mean number of busy servers is a function of service time distribution (in fact, also of inter-arrival time distribution) only through its mean.

4.5 Question 5

The proof is straightforward and follows the definition of N. The point of this question is that the number to arrive during a busy period is a stopping time with respect to the arrival process, commencing at the first interarrival time which follows the first arrival. Note that the issue of stopping time is not dealt elsewhere in the textbook.

5 Chapter 5

5.1 Question 1

First,

$$\frac{1}{1-\sigma_i} - \frac{1}{1-\sigma_{i-1}} = \frac{1-\sigma_{i-1}-(1-\sigma_i)}{(1-\sigma_i)(1-\sigma_{i-1})} = \frac{\rho_i}{(1-\sigma_i)(1-\sigma_{i-1})}.$$

Then, by (5.3),

$$\sum_{i=1}^{N} q_i W_i^q = \sum_{i=1}^{N} \frac{\rho_i}{(1-\sigma_i)(1-\sigma_{i-1})} = W_0 \sum_{i=1}^{N} (\frac{1}{1-\sigma_i} - \frac{1}{1-\sigma_{i-1}}) =$$

$$W_0(\frac{1}{1-\sigma_N} - \frac{1}{1-\sigma_0}) = W_0(\frac{1}{1-\rho} - 1) = W_0 \frac{\rho}{1-\rho},$$

as required.

5.2 Question 2

a) We have an M/G/1 model with non-preemptive priorities. The number of classes, N, equals 2. The arrival rate of class 1, λ_1, equals $\lambda G(x_0)$. Thus,

$$\sigma_1 = \rho_1 = \lambda G(x_0) \int_{x=0}^{x_0} x \frac{g(x)}{G(x_0)} \, dx = \lambda \int_{x=0}^{x_0} x g(x) \, dx.$$

In a similar way,

$$\rho_2 = \lambda \int_{x=x_0}^{x=\infty} x g(x) \, dx.$$

For these values,

$$W_1^q = \frac{W_0}{1-\rho_1} \quad \text{and} \quad W_2^q = \frac{W_0}{(1-\rho)(1-\rho_1)}.$$

b) In order to find the mean overall waiting time, we need to average between these two values giving them weights $G(x_0)$ and $(1-G(x_0))$, respectively. Hence,

$$W^q = G(x_0) W_0 \frac{1}{1-\rho_1} + (1-G(x_0)) W_0 \frac{1}{(1-\rho)(1-\rho_1)},$$

which by trivial algebra is shown to equal

$$\frac{W_0}{1-\rho}\frac{1-\rho G(x_0)}{1-\rho_1}.$$

Since $W^q_{FCFS} = W_0/(1-\rho)$ (see (4.5)), we easily get that

$$W^q = W^q_{FCFS}\frac{1-\rho G(x_0)}{1-\rho_1}.$$

As for the rightmost inequality, we need to show that $1-\rho G(x_0) < 1-\rho_1$, or, equivalently, to $\rho G(x_0) > \rho_1$, or, recalling the definition for ρ_1 above, to $\bar{x} > \bar{x}_1$. This is certainly the case, since $\bar{x}$ is the overall mean waiting time, while $\bar{x}_1$ is the mean only through those service time is smaller than or equal to x_0.

c) In the case where $x_0 = 0$, $G(x_0) = \rho_1 = 0$ and hence $W^q = W_0/(1-\rho) = W^q_{FCFS}$. Likewise, in the case where $x_0 = \infty$, $\rho_1 = \rho$ and $G(x_0) = 1$, and hence (and again) $W^q = W^q_{FCFS}$. The explanation is as follows: In both case, there is basically only one class (the other class, regardless it is it of premium or of disadvantage customers is of measure zero) and hence the mean waiting time is as in a standard FCFS queue.

d) The optimization problem we face here is that of minimizing W^q with respect to x_0. In other words, we look for

$$\min_{0\le x_0<\infty}\frac{1-\rho G(x_0)}{1-\rho_1}.$$

e) Denote $\lambda\int_{x=0}^{x_0} xg(x)\,dx$ by $\rho(x_0)$, which now is looked as a function of x_0. Recall that our goal is

$$\min_{x_0}\frac{1-\rho G(x_0)}{1-\rho(x_0)}.$$

Taking derivative with respect to x_0, we get that the numerator of the derivative equals

$$-\rho g(x_0)(1-\rho(x_0)) + (1-\rho G(x_0))\lambda g(x_0).$$

Replacing ρ here by $\lambda\bar{x}$ and equating it to zero, leads to the fact that for the optimal x_0,

$$\bar{x}(1-\rho(x_0)) = x_0(1-\rho G(x_0)).$$

From which we easily get

$$\frac{\bar{x}}{x_0} = \frac{1-\rho G(x_0)}{1-\rho(x_0)} = \frac{W^q}{W^q_{FCFS}} < 1.$$

5.3 Question 3

a) (i) This item's result follows Theorem 4.8. Specifically, when a class i customers arrives he faces an mount of work of R_i which, by definition, needs to be completed before he enters to service. He indeed enters to service at the instant in which the server would have been idle for the first time, had classes $1, \ldots, i$ would have been the only classes which exist. Note that this includes class i since all this class customers who arrive while the tagged customer is in queue, overtake him. This implies that $W_i^q = R_i/(1-\sigma_i)$, $1 \le i \le N$. In the case where $i = 1$, the amount of work that such a customer faces in the system and needs to be processed prior to it entrance, is the residual service time of the one in service (if there is one). This amount of work has a mean of W_0. In other words, $R_1 = W_0$. We need to show that $W_i^q = W_0/[(1-s_i)(1-s_{i-1})]$, $1 \le i \le N$. Since, $\sigma_0 = 0$, we are done for the case where $i = 0$, and this fact is the anchor of our induction argument.

(ii) We next claim that $R_{i+1} = R_i + L_i\bar{x}_i$, $1 \le i \le N-1$. Indeed, the amount of work in the system faced by a class i+1 customers is as much a class i customer with the additional work due to class-i customers who are in the queue upon once arrival (which are overtaken by a class i arrival). The mean number of such customers is by definition L^q, each of which adding mean value of work of $\bar{x}_i$. Note, by Little's, that $L_i^q\bar{x}_i = \rho_i W_i^q$, $1 \le i \le N$.

(iii) Using all of the above, we get that for $1 \le i \le N-1$.

$$W_{i+1}^q = \frac{R_{i+1}}{1-\sigma_{i+1}} = \frac{R_i + \rho_i W_i^q}{1-\sigma_{i+1}} = \frac{W_i^q(1-\sigma_i) + \rho_i W_i}{1-\sigma_{i+1}}$$

$$= \frac{W_i^q(1-\sigma_{i-1})}{1-\sigma_{i+1}}.$$

Finally, invoke the induction hypothesis that $W_i^q = W_0/[(1-\sigma_i)(1-\sigma_{i-1})]$ and get that $W_{i+1}^q = W_0/[(1-\sigma_{i+1})(1-\sigma_i)]$, as required.

b) The proof below has much similarities with the proof for the case where LCFS without preemption is used. There in a preliminary observation needed here: If two customers of the same class turned out to be in the queue at the same time, then each one of them enters to service before the other with a probability of 1/2, regardless of who arrived earlier.

(i) In the previous case R_1 was equal to W_0. In the random order regime, a class 1 arrival will find (on average) half of class 1 who are there upon his arrival, entering to service ahead of him. Hence, $R_1 = W_0 + \bar{x}_1 \mathrm{Ł}_1^q/2$.

(ii) We use again Theorem 4.8. The reasoning is as in the previous model with the exception that the traffic intensity of those who overtake a tagged class i customer is $1-\sigma_{i-1}-\rho_i/2$. In particular, the rational behind $\rho_i/2$ is due to the fact that the arrival rate of those of his class who overtake him is $\lambda_i/2$.

(iii) Note: the r in the righthand side is a typo. We are asked here to compare R_{i+1} with R_i. In fact, we need to check when needs to be added to R_i in order to get R_{i+1}. Indeed, this added work has two sources. The first is those of class i which are overtaken a tagged class i customer but of course enter before a class i+1 customer. This has a mean value of $L_i^q \bar{x}_i/2$. The second source is half of those of his class he finds in the queue upon arrival (the second half is being overtaken by him). This contributes an additional value of $L_{i+1}^q \bar{x}_{i+1}/2$.

(iv) We proof the claim with the help of an induction argument. We start with $i = 1$. From the first item here, coupled with Little, we get that

$$R_1 = W_0 + \rho_1 W_1/2.$$

From the second item we learn (since σ_0) that

$$R_1 = W_1^q(1 - \rho_1/2).$$

Combining the two and solving for W_1^q, we get that

$$W_1^q = \frac{W_0}{1-\rho_1} = \frac{W_0}{(1-\sigma_1)(1-\sigma_0)},$$

concluding the proof for the case $i = 1$. Next we assume the result to hold for i and establish for the case $i + 1$. Indeed, from the previous item we get that

$$W^q_{i+1}(1-\sigma_i-\rho_{i+1}/2) = W^q_i(1-\sigma_{i-1}-\rho_i/2)+\rho_i W^q_i/2+\rho_{i+1}W^q_{i+1}/2,$$

from which we easily get W^q_{i+1} in terms of W^q_i. Specifically,

$$W^q_{i+1} = W^q_i \frac{1-\sigma_{i-1}}{1-\sigma_{i+1}}.$$

Using the induction hypothesis that $W^q_i = W_0/[(1-\sigma_i)(1-\sigma_{i-1})]$, completes the proof.

5.4 Question 4

a) Under any possible scenario of a state $\underline{n} = (n_1, \ldots, n_M)$, it is possible to see that given the event that one of two two tagged customers, one an class i customer, the other a class j customer, just entered service, it is actually the class i customer with probability $p_i/(p_i+p_j)$, $1 \le i, j \le M$. Note that this is also the case when $i = j$, namely the two belong to the same class, making the corresponding probability 1/2.

b) A class i arrival find the following mean work in the system he/she as to wait for before commencing service. First, the standard W_0. Second, each class-j customer presents in queue will enter to service prior to him/her with probability $p_j/(p_i + p_j)$ (see previous item), leading to a work of $\bar{x}_j$. Since the expected number of such class-j customers is denoted by L^q_j we can conclude that

$$R_i = W_0 + \sum_{j=1}^{N} \frac{p_j}{p_i+p_j} L^q_j \bar{x}_j, \quad 1 \le i \le N. \tag{10}$$

c) The value for τ_i as defined in the question, i.e.,

$$\tau_i = \sum_{j=1}^{N} \frac{p_j}{p_i+p_j}\rho_j, \quad 1 \le i \le N,$$

is the traffic intensity of those who in fact have priority over a tagged class i customer and overtake him even if they arrive while he/she is in line. Then, again with the use of Theorem 4.8, his/her instant of service commencement can be looked at as the first time of idleness when R_i is the amount of work in the system and τ_i is the traffic intensity.

d) Observe (10). Replace R_i on lefthand side by $W_i^q(1-\tau_i)$ (which was proved in the previous item) and $L_j\bar{x}_j$ on the righthand side by $\rho_j W_j^q$ (by Little's rule), and get

$$(1-\tau_i)W_i^q = W_0 + \sum_{j=1}^{N} \frac{p_j\rho_j}{p_i+p_j} W_j^q, \quad 1 \le i \le N.$$

This is a system of $N \times N$ linear equations in the unknown W_i^q, $1 \le i \le N$. Note that in the system the righthand side is a vector of identical entries, all equal to W_0. Thus, one can solve for the system assuming all entries on the righthand side equal to one, and then multiply the solution by W_0. As for the matrix which needs to be inverted in order to solve the equations, note that its off-diagonal ij entry equals $-p_j\rho_j/(p_i+p_j)$, $1 \le i \ne j \le n$, while its i-th diagonal entry equals $(1-\tau_i-\rho_i/2)$, $1 \le i \le N$.

5.5 Question 5

Consider a customer whose service time is x. All customers with service time of y with $y < x$ who arrive while he/she are in the system leave before he/she does from his/her point of his/her delay, it is as all these customers get preemptive priority over him/her. As for customers with $y > x$ they also inflict additional waiting time on him. Yet, they all do as they were all x customers who has preemptive priority on the tagged customer. Hence, we can apply the formula for T_x which appears in the middle of page 77 but with the update of σ_x to σ'_x and $\sigma_x^{(2)}$ to $\sigma_x^{'(2)}$ to reflect the fact that an additional set of customers, those with a rate of arrival $\lambda\overline{G}(x)$ and service rate of (practically) x is to be of concern as they practically overtake the tagged customers. Note that the fact that he/she also inflicts waiting time on them is irrelevant.

5.6 Question 6

(The question is not phrase well.) Assuming the SJF policy, let x be the (infinitesimal) class of all those whose service time equals x. Their arrival rate is $\lambda g(x)$, their mean service time is (of course) x and hence their traffic intensity equals $\lambda g(x)x$. Finally, their mean time in the queues, denoted by W_x^q equals (see above (5.5)) to $W_0/(1-\sigma_x)^2$. Hence, using the continuous version of (5.1), we get that

$$\int_{x=0}^{\infty} \lambda x g(x) \frac{W_0}{(1-\sigma_x)^2}\, dx = W_0 \frac{\rho}{1-\rho},$$

from which we can deduce that

$$\int_{x=0}^{\infty} \frac{xg(x)}{(1-\sigma_x)^2}\,dx = \frac{\bar{x}}{1-\rho}$$

6 Chapter 6

6.1 Question 1

(By Yoav)

a) Given the service time X, $v_n \sim Po(\lambda X)$. Thus,

$$\mathrm{E}(v_n) = \mathrm{E}(\lambda X) = \lambda \bar{x} = \rho.$$

Also,

$$\mathrm{E}(v_n^2) = \mathrm{E}((\lambda X)^2 + \lambda X) = \lambda^2(\bar{x}^2) + \rho.$$

b) The difference between the number of customers left behind in two consecutive departures is the number of arrivals minus the one who has been served, if there was any.

c)

$$q_{n+1}^2 = (q_n - \triangle q_n + v_{n+1})^2 = q_n^2 + \triangle q_n^2 + v_{n+1}^2 - 2q_n \triangle q_n + 2q_n v_{n+1} - 2v_{n+1} \triangle q_n$$

$$= q_n^2 + \triangle q_n + v_{n+1}^2 - 2q_n + 2q_n v_{n+1} - 2v_{n+1} \triangle q_n.$$

d) Taking expected value of the above yields

$$\mathrm{E}\left(q_{n+1}^2\right) = \mathrm{E}\left(q_n^2\right) + \mathrm{E}(\triangle q_n) + \lambda^2 \overline{x^2} + \rho - 2\mathrm{E}(q_n)(1-\rho) - 2\rho \mathrm{E}(\triangle q_n).$$

Taking the limit $n \to \infty$, using $\lim_{n\to\infty} \mathrm{E}(\triangle q_n) = \rho$, and erasing the second moment from both hand sides we get

$$0 = \lambda^2 \overline{x^2} + 2\rho(1-\rho) - 2q(1-\rho).$$

Dividing both hand sides by $2(1-\rho)$ and rearranging yields the result.

e) By little's formula

$$W = L/\lambda = \frac{\lambda \overline{x^2}}{2(1-\rho)} + \bar{x}.$$

Since $W = W_q + \bar{x}$ we have

$$W_q = \frac{\lambda \overline{x^2}}{2(1-\rho)}.$$

6.2 Question 2

(By Yoav)

a) First, each one of the j present customers waits $\bar{x}$ (on expectation). During the service time, each arriving customer (who arrived in the middle) waits $\frac{\overline{x^2}}{2\bar{x}}$. The number of arriving customers is $\lambda\bar{x}$ (on expectation) and hence $w_j = j\bar{x} + \lambda\overline{x^2}/2$.

b) If 0 customers were left behind, there is no waiting until the first arrival. Since then, the behavior is the same as one customer was left behind.

c)

$$\phi = \sum_{j=0}^{\infty} jw_j = \sum_{j=0}^{\infty} \left(j\bar{x} + \frac{\lambda\overline{x^2}}{2} \right) u_j = \sum_{j=0}^{\infty} ju_j\bar{x} + \frac{\lambda\overline{x^2}}{2} = L + \frac{\lambda\overline{x^2}}{2}$$

d) Each departure implies a waiting time ϕ. As there are λ departures per time unit, the product $\lambda\phi$ is the expected waiting time added each time unit.

e) The latter is also the expected number of customers in the system. Hence,

$$L = \lambda\phi = \lambda L\bar{x} + \frac{\lambda^2\overline{x^2}}{2} = \rho L + \frac{\lambda^2\overline{x^2}}{2}$$

Note: the solution is L_q and not L? what am I missing?

6.3 Question 3

(By Yoav) Let $Y, Y_1, Y_2, \ldots$ be i.i.d. with $\mathrm{E}(t^Y) = N(t)$. Let X be the service time that opens the busy period and let $A(X)$ be the number of arrival during X. Note that $A(X)|X \sim Po(\lambda X)$. Since each arrival during X open a new busy period, we have that the distribution of Y is the same as the distribution of $1 + \sum_{i=1}^{A(X)} Y_i$ Thus we have,

$$\mathrm{E}(t^Y) = \mathrm{E}\left(t^{1+\sum_{i=1}^{A(X)} Y_i} \right) = t\mathrm{E}\left(N(t)^{A(X)} \right) = t\mathrm{E}\left[\mathrm{E}\left(N(t) \right)^{A(X)} |X \right]$$

$$= t\mathrm{E}\left[e^{-\lambda X(1-N(t))} \right] = tG^*(\lambda(1 - N(t)))$$

In the $M/M/1$ case, with $G^*(s) = \frac{\mu}{\mu+s}$, we have

$$N(t) = \frac{t\mu}{\mu + \lambda(1 - N(t))} \quad or \quad (\mu + \lambda)N(t) - \lambda N^2(t) = t\mu$$

which implies our result.

6.4 Question 4

(By Yoav)

a) The second item in (6.2) is now Z_{n+1} instead of Y_{n+1}, where Z_{n+1} follows the probabilities b_j.

b) The first row represents the transition probabilities from state 0. That is, transitions from an empty system. Thus, after a departure who leaves and empty system behind him, the next service time will be the first in a busy period and hence the number of arrivals will be according to the b_j probabilities. Thus, the first row in (6.3) is with b_j replacing a_j and the rest or the rows are the same.

c) In each equation in (6.4), the first addend in the right hand side stands for the product of u_0 and the relevant a_j from the first row. In our model the a_j is replaced by the corresponding b_j. The rest of the elements of course remain the same.

d) Assuming FCFS, the number of customers left behind upon departure, given sojourn time W of the the departing customer, follows the distribution $Po(\lambda W)$. Hence, $\Pi(t) = \mathrm{E}\left(e^{-\lambda(1-t)W}\right) = W^*(\lambda(1-t)$. Inserting $s = \lambda(1-t)$ implies (6.11).

e)

$$\Pi(t) = \sum_{j=0}^{\infty} u_j z^j = \sum_{j=0}^{\infty} z^j \sum_{i=0}^{\infty} u_i P_{ij} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} z^j u_i P_{ij} = u_0 \sum_{j=0}^{\infty} z^j b_j + \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} z^j u_i P_{ij}$$

$$= u_0 B(t) + \sum_{i=1}^{\infty} \sum_{j=i-1}^{\infty} z^j u_i a_{j-i+1} = u_0 B(t) + \sum_{i=1}^{\infty} u_i z^{i-1} \sum_{k=0}^{\infty} a_k z^k$$

$$u_0 B(t) + \frac{1}{t} \left(\Pi(t) - u_0\right) A(t) = u_0 \left(B(t) - \frac{A(t)}{t}\right) + \Pi(t) \frac{A(t)}{t}.$$

Hence

$$\Pi(t)\left(1-\frac{A(t)}{t}\right)=u_0\left(B(t)-\frac{A(t)}{t}\right)$$

or

$$\Pi(t)=u_0\frac{tB(t)-A(t)}{t-A(t)}$$

f) As in the standard $M/G/1$ we insert $t=1$ in both sides of the latter and apply L'Hopital's rule:

$$1=u_0\frac{B(1)+B'(1)-A'(1)}{1-A'(1)}=u_0\frac{1+\lambda b-\rho}{1-\rho}$$

and hence

$$u_0=\frac{1-\rho}{1+\lambda b-\rho},$$

where $b=\sum_{j=0}^{\infty} jb_j$.

g) We can refer to the sum of setup time and the first service time as the "different" service time that opens a busy period.

Let $c_j=\int\limits_{x=0}^{\infty} e^{-\lambda x}\frac{(\lambda x)^j}{j!}f(x)dx$ The number of arrivals during the "different" service time is the sum of the number of arrivals during the setup time and the number of arrivals during the first service time. Thus,

$$b_j=\sum_{i=0}^{j} a_i c_{j-i} \quad \mathit{and} \quad B(t)=C(t)A(t)=F^*(\lambda(1-t))G^*(\lambda(1-t))$$

6.5 Question 5

(By Yoav)

a) The time between the first arrival during a vacation is considered "setup time" in terms of 4(g).

b) At the beginning of the last vacation, a vacation time V and an inter-arrival time A started competing, and the inter-arrival time occurred first. The remaining vacation time is the difference between the two, $V-A$. its distribution is the distribution of $V-A|V>A$. First note that $P(V>A)=\mathrm{E}\left(1-e^{-\lambda V}\right)=1-V^*(\lambda)$. Second, we have

$$F^*(s)=\mathrm{E}\left(e^{-s(V-A)}|V>A\right)=\frac{\mathrm{E}\left(e^{-s(V-A)}I_{\{A<V\}}\right)}{1-V^*(\lambda)}$$

$$= \frac{1}{1-V^*(\lambda)} \int\limits_{x=0}^{\infty} \int\limits_{a=0}^{x} e^{-s(x-a)} \lambda e^{-\lambda a} v(x) da dx$$

$$= \frac{\lambda}{1-V^*(\lambda)} \int\limits_{x=0}^{\infty} e^{-sx} v(x) \int\limits_{a=0}^{x} e^{-a(\lambda-s)} da dx$$

$$= \frac{\lambda}{(1-V^*(\lambda))(\lambda-s)} \int\limits_{x=0}^{\infty} e^{-sx} v(x) \left(1-e^{-(\lambda-s)x}\right) dx$$

$$= \frac{\lambda}{(1-V^*(\lambda))(\lambda-s)} \int\limits_{x=0}^{\infty} v(x) \left(e^{-sx}-e^{-\lambda x}\right) dx = \frac{\lambda}{\lambda-s} \frac{V^*(s)-V^*(\lambda)}{1-V^*(\lambda)}$$

6.6 Question 6

(By Benny)

a) Let X be the number of arrivals during one service period.

$$P_{0j} = P(X = j+1 | X > 0) = \frac{a_{j+1}}{1-a_0}$$

b) The balance equations are:

$$\begin{array}{rcl} u_0 & = & u_0 \frac{a_1}{1-a_0} + u_1 a_0 \\ u_1 & = & u_0 \frac{a_2}{1-a_0} + u_1 a_1 + u_2 a_0 \\ u_2 & = & u_0 \frac{a_3}{1-a_0} + u_1 a_2 + u_2 a_1 + u_3 a_0 \\ & \vdots & \end{array}$$

Multiplying equation j with t^j and summing up gives

$$\begin{array}{rcl} \Pi(t) & = & u_0 \frac{\sum_{j=0}^{\infty} t^j a_{j+1}}{1-a_0} + \frac{1}{t} A(t)(\Pi(t) - u_0) \\ & = & u_0 \frac{\sum_{j=0}^{\infty} t^{j+1} a_{j+1}}{t(1-a_0)} + \frac{1}{t} A(t)(\Pi(t) - u_0) \\ & = & u_0 \frac{A(t)-a_0}{t(1-a_0)} + \frac{1}{t} A(t)(\Pi(t) - u_0) \end{array}$$

Now

$$\Pi(t)\left[t - A(t)\right] = u_0 \frac{a_0}{1-a_0} \left[A(t) - 1\right]$$
$$\Pi(t) = C \frac{A(t)-1}{t-A(t)}$$

where C is constant. Since $\Pi(t)$ is z-transform

$$\lim_{t\to 1} C\frac{A(t)-1}{t-A(t)} = 1 \implies C = \lim_{t\to 1}\frac{t-A(t)}{A(t)-1}$$

Using L'Hopital's rule

$$C = \lim_{t\to 1}\frac{1-A'(t)}{A'(t)} = \frac{1-\lambda\bar{x}}{\lambda\bar{x}} = \frac{1-\rho}{\rho}$$

c) In the above item, we showed that

$$u_0\frac{a_0}{1-a_0} = C = \frac{1-\rho}{\rho} \implies u_0 = \frac{1-\rho}{\rho}\frac{1-a_0}{a_0}$$

If service times are exponentially distributed

$$a_0 = \int_{t=0}^{\infty} e^{-\lambda t}\mu e^{-\mu t}dt = \frac{\mu}{\lambda+\mu} = \frac{1}{\rho+1}$$

hence

$$u_0 = \frac{1-\rho}{\rho}\frac{\frac{\rho}{\rho+1}}{\frac{1}{\rho+1}} = 1-\rho$$

d)

$$\begin{aligned}
\Pi'(1) &= \lim_{t\to 1}\tfrac{1-\rho}{\rho}\tfrac{A'(t)(t-A(t))-(A(t)-1)(1-A'(t))}{(t-A(t))^2}\\
&= \lim_{t\to 1}\tfrac{1-\rho}{\rho}\tfrac{(t-1)A'(t)-A(t)+1}{(t-A(t))^2}\\
&= \lim_{t\to 1}\tfrac{1-\rho}{\rho}\tfrac{A'(t)+(t-1)A''(t)-A'(t)}{2(t-A(t))(1-A'(t))}\\
&= \lim_{t\to 1}\tfrac{1-\rho}{\rho}\tfrac{t-1}{t-A(t)}\tfrac{A''(t)}{2(1-A'(t))}\\
&= \lim_{t\to 1}\tfrac{1-\rho}{\rho}\tfrac{1}{1-A'(t)}\tfrac{A''(t)}{2(1-A'(t))}\\
&= \tfrac{1-\rho}{\rho}\tfrac{A''(1)}{2(1-A'(1))^2}
\end{aligned}$$

e) The limit distributions at arrival and departure instants coincide and by the PASTA property this is also the distribution at arbitrary instants. Hence, $\Pi'(1)$ is the expected number of customers in the system, L, and by Little's law we get

$$\begin{aligned}
W &= L/\lambda = \tfrac{1}{\lambda}\tfrac{1-\rho}{\rho}\tfrac{A''(1)}{2(1-A'(1))^2}\\
&= \tfrac{1}{\lambda}\tfrac{1}{\rho}\tfrac{\lambda^2\overline{x^2}}{2(1-\rho)}\\
&= \tfrac{\lambda\overline{x^2}}{2\rho(1-\rho)}
\end{aligned}$$

6.7 Question 7

a) The distribution of B_{ex} is the distribution of $\sum_{i=1}^{A(X)} B_i$, where B_i are standard busy periods.

$$B^*_{ex}(s) = \mathrm{E}\left(e^{-sB_{ex}}\right) = \mathrm{E}\left[(B^*(s))^{A(X)}\right] = \mathrm{E}\left(e^{-\lambda X(1-B^*(s))}\right) = G^*_0(\lambda(1-B^*(s)))$$

b) Since $B_{in} = X + B_{ex}$, we have

$$\mathrm{E}\left(e^{-sB_{in}}\right) = \mathrm{E}\left(e^{-sX-sB_{ex}}\right) = \mathrm{E}\left(e^{-sX+\lambda X(1-B^*(s))}\right) = G^*_0(s+\lambda(1-B^*(s)))$$

6.8 Question 8

THERE IS AN ERROR IN THE QUESTION. IT SHOULD BE: The queueing time (exclusive of service) is as a busy period where the first service time is the residual service time. The latter has the LST of $(1-G^*(s))/\bar{x}s$. Hence, we are in the situation described in Q7a above with $G^*_0(s)$ having this expression. Using this for $G^*_0(s)$, we get that the LST of this queueing time is derived by inserting $(\lambda - \lambda B^* s(s))$ for s in $(1 - G^*(s))/\bar{x}s$. The final result is thus

$$\frac{(1 - G^*(\lambda - \lambda B^*(s)))}{(\lambda - \lambda B^*(s))\bar{x}}.$$

6.9 Question 9

By definition,

$$a_0 = \int_{x=0}^{\infty} e^{-\lambda x} g(x)dx = G^*(\lambda).$$

Using the first equation in (6.4) and the fact that $u_0 = 1 - \rho$, we have $1 - \rho = (1 - \rho)G^*(\lambda) + u_1 G^*(\lambda)$ and hence

$$u_1 = \frac{(1 - \rho)(1 - G^*(\lambda))}{G^*(\lambda)}$$

6.10 Question 10

$$\frac{d}{ds}W^*_q(s) = \frac{d}{ds}\frac{1 - \rho}{1 - \rho G^*_r(s)} = \frac{(1 - \rho)\frac{d}{ds}G^*_r(s)}{(1 - \rho G^*_r(s))^2}$$

Multiplying by -1 and inserting $s = 0$ yields

$$\mathrm{E}(W_q) = \frac{(1-\rho)\mathrm{E}(R)}{(1-\rho)^2} = \frac{\mathrm{E}(R)}{1-\rho}$$

$$\frac{d^2}{ds^2}W_q^*(s) = \frac{d^2}{ds^2}\frac{1-\rho}{1-\rho G_r^*(s)} = \frac{d}{ds}\frac{(1-\rho)\frac{d}{ds}G_r^*(s)}{(1-\rho G_r^*(s))^2}$$
$$= \frac{(1-\rho)\left(2\rho\left(\frac{d}{ds}G_r^*(s)\right)^2 + (1-\rho G_r(s))\frac{d^2}{ds^2}G_r^*(s)\right)}{(1-\rho G_r(s))^3}.$$

Inserting $s = 0$ yields

$$\frac{(1-\rho)\left(2\rho(\mathrm{E}(R))^2 + (1-\rho)\mathrm{E}(R^2)\right)}{(1-\rho)^3} = \frac{2\rho(\mathrm{E}(R))^2}{(1-\rho)^2} + \frac{\mathrm{E}(R^2)}{1-\rho}$$

6.11 Question 11

(Benny)

a) First note that

$$a_0 = \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^0}{0!}g(t)dt = \int_0^\infty e^{-\lambda t}g(t)dt = G^*(\lambda)$$

and by (6.7) and (6.5) we get

$$u_1 = \frac{1}{a_0}(u_0 - (1-\rho)a_0) = \frac{1}{G^*(\lambda)}(1-\rho-(1-\rho)G^*(\lambda)) = (1-\rho)\frac{1-G^*(\lambda)}{G^*(\lambda)}$$

Now, from (6.17) it follows that

$$\bar{r}_1 = \frac{1-\rho}{\lambda}\frac{1-h_1}{h_1} = \frac{1-\rho}{\lambda}\frac{1-u_1/(1-u_0)}{u_1/(1-u_0)}$$

Substituting u_0 and u_1 gives

$$\bar{r}_1 = \frac{\rho G^*(\lambda)}{\lambda(1-G^*(\lambda))} - \frac{1-\rho}{\lambda} = \frac{\bar{x}}{1-G^*(\lambda)} - \frac{1}{\lambda}$$

b) Using (6.17) again

$$\begin{aligned}
\bar{r}_{n+1} &= \tfrac{1-\rho}{\lambda}\tfrac{1-h_{n+1}}{h_{n+1}} = \tfrac{1-\rho}{\lambda}\tfrac{\sum_{i=n+1}^\infty u_i - u_{n+1}}{u_{n+1}} \\
&= \tfrac{1-\rho}{\lambda}\tfrac{\sum_{i=n}^\infty u_i - u_n - u_{n+1}}{u_{n+1}} = \tfrac{1-\rho}{\lambda}\left[\tfrac{\sum_{i=n}^\infty u_i/u_n - 1}{u_{n+1}/u_n} - 1\right] \\
&= \tfrac{1-\rho}{\lambda}\tfrac{u_n}{u_{n+1}}\tfrac{1-h_n}{h_n} - \tfrac{1-\rho}{\lambda} = \tfrac{u_n}{u_{n+1}}\bar{r}_n - \tfrac{1-\rho}{\lambda}
\end{aligned}$$

6.12 Question 12

a) Let S_n be the number of customers present in the system upon the commencement of the n^{th} service and let X_n be the number of customers left behind upon the n^{th} departure. We have $S_n = X_{n-1} + I_{\{X_{n-1}=0\}}$. That is, X and S coincide unless $X = 0$ (then $S = 1$). Since in the transition matrix of X $P_{0j} = P_{1j}$ for all $j \geq 0$, we can merge states 0 and 1. Hence, S_n is a Markov chain as well.

b) By the latter, after merging the states 0 and 1, we get the desired result.

6.13 Question 13

First note that the u_n/ρ is the probability of n customers in the queue, given it is greater than or equal to one, $n \geq 1$. Denote such a random variable by L^+. Second, from the previous question we learn that upon entrance to service the probability of n customers (denoted by e_n) is also u_n but only for $n \geq 2$. For $n = 1$ it is $u_0 + u_1$ while it is zero with a probability zero, of course. Then, by Bayes' rule we get for $L \geq 1$,

$$f_{A|L^+=n}(a) = \frac{f_A(a)\mathrm{P}(L^+ = n|A = a)}{\mathrm{P}(L^+ = n)}. \tag{11}$$

Clearly, $f_A(a) = (1 - G(a))/\bar{x}$ (see (2.4) in page 25) and, as said, $\mathrm{P}(L^+ = n) = u_n/\rho$, $n \geq 1$. Next, note that L^+ is the sum of two random variables, how many where in the system upon the current service commencement (which is independent of the age of this service) plus how many had arrive during it (which, given an age of service of a follows a Poisson distribution with parameter λa. Hence,

$$\mathrm{P}(L^+ = n|A = a) = \sum_{i=1}^{n} e_i e^{-\lambda a} \frac{(\lambda a)^{n-i}}{(n-i)!}$$

$$= u_0 \frac{(\lambda a)^{n-1}}{(n-1)!} + \sum_{i=1}^{n} u_i e^{-\lambda a} \frac{(\lambda a)^{n-i}}{(n-i)!},\ n \geq 1.$$

Putting all in 11 (recall that $u_0 = 1 - \rho$) concludes the proof. For the case where $n = 1$, one needs to use the expression for u_1 as given in Question 9 above, coupled with some minimal algebra, in order to get that

$$f_{A|L=1} = \lambda \frac{1 - G(a)}{1 - G^*(\lambda)} e^{-\lambda a}.$$

6.14 Question 14

By (2.6) (see page 28), we get that $f_{A,R}(a,r) = g(a+r)/\bar{x}$. Likewise, by (2.4) (see page 25) we get that $f_A(a) = (1 - G(a))/\bar{x}$. Hence,

$$f_{R|A=a}(r) = f_{A,R}\bar{(}a,r)f_A(a) = \frac{g(a+r)}{1 - G(a)}.$$

Insert this and (6.26) into (6.28) and conclude the proof.

6.15 Question 15

(Benny) The second moment of of the queueing time equals $\left.\frac{d^2 W_q^*(s)}{ds^2}\right|_{s=0}$. Define $\phi(s) := \lambda G^*(s) + s - \lambda$. From (6.13) we get

$$\frac{dW_q^*(s)}{ds} = (1-\rho)\frac{\phi(s) - s\phi'(s)}{\phi^2(s)}$$

and

$$\begin{aligned}
\frac{d^2 W_q^*(s)}{ds^2} &= (1-\rho)\tfrac{(\phi'(s)-\phi'(s)-s\phi''(s))\phi^2(s)-(\phi(s)-s\phi'(s))(2\phi(s)\phi'(s))}{\phi^4(s)} \\
&= (1-\rho)\tfrac{-s\phi''(s)\phi^2(s)-2\phi^2(s)\phi'(s)+2s\phi(s)(\phi'(s))^2}{\phi^4(s)} \\
&= (1-\rho)\tfrac{-s\phi''(s)\phi(s)-2\phi(s)\phi'(s)+2s(\phi'(s))^2}{\phi^3(s)}
\end{aligned} \tag{12}$$

Note that $\phi(0) = 0$ and hence $\lim_{s\to 0}\frac{d^2 W_q^*(s)}{ds^2}$ has $\frac{0}{0}$ form. We use L'Hopital's rule three times

$$\begin{aligned}
\lim_{s\to 0}\tfrac{d^2 W_q^*(s)}{ds^2} &= \lim_{s\to 0}(1-\rho)\tfrac{-s\phi''(s)\phi(s)-2\phi(s)\phi'(s)+2s(\phi'(s))^2}{\phi^3(s)} \\
&= \lim_{s\to 0}(1-\rho)\frac{\begin{array}{l}-\phi''(s)\phi(s) \quad -s(\phi^{(3)}(s)\phi(s) + \phi''(s)\phi'(s)) \\ \quad -2((\phi'(s))^2 + \phi(s)\phi''(s)) + 2((\phi'(s))^2 + 2s\phi'(s)\phi''(s))\end{array}}{3\phi^2(s)\phi'(s)} \\
&= \lim_{s\to 0}(1-\rho)\tfrac{-3\phi''(s)\phi(s)-s\phi^{(3)}(s)\phi(s)+3s\phi'(s)\phi''(s)}{3\phi^2(s)\phi'(s)} \\
&= \lim_{s\to 0}(1-\rho)\frac{\begin{array}{l}-3(\phi^{(3)}(s)\phi(s)+ \quad \phi''(s)\phi'(s)) - \phi^{(3)}(s)\phi(s) \\ \quad -s(\phi^{(4)}(s)\phi(s) + \phi^{(3)}(s)\phi'(s)) \\ \quad +3(\phi'(s)\phi''(s) + s((\phi''(s))^2 + \phi'(s)\phi^{(3)}(s)))\end{array}}{3(2\phi(s)(\phi'(s))^2+\phi^2(s)\phi''(s))} \\
&= \lim_{s\to 0}(1-\rho)\tfrac{-4\phi^{(3)}(s)\phi(s)-s\phi^{(4)}(s)\phi(s)+3s(\phi''(s))^2+2s\phi'(s)\phi^{(3)}(s)}{6\phi(s)(\phi'(s))^2+3\phi^2(s)\phi''(s)} \\
&= \lim_{s\to 0}(1-\rho)\frac{\begin{array}{l}-4(\phi^{(4)}(s) \quad \phi(s) + \phi^{(3)}(s)\phi'(s)) - \phi^{(4)}(s)\phi(s) \\ \quad -s(\phi^{(5)}(s)\phi(s) + \phi^{(4)}(s)\phi'(s)) + 3((\phi''(s))^2 + 2s\phi''(s)\phi'(s)) \\ \quad +2(\phi'(s)\phi^{(3)}(s) + s(\phi''(s)\phi^{(3)}(s) + \phi'(s)\phi^{(4)}(s)))\end{array}}{6((\phi'(s))^3+\phi(s)2\phi'(s)\phi''(s))+3(2\phi(s)\phi'(s)\phi''(s)+(\phi(s))^2\phi^{(3)}(s))} \\
&= \lim_{s\to 0}(1-\rho)\tfrac{-2\phi^{(3)}(s)\phi'(s)+3(\phi''(s))^2}{6(\phi'(s))^3}
\end{aligned}$$

$$\begin{aligned}
\phi'(s) &= \lambda\tfrac{dG^*(s)}{ds} + 1\\
\phi'(0) &= \lambda\left.\tfrac{dG^*(s)}{ds}\right|_{s=0} + 1 = \lambda(-\overline{x}) + 1 = 1-\rho\\
\phi^{(k)}(s) &= \lambda\tfrac{d^kG^*(s)}{ds^k}, \quad k \geq 2\\
\phi^{(k)}(0) &= \lambda\left.\tfrac{d^kG^*(s)}{ds^k}\right|_{s=0} = (-1)^k\lambda\overline{x^k}, \quad k \geq 2
\end{aligned}$$

$$\begin{aligned}
\lim_{s\to 0}\tfrac{d^2W_q^*(s)}{ds^2} &= (1-\rho)\tfrac{-2(-\lambda\overline{x^3})(1-\rho)+3(\lambda\overline{x^2})^2}{6(1-\rho)^3}\\
&= \tfrac{2\lambda\overline{x^3}(1-\rho)+3(\lambda\overline{x^2})^2}{6(1-\rho)^2}
\end{aligned}$$

6.16 Question 16

We have here an M/G/1 queue. Denote the arrival rate by λ. Service times follow an Erlang distribution with parameters 2 and μ. Then, by (1.24) (see page 16), $G^(s) = \mu^2/(s+\mu)^2$. Also, the mean service time equals $2/\mu$ and hence $\rho = 2\lambda/m$. The rest follows (6.10) (see page 85). Specifically,

$$G^*(\lambda(1-t)) = \frac{\mu^2}{(\mu+\lambda(1-t))^2}$$

and hence

$$\Pi(t) = (1-\rho)\frac{(t-1)\mu^2}{t(\mu+\lambda(1-t))^2-\mu^2}.$$

Some algebra shows that the denominator here equals $\lambda^2(1-t)^2t + 2\lambda\mu t(1-t) + \mu^2(1-t)$. Inserting this and recalling dividing all by μ^2, concludes the proof.

6.17 Question 17

a) Some thoughts leads to the conclusion that the event $A \geq W$ coincides with the event that one who just concluded service in a FCFS M/G/1 queue, leaves behinds an empty system. This probability is $1-\rho$. See (6.5) on page 83.

b) Denote by $f_W(w)$ the density function of the waiting time (service inclusive). Then,

$$\mathrm{P}(A \geq W) = \int_{w=0}^{\infty} f_W(w)\mathrm{P}(A \geq w)\,dw = \int_{w=0}^{\infty} f_W(w)e^{-\lambda w}\,dw = W^*(\lambda).$$

The rest of the proof is by the previous item.

c) Insensitivity property was defined as some parameter of a queueing system which is a function of the service distribution only through its mean value which was denoted by $\bar{x}$. See first main paragraph on page 56. In the previous two items of this question, the lefthand sides are functions of $G(x)$ while the righthand side only of $\bar{x}$.

Note that the order of proving the first two items can be reversed. Specifically, inserting $s = \lambda$ in (6.12) on page 86, immediately leads to the fact that $W^*(\lambda) = 1 - \rho$.

An alternative proof: (Yoav's)

a) Consider an arbitrary customer, who sojourn time is W. The event $\{A \geq W\}$ is the event the customer who arrives after the arbitrary customer will find an empty system (because his inter arrival time is larger than the sojourn time of the previous customer). The probability of finding an empty system is $1 - \rho$.

b) Recall that for any random variable Y, $\mathrm{P}(Y \leq A) = Y^*(\lambda)$. This fact is sufficient for the result. Yet, direct derivation, by recalling that

$$W^*(s) = \frac{(1-\rho)G^*(s)}{1-\rho G_r^*(s)} = \frac{s(1-\rho)G^*(s)}{s-\lambda(1-G^*(s))}$$

and inserting $s = \lambda$ we get $W^*(\lambda) = 1 - \rho$.

c) These results actually implies that the value of $W^*(\lambda)$ depends on the service time distribution on through its mean and insensitive to the other properties of the distribution.

6.18 Question 18

The corollary is referring on an M/G/1 queue. We first prove (6.23) although it was already established as Theorem 4.7 (page 62). Our point of departure is Equation (6.19). Taking derivative with respect to s is both hand sides we get that

$$(B^*)'(s) = (G^*)'(s + \lambda(1 - B^*(s)))(1 - \lambda(B^*)'(s)). \tag{13}$$

Recall at any LST gets the value of 1 when the variable s gets the value of zero and the negative of its derivative there coincides with the corresponding mean value we get that

$$-\bar{b} = -\bar{x}(1 + \lambda\bar{b}).$$

Solving this for $\bar{b}$ we get (6.23). Our next mission is to prove (6.24). Taking derivative with respect to s in (13) we get

$$(B^*)"(s) = (G^*)"(s+\lambda(1-(B^*)(s))(1-\lambda(B^*)'(s))^2 - \lambda(B^*)"(s)(G^*)'(s+\lambda(1-(B^*)(s)).$$

Recall that in the value of the second derivative of an LST at zero yields the corresponding second moment, we get

$$\bar{b^2} = \bar{x^2}(1+\lambda\bar{b})^2 + \lambda\bar{b^2}\bar{x}.$$

Using the value for $\bar{b}$ just derived and solving for $\bar{b^2}$ gives the expression we are after.

6.19 Question 19

a) Consider the hint. The LST in case $T = 0$ and it is the residual of a busy period in case $T > 0$. Using Equation (2.10) (page 30), we get the later equals $(1 - B^*(s))/(\bar{b}s)$. Hence,

$$T^*(s) = (1-\rho) + \rho\frac{1 - B^*(s)}{\bar{b}s}.$$

Since $\bar{b} = [(1-\rho)\mu]^{-1}$ (see Equation (4.10)), we get that

$$T^*(s) = (1-\rho) + (1-\rho)\lambda\frac{1 - B^*s}{s}.$$

Note that this derivation holds for any $M/G/1$.

b) The first thing to observe here is that in an M/M/1 the time to reduce the number in the queue from n to $n-1$ is as the length of a busy period.[1] Hence, conditioning on L, T is distributed as the sum of L independent busy periods. The random variable $L \geq 0$ itself follows a geometric distribution with parameter $(1-\rho)$: $\mathrm{P}(L = l) = (1-\rho)\rho^l$, $l \geq 0$. Hence,

$$T^*(s) = \mathrm{E}(e^{-sT}) = \mathrm{E}(\mathrm{E}(e^{-sT}|L)) = \sum_{l=0}^{\infty}(1-\rho)\rho^l(B^*(s))^l = \frac{1-\rho}{1-\rho B^*(s)},$$

as required.

c) Equating the two expressions for $T^*(s)$ derived above, leads to an equation in the unknown $B^*(s)$. Some algebra leads to its equivalence to Equation (6.21).

[1] The argument in the beginning of Section 6.3.2 (page 89) explains in fact why this result does not extend to an M/G/1.

6.20 Question 20

There are two typos here. First, the recursion should be read as

$$u_i = \frac{1}{a_0}(u_0\beta_i + \sum_{j=0}^{i-1} \beta_{i-j+1}).$$

Second, the definition of β_i is $\Sigma_{j=i}^{\infty} a_i$. Finally, this recursion is in fact given in Equation (6.6) (see page 84). The derivation which proceeds it proves it.

6.21 Question 21

a) The sojourn time in an M/M/1 queue follows an exponential distribution with parameter $(1-\rho)\mu$ (see Theorem 6.5 on page 87). Hence the residual and the age of Mr. Smith time in the system follows the same distribution (see Example 1 on page 27). The total time in the system, given one is there, is the corresponding length bias distribution which is the sum of two independent such exponential random variables, namely Erlang with parameters 2 and $(1-\rho)\mu$) (see Example 2 in page 22).

b) The number in an M/M/1 queue (service inclusive) follows a geometric distribution with parameter $(1-\rho)$. Those ahead of Mr. Smith (inclusive of him) can be looked as the residual and those behind him (inclusive of him) as the age of this number. Hence, by Example 4 on page 33, we learn that both follow the same distribution which is hence geometric with parameter $(1-\rho)$. It is also shown there that these two are independent. The total number is hence their sum minus one (in order not to count Mr. Smith twice). This sum, again see Example 4 on page 33, follows a negative binomial distribution with parameters 2 and $(1-\rho)$.

7 Chapter 7

7.1 Question 1

a) The LST of the random variable corresponding to an interarrival time is
$$\frac{\lambda^n}{(\lambda+s)^n}$$
(see (1.24) on page 16). This, coupled with (7.4) from page 101, lead to the value we are after solves
$$\sigma = \frac{\lambda^n}{(\lambda+\mu(1-\sigma))^n}.$$
Looking at this an equation who might have more than one solution, we are interested in the unique root which is a fraction between zero and one (see Theorem 7.1 on page 100).

b) In the case where $n = 1$ we the quadratic equation $\sigma(\lambda+\mu(1-\sigma)) = \lambda$. This is a quadratic equation. It has two solutions 1 and λ/μ. We are after the latter solution (which is assumed to be smaller than one). Indeed, in the where $n = 1$ we in fact have an M/M/1 model and for example (7.6) on page 103 hence coincides with what we know for $M/M/1$ (see page 85 and or Example 2 on page 120 (note that the latter corresponds to a later chapter). In the case where $n = 2$ the quadratic equation
$$\sigma(\lambda+\mu(1-\sigma))^2 = \lambda^2,$$
or, equivalently,
$$\mu^2\sigma^3 - 2\mu(\lambda+\mu)\sigma^2 + (\lambda+\mu)^2\sigma - \lambda^2 = 0.$$
This cubic equation has a root which equals one, so what we look of solves
$$\mu^2\sigma^2 - (2\lambda\mu+\mu^2) + \lambda^2 = 0.$$
This quadratic equation has two positive roots. We look for the smallest among them (it is impossible to have two fractions as solutions). Hence, the value we are looking for equals
$$\frac{2\lambda\mu+\mu) - \sqrt{(2\lambda\mu+\mu)^2 - 4\mu^2\lambda^2}}{2\mu^2}.$$

7.2 Question 2

a) (i) Initiate with Bayes' rule:

$$f_{A|L=n}(a) = \frac{f_A(a)}{\mathrm{P}(L=n)}\mathrm{P}(L=n|A=0).$$

Next, by (2.4) (see page 25), $f_A(a) = \overline{G}(a)/\bar{t}$, and by (7.6) (see page (103)), $\mathrm{P}(L=n) = \rho(1-\sigma)\sigma^{n-1}$, for $n \geq 1$. Thus, all needed is to derive $\mathrm{P}(L=n|A=a)$ for $n \geq 1$. This value can be looked by conditioning on the number in the system upon the beginning of the current inter-arrival time. This number, aught to equal some value m with $n-1 \leq m < \infty$ (where $m-(n-1)$ is the number of service completions during the current (age of) inter-arrival time). This number, given an age of a equals k for $0 \leq k < m$ with probability $\eta^{-\mu a}(\mu a)^k/k!$ (Note that this expression does not hold for the case $k = m$ as $k = m$ gets the complementary value of all the cases $k < m$). Hence,

$$\mathrm{P}(L=n) = \sum_{m=n-1}^{\infty}(1-\sigma)\sigma^m e^{-\mu a}\frac{(\mu a)^{m-n+1}}{(m-n+1)!}.$$

This value equals

$$(1-\sigma)\sigma^{n-1}e^{-\mu a}\sum_{i=0}^{\infty}\frac{(\sigma\mu a)^i}{i!} = (1-\sigma)\sigma^{n-1}e^{-\mu a}e^{\sigma\mu a} = (1-\sigma)\sigma^{n-1}e^{-\mu a(1-\sigma)}.$$

Putting it all together and some minimal algebra concludes the proof.

(ii) First, trivially,

$$\int_{a=0}^{\infty}\overline{G}(a)\mu e^{-\mu(1-\sigma)}\,da = \bar{t}\mu\int_{a=0}^{\infty}\frac{\overline{G}(a)}{\bar{t}}e^{-\mu(1-\sigma)}\,da.$$

Since $\overline{G}/\bar{t}$ is the density of the age of service, we get that the last expression equals $\bar{t}\mu$ times the LST of this age at the point $\mu(1-\sigma)$. Using (2.10) (see page 30). Hence, we get

$$\bar{t}\mu\frac{1-G^*(\mu(1-\sigma))}{\bar{t}\mu(1-\sigma)} = \frac{1-G^*(\mu(1-\sigma))}{1-\sigma}.$$

This by the definition of σ (see (7.4) on page 101), equals

$$\frac{1-\sigma}{1-\sigma} = 1,$$

as required.

(iii)

$$\mathrm{P}(S \le s | S \le Y) = \frac{\mathrm{P}(S \le s, S \le Y)}{\mathrm{P}(S \le Y)} = \frac{\mathrm{P}(S \le \{s, Y\})}{\mathrm{P}(S \le Y)}$$
$$= \frac{\int_{t=0}^{s} \mu(1-\sigma) e^{\mu(1-\sigma)} \overline{G}(t)\, dt}{\int_{t=0}^{\infty} \mu(1-\sigma) e^{\mu(1-\sigma)} \overline{G}(t)\, dt}.$$

Consider the numerator. Take a derivative with respect to s and get $\mu(1-\sigma)e^{-\mu(1-\sigma)s}\overline{G}(s)$. This function is proportional to the density we are looking for and in particular, there is no need to consider further the denominator. This value up to a multiplicative constant is what appears in (7.7) and of course (7.7) is hence the density we are after.

b) First, note that $\mathrm{P}(L = 0) = 1 - \rho$ and $\mathrm{P}(L \ge 1) = \rho$. Second, note that the unconditional density of A, the age of of the arrival, is $\overline{G}(a)/\bar{t}$ (see (2.4) on page 25). Hence,

$$\frac{\overline{G}(a)}{\bar{t}} = (1-\rho) f_{A|L=0}(a) + \rho f_{A|L\ge 1}(a).$$

The value of $f_{A|L\ge 1}(a)$ is given in fact in (7.7) (as said above, the expression there is free of n as long as $n \ge 1$ and hence it coincide with the corresponding expression when $L \ge 1$). All left is to solve for $f_{A|L=0}(a)$ is can be done by some trivial algebra.

c) The point of departure should be that

$$f_R(r) = \int_{a=0}^{\infty} f_A(a) \frac{g(a+r)}{\overline{G}(a)}\, da.$$

This can be seen by integrating (2.6) (see page 28) with respect to a. The same is the case if the density of A is replace by some conditional density of A, say $A|X$ as long as given A, R and X are independent. This is for example the case here: L and R are certainly not independent. Yet, given A, L and R are independent.

7.3 Question 3

(Benny)

a) In order to determine the distribution of the number in the system at the next arrival, all is needed is the number in the system at the current arrival. This is true since the service times are memoryless.

b) Let Y_n be the (random) number of customers served during the n-th interarrival time. Clearly,

$$X_{n+1} = X_n + 1 - Y_n$$

c) When s servers serve s customers simultaneously, the time till next service completion follows exponential distribution with rate $s\mu$. Hence, assuming there is no shortage of customers in the queue, i.e., all service completions during this interarrival time occurred while all s servers were busy,

$$b_k = \int_{t=0}^{\infty} e^{-s\mu t} \frac{(s\mu t)^k}{k!} g(t)\, dt$$

d) In order that all service completions during this interarrival time occurred while all s servers were busy we need that $j \geq s-1$ which is assumed. Hence,

$$P(X_{n+1} = j | X_n = i) = P(Y_n = i+1-j | X_n = i) = b_{i+1-j}$$

e) The corresponding balance equations are

$$u_i = \sum_{j=i-1}^{\infty} u_j b_{j+1-i}, \quad i \geq s,$$

or alternatively

$$u_i = \sum_{n=0}^{\infty} u_{i-1+n} b_n, \quad i \geq s.$$

Now, it can be checked that $u_i = u_s \sigma^{i-s}$, $i \geq s$ for σ that solves $\sigma = \sum_{n=0}^{\infty} b_n \sigma^n$, solve the above equations.

$$\sum_{n=0}^{\infty} u_{i-1+n} b_n = \sum_{n=0}^{\infty} u_s \sigma^{i-1+n-s} b_n = u_s \sigma^{i-1-s} \sum_{n=0}^{\infty} \sigma^n b_n = u_s \sigma^{i-s} = u_i$$

f) The proof is the same as the proof of Theorem 7.1, but with $s\mu\bar{t} > 1$.

8 Chapter 8

8.1 Question 1

a) The state space is $\{0, 1, \ldots, N\}$, representing the number of customers in the system. It is a birth and death process with $\lambda_{i,i+1} = \lambda$, $1 \le i \le N-1$, and $\mu_{i,i-1} = \mu$, $1 \le i \ne N$. All other transition rates equal zero.

b) The limit probabilities can be derived as leading to equation (8.17) but now $1 \le i \le N-1$. Using induction, this leads to $\pi_i = \pi_0 \rho^i$, $0 \le i \le N$ where $\rho = \lambda/\mu$. Using the fact that the sum of the limit probabilities is one, we get that $\pi_0 = [\sum_{i=0}^{N} \rho^i]^{-1}$. This completes the proof.

c) The fact that $L(0) = 1$ is trivial. The rest is algebra:

$$L(N+1) = \frac{\rho^{N+1}}{\sum_{i=0}^{N+1} \rho^i} = \frac{\rho^{N+1}}{\sum_{i=0}^{N} \rho^i + \rho^{N+1}}$$

$$= \frac{\rho^{N+1}}{\frac{\rho^N}{L(N)} + \rho^{N+1}} = \frac{\rho}{\frac{1}{L(N)} + \rho} = \frac{\rho L(N)}{1 + \rho L(N)},$$

as required.

d) Note first that

$$\lambda \pi_i = \mu \pi_{i+1}, \quad 0 \le i \le N-1.$$

Summing up both sides from $i = 0$ through $i = N-1$, we get

$$\lambda(1 - \pi_N) = \mu(1 - \pi_N),$$

as required

e) No conditions are required on the transitions rates as long as they are positive.

f) As any birth and death process, this is a time-reversible Markov process and the detailed balance equations hold. They are

$$\lambda \pi_i = \mu \pi_{i+1}, \quad 0 \le i \le N-1.$$

8.2 Question 2

(Liron) Proof of formula formula (8.14).
For the case were the service rate is equal for all classes, i.e. $\mu_c = \mu \ \forall c \in \mathcal{C}$ (where $\mathcal{C}$ is the set of classes), the balance equations can be re-written as:

$$\begin{aligned}
\pi(\emptyset) \sum_{c\in\mathcal{C}} \rho_{c\in\mathcal{C}} &= \sum_{c} \pi(c) \\
\left(\sum_{c\in\mathcal{C}} \rho_c + 1\right) \pi(c_1, \ldots, c_n) &= \rho_{c_n} \pi(c_1, \ldots, c_{n-1}) + \sum_{c\in C} \pi(c, c_1, \ldots, c_n),
\end{aligned}$$

where $\rho_c = \frac{\lambda_c}{\mu}$. The solution is of a product form:

$$\pi(c_1, \ldots, c_n) = \pi(\emptyset) \prod_{i=1}^{n} \rho_{c_i}. \tag{14}$$

We find the multiplicative constant by summing over the probabilities and equating to one. One must be careful in the summation in order to cover all of the possible state space, which can be stated as follows:

$$\emptyset \cup \left\{ n \in \{1, 2, \ldots\} : \sum_{c\in\mathcal{C}} \sum_{i=1}^{n} \mathbb{1}(c_i = c) = n \right\},$$

i.e. all the possible queue sizes, and all the possible class divisions for every queue size. If we denote $m_c := \sum_{i=1}^{n} \mathbb{1}(c_i = c)$, then the number of possible vectors $(c_1, \ldots, c_n)$ is $\frac{n!}{\prod_{c\in\mathcal{C}} m_c!}$ (because the order within a class does not matter). Finally the normalization equation is:

$$\begin{aligned}
1 &= \pi(\emptyset) + \sum_{n=1}^{\infty} \sum_{\{\sum_{c\in\mathcal{C}} m_c = n\}} \frac{n!}{\prod_{c\in\mathcal{C}} m_c!} \pi(c_1, \ldots, c_n) \\
&= \pi(\emptyset) + \sum_{n=1}^{\infty} \sum_{\{\sum_{c\in\mathcal{C}} m_c = n\}} \frac{n!}{\prod_{c\in\mathcal{C}} m_c!} \pi(\emptyset) \prod_{c\in\mathcal{C}} \rho_c^{m_c},
\end{aligned}$$

which yields:

$$\pi(\emptyset) = (1 + A)^{-1}, \tag{15}$$

where:

$$A = \sum_{n=1}^{\infty} \sum_{\{\sum_{c\in\mathcal{C}} m_c = n\}} \frac{n!}{\prod_{c\in C} m_c!} \prod_{c\in\mathcal{C}} \rho_c^{m_c}. \tag{16}$$

8.3 Question 3

This exercise goes verbatim with Exercise 3 from Chapter 3.

8.4 Question 4

From (8.23) (swapping the roles of the indices) we get that $q^*_{ij} = \pi_j q_{ji}/\pi_i$. Substituting this in (8.22), we get

$$\sum_{j \in N} q_{ij} = \sum_{j \in N} \frac{\pi_j q_{ji}}{\pi_i},$$

from which the balance equation

$$\pi_i \sum_{j \in N} q_{ij} = \sum_{j \in N} \pi_j q_{ji}$$

is immediate. The symmetry between the original and the time-reversed processes, namely the fact that $(q^*)^* = q$, takes care of the second part of the exercise.

8.5 Question 5

a) In the case where the server is busy, the next departure takes time which follows an exponential distribution with parameter μ.

b) In case where the server is idle the time until first departure is the sum of two independent and exponential random variables, one with parameter λ (arrival) and one with parameter μ (service). Denote the first by X and the second by Y. Then,

$$f_{X+Y}(x) = \int_{t=0}^{x} f_X(t) f_y(x-t)\,dt = \int_{t=0}^{x} \lambda e^{-\lambda t} \mu e^{-\mu(x-t)}\,dt = \frac{\lambda \mu e^{-\mu x}}{\lambda - \mu} \int_{t=0}^{x} (\lambda - \mu) e^{-(\lambda-\mu)t}\,dt$$

$$= \frac{\lambda\mu}{\lambda - \mu} e^{-\mu x}(1 - e^{-(\lambda-\mu)x}) = \frac{\lambda\mu}{\lambda - \mu}(e^{-\mu x} - e^{-\lambda x}),$$

as required.

c) We consider in the previous two items, the two possible cases, leading to the next departure. The former is the case with probability ρ (which is the probability that the server is busy) while it is the second with probability $1-\rho$ (which is the probability that the server is idle). Thus, the density function at point x for the time until the next departure is

$$\rho\mu e^{-\mu x} + (1-\rho)\frac{\lambda\mu}{\lambda-\mu}(e^{-\mu x} - e^{-\lambda x}).$$

Minimal algebra shows that this equals

$$\lambda e^{-\lambda x},$$

which is the exponential density with parameter λ. Note that the departure rate is λ (and not μ): what's come in, must come out. So the value of the parameter was quite expected.

An alternative proof using LSTs: Let T be the random time between two departures. It's LST is

$$\mathrm{E}(e^{-sT}) = (1-\rho)\mathrm{E}(e^{-sT}|L=0) + \rho E(e^{-sT}|L \geq 1).$$

In the case where $L = 0$, $T = X + Y$ where X is exponential with λ and Y is exponential with λ. In the case where $L \geq 1$, $T = Y$. Hence.

$$\mathrm{E}(e^{-sT}) = (1-\rho)\frac{\lambda}{\lambda+s}\frac{\mu}{\mu+s} + \rho\frac{\mu}{\mu+s}.$$

Some algebra shows that it equals $\lambda/(\lambda+s)$ which is the LST of an exponential random variable with parameter λ is required.

d) There is one more thing which needs to be established and the analysis above does not address it: In order to show that the departure process is Poisson, we need to show that consecutive departure times are independent.

8.6 Question 6

(Liron)

a) The only transition rates that need to be updated are $q_{,0,1A} = p\lambda$ anfd $q_{,0,1B} = (1-p)\lambda$. The new flow diagram:

b) The balance equations:

$$\begin{aligned}
\pi_0\left(\lambda p + \lambda(1-p)\right) &= \pi_{1A}\mu_1 + \pi_{1B}\mu_2 && (17)\\
\pi_{1A}\left(\mu_1 + \lambda\right) &= \pi_0\lambda p + \pi_2\mu_2 && (18)\\
\pi_{1B}\left(\mu_1 + \lambda\right) &= \pi_0\lambda(1-p) + \pi_2\mu_1 && (19)\\
\pi_2\left(\lambda + \mu_1 + \mu_2)\right) &= \pi_{1A}\lambda + \pi_{1B}\lambda + (\mu_1+\mu_2)\pi_3 && (20)\\
\pi_i\left(\lambda + \mu_1 + \mu_2)\right) &= \pi_{i-1}\lambda + (\mu_1+\mu_2)\pi_{i+1},\ i \geq 3. && (21)
\end{aligned}$$

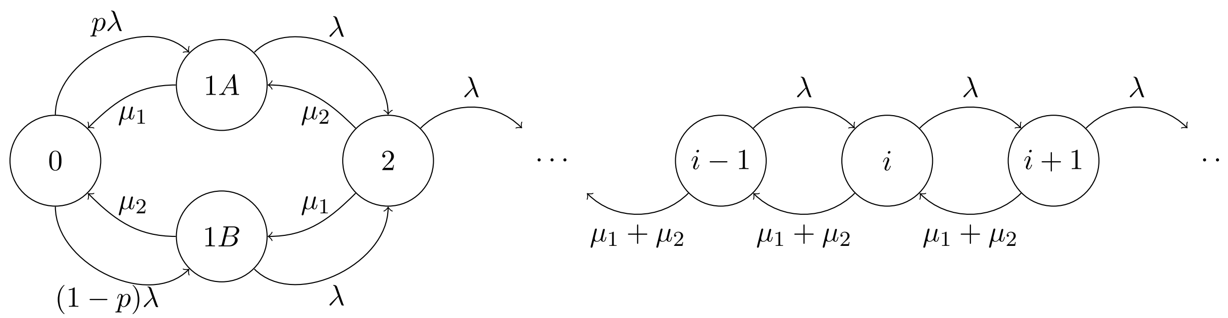


Figure 1: Two non-identical servers. First server in busy period chosen according to p.

We can apply the cut-balancing Theorem 8.3 to states $i \geq 2$:

$$\pi_i \lambda = \pi_{i+1}(\mu_1+\mu_2) \Leftrightarrow \pi_{i+1} = \pi_i \frac{\lambda}{\mu_1+\mu_2}. \tag{22}$$

Which is also to equivalent to $\pi_i = \pi_2 \left(\frac{\lambda}{\mu_1+\mu_2}\right)^{i-2}$ $\forall i \geq 2$. We are left with computing $\pi_0, \pi_{1A}, \pi_{1B}$ and π_2. From equations (18) and (19) we derive:

$$\pi_{1A} = \pi_0 \frac{\lambda p}{\lambda+\mu_1} + \pi_2 \frac{\mu_2}{\lambda+\mu_1} \tag{23}$$

$$\pi_{1B} = \pi_0 \frac{\lambda(1-p)}{\lambda+\mu_2} + \pi_2 \frac{\mu_1}{\lambda+\mu_2}. \tag{24}$$

By plugging in the above equations into (17) we get:

$$\pi_0 \lambda = \pi_0 \frac{\lambda p \mu_1}{\lambda+\mu_1} + \pi_2 \frac{\mu_1 \mu_2}{\lambda+\mu_1} + \pi_0 \frac{\lambda(1-p)\mu_2}{\lambda+\mu_2} + \pi_2 \frac{\mu_1\mu_2}{\lambda+\mu_2}. \tag{25}$$

Simple algebra yields $\pi_2 = \pi_0 C$, where:

$$A = \frac{\lambda^2 \left(\lambda + (1-p)\mu_1 + p\mu_2\right)}{\mu_1\mu_2(2\lambda+\mu_1+\mu_2)}. \tag{26}$$

We can now conclude that the solution to the balance equations is given by:

$$\begin{aligned}
\pi_{1A} &= \pi_0 \frac{\lambda p + \mu_2 C}{\lambda + \mu_1} && (27)\\
\pi_{1B} &= \pi_0 \frac{\lambda(1-p) + \mu_1 C}{\lambda + \mu_2} && (28)\\
\pi_i &= \pi_0 C \left(\frac{\lambda}{\mu_1+\mu_2}\right)^{i-2}, \ i \geq 2. && (29)
\end{aligned}$$

Finally, to derive π_0 we sum the probabilities to and equate them to one:

$$\begin{aligned}
1 &= \pi_0 + \pi_{1A} + \pi_{1B} + \sum_{i=2}^{\infty} \pi_i \\
&= \pi_0 \left(1 + \frac{\lambda p + \mu_2 C}{\lambda + \mu_1} + \frac{\lambda(1-p) + \mu_1 C}{\lambda + \mu_2} + A \sum_{i=2}^{\infty} \left(\frac{\lambda}{\mu_1+\mu_2}\right)^{i-2}\right) \\
&= \pi_0 \left(1 + \frac{\lambda p + \mu_2 C}{\lambda + \mu_1} + \frac{\lambda(1-p) + \mu_1 C}{\lambda + \mu_2} + A \frac{\mu_1+\mu_2}{\mu_1+\mu_2-\lambda}\right),
\end{aligned}$$

yielding:

$$\pi_0 = \left(1 + \frac{\lambda p + \mu_2 C}{\lambda + \mu_1} + \frac{\lambda(1-p) + \mu_1 C}{\lambda + \mu_2} + A \frac{\mu_1+\mu_2}{\mu_1+\mu_2-\lambda}\right)^{-1}. \quad (30)$$

c) From Theorem 8.4 we know that the transition rates of the time-reversed process are: $q^*_{ij} = \frac{\pi_j}{\pi_i} q_{ji}, \ i \geq 0$. We can apply the results of (b):

$$
\begin{aligned}
q^*_{0,1A} &= \mu_1 \frac{\lambda p + \mu_2 C}{\lambda + \mu_1} \\
q^*_{0,1B} &= \mu_2 \frac{\lambda(1-p) + \mu_1 C}{\lambda + \mu_2} \\
q^*_{1A,0} &= \lambda \frac{\lambda + \mu_1}{\lambda p + \mu_2 C} \\
q^*_{1B,0} &= \lambda \frac{\lambda + \mu_2}{\lambda(1-p) + \mu_1 C} \\
q^*_{1A,2} &= \mu_2 C \lambda \frac{\lambda + \mu_1}{\lambda p + \mu_2 C} \\
q^*_{1B,2} &= \mu_1 C \lambda \frac{\lambda + \mu_2}{\lambda(1-p) + \mu_1 C} \\
q^*_{2,1A} &= \lambda \frac{\lambda p + \mu_2 C}{C(\lambda + \mu_1)} \\
q^*_{2,1B} &= \lambda \frac{\lambda(1-p) + \mu_1 C}{C(\lambda + \mu_2)} \\
q^*_{i,i+1} &= \lambda,\ i \geq 2 \\
q^*_{i,i-1} &= \mu_1 + \mu_2,\ i \geq 3
\end{aligned}
$$

d) The process is not time-reversible. The Kolmogorov criterion on the product of the transition rates is a necessary and sufficient condition for time reversibility, so it is enough to show one cycle where the criterion is not met. Consider the cycle $(0, 1A, 2, 1B, 0)$. The product of the transition rates in the original order along this cycle equals $p\lambda\lambda\mu_1\mu_2$. Reversing the order one gets $(1-p)\lambda\lambda\mu_2\mu_1$. These two products are equal if and only if $p = \frac{1}{2}$. This shows that $p = \frac{1}{2}$ is a necessary condition for time-reversibility.

e) As it turns out it is also sufficient: All other cycles are as this one or they are based on simply traversing the order of the states which trivially obeys the condition.

8.7 Question 7

Inspect Figure 8.11 in page 117. The states are the same. Likewise is the case with regard to the service rates μ_i, $i = 1, 2$, where applicable. The difference are as follows: $q_{00,10} = q_{00,01} = \lambda/2$ and all other λ_1 and λ_2 need to be replaced with the common value of λ. The process is certainly time-reversible: The Markov process has a tree-shape and in fact, by re-labeling the states, we have here a birth-and-death process. See the discussion on the top of page 132. As for the limit probabilities, keep the definition for π_{ij} as stated in Example 12 in page 126. By the cut balancing theorem, we get that $\pi_{i+1,0} = \rho_1 \pi_{i,0}$ for $i \geq 1$ and $\rho_1 = \lambda/\mu_1$. Hence, $\pi_{i,0} = \rho^{i-1}\pi_{1,0}$, $i \geq 1$. Likewise, $\pi_{1,0} = \rho_1 \pi_{0,0}/2$ and hence $\pi_{i,0} = \rho_1^i \pi_{00}/2$. Similarly, $\pi_{0,i} = \rho_2^i \pi_{00}/2$ for $i \geq 1$ and $\rho_2 = \lambda/\mu_2$. Since $\pi_{00} + \Sigma_{i=1}^{\infty}\pi_{i0} + \Sigma_{i=1}^{\infty}\pi_{0i} = 1$, we get that

$$\pi_{00}(1 + \frac{\rho_1}{2(1-\rho_1)} + \frac{\rho_2}{2(1-\rho_2)} = 1,$$

from which the value for π_{00} can be easily found to equal

$$\frac{2(1-\rho_1)(1-\rho_2)}{2-\rho_1-\rho_2-\rho_1\rho_2}.$$

8.8 Question 8

Assume for all pairs of i and j, $\pi_i q_{ij} = \pi_j q_{ij}$, namely the vector π obeys

$$\sum_j \pi_i q_{ij} = \sum_j \pi_j q_{ij}$$

or

$$\pi_i \sum_j q_{ij} = \sum_j \pi_j q_{ij},$$

which is the balance equation corresponding to state i.

8.9 Question 9

a) Exponential with parameter μ. This is by the fact that the distribution of the age of an exponential distribution coincides with the distribution of the original random variable. See Example 1 in Section 2.2.2.

b) Consider the time epoch of the current service commencement. Looking from this point of time, no arrival took place and likewise a not service completion. This means that time since service commencement, namely its age, is the minimum between two independent exponential random variables with parameters λ and μ. Since this minimum is also exponential with parameter $\lambda + \mu$ and its age follows the same distribution (since it is exponential) we conclude with the answer: Exponential with parameter $\lambda + \mu$.

c) The answer was in fact given at the previous item.

d) Using the results of the previous item and replacing densities with probabilities

$$\mathrm{P}(empty|Age = t) = \frac{\mathrm{P}(empty)\mathrm{P}(Age = t|empty)}{\mathrm{P}(Age = t)} = (1-\rho)\frac{(\lambda+\mu)e^{-(\lambda+\mu)}}{\mu e^{-\mu t}} = (1-\rho^2)e^{-\lambda t}.$$

Another possible question here is: Given one is in service, what is the distribution of one's service time. The answer: it an Erlang distribution with parameters 2 and μ. One way to look at this is by the fact that the length distribution of an exponential distribution is an Erlang with these parameters. See Example 1 in Section 2.2.1. Another way is to notice that the future (ie., residual) service time is exponential with μ by the memoryless property. Likewise, by the time-reversibility of the process, this is also the distribution of the age. Both are independent. Finally, the sum of two iid exponentially distributed random variable with parameter μ follows an Erlang distribution with parameters 2 and μ. See Section 1.2.5.

8.10 Question 10

a) The amount of work at the system is less than or equal than in the corresponding FCFS (without reneging) due to work who leaves without being served. Hence this is not a work-conversing system.

b) Since given the number in the system at some time determines the future (statistically) in the same way as in the case when further past history is given, the number in the system presents a state in a Markov chain.

c) We have a birth and death process with $\{0, 1, 2, \ldots\}$ as the state space. The transition rates are $\lambda_{i,i+1} = \lambda$, $i \geq 0$, and $\mu_{i,i-1} = \mu + i\theta$, $i \geq 1$.

d) The balance equations are

$$\lambda\pi_0 = \mu\pi_1$$

and

$$(\lambda + i\theta)\pi_i = \lambda\pi_{i-1} + (\mu + (i+1)\theta)\pi_{i+1}, \quad i \geq 1$$

The detailed balance equations are

$$\lambda\pi_i = (\lambda + (i+1)\theta)\pi_{i+1}, \quad i \geq 0.$$

Their solution is

$$\pi_i = \pi_0 \frac{\lambda^i}{\Pi_{j=1}^{i}(\mu + j\theta)}, i \geq 0.$$

π_0 is the constant which makes the summation across limit probabilities equal to one:

$$\pi_0 = [\sum_{i=0}^{\infty} \frac{\lambda^i}{\Pi_{j=1}^{i}(\mu + j\theta)}]^{-1}.$$

This summation is always well defined (as long as all parameters λ, μ and θ are positive. The model resembles mostly the $M/M/\infty$ model.

e) It is only the third process which is Poisson. The argument is that the abovementioned Markov process is time-reversible as any birth and death process. The arrival process in the time-reversible process is λ but any such arrival corresponds to a departure in the original process (without 'telling' the reason for this departure, service completion or abandonment).

f) λP is the long run abandonment rate. θ is the individual abandonment rate and since L is the mean number in the system, θL is the long run abandonment rate. Hence $\lambda P = \theta L$. Since $L = \lambda W$ by Little's, we get $\lambda P = \theta \lambda W$ and hence $P = \theta W$.

8.11 Question 11

a) Consider the time-reversed process and recall that an $M/M/1$ is time-reversible. Looking from the time-reversed process angle, the age of service (in the original process) is the time first arrival or until the system is empty, whatever comes first. The next event will take within time whose mean equals $1/(\lambda + \mu)$. It is a departure with probability $\lambda/(\lambda + \mu)$, a case which adds zero to the mean value. It is a service completion with probability $\mu/(\lambda + \mu)$, a case which adds a_{n-1} to the mean. Hence,

$$a_n = \frac{1}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} a_{n-1}, \quad n \geq 1$$

with $a_0 = 0$.

b) Note first that the case $n = 1$ was already dealt with in Q9(c). From there we can conclude that $a_1 = 1/(\lambda + \mu)$ which will be used as our anchor for the induction argument. Using the previous item, coupled with the induction hypothesis, we get that

$$a_n = \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}(\frac{1}{\lambda} - \frac{1}{\lambda}\frac{1}{(1+\rho)^{n-1}}$$

$$= \frac{1}{\lambda} - \frac{1}{\lambda+\mu}\frac{\mu}{\lambda}\frac{1}{(1+\rho)^n} = \frac{1}{\lambda} - \frac{1}{\lambda}\frac{1}{(1+\rho)^n},$$

as required.

c) This item is trivial.

8.12 Question 12

a) This is a birth and death process with state space $\{0, 1, \ldots, s\}$. The birth rates $\lambda_{i,i+1} = \lambda$, $0 \le i \le s-1$, and death rates $\mu_{i,i-1} = i\mu$. $1 \le i \le s$. The detailed balance equations are

$$\lambda\pi = (i+1)\mu\pi_{i+1}, \quad 0 \le i \le s-1.$$

From that we get that

$$\pi_{i+1} = \frac{\lambda}{(i+1)\mu}\pi_i, \quad 0 \le i \le s-1.$$

and then by induction that

$$\pi_i = \frac{\rho^i}{i!}\pi_0, \quad 0 \le i \le s.$$

The value of π_0 is found by noting the the sum of the probabilities equals one. Hence,

$$\pi_0 = [\sum_{i=0}^{s}\frac{\rho^i}{i!}]^{-1}.$$

This concludes our proof. Finally, note that both numerator and denominator can be multiplied by $e^{-\rho}$. Then, In the numerator one gets $\mathrm{P}(X = i)$ when X follows a Poisson distribution with parameter ρ. The denominator equals $\mathrm{P}(X \le s)$.

b) $B(0) = 1$ as this is the loss probability in case of no servers.

$$B(s{+}1) = \frac{\rho^{s+1}/(s+1)!}{\sum_{i=0}^{s+1} \rho^i/i!} = \frac{\rho^{s+1}/(s+1)!}{\sum_{i=0}^{s} \rho^i/i! + \rho^{s+1}/(s+1)} = \frac{\rho^{s+1}/(s+1)!}{\frac{r^s/s!}{B(s)} + \rho^{s+1}/(s+1)}$$

$$\frac{\rho/(s+1)}{\frac{1}{B(s)} + 1 + \rho/(s+1)} = \frac{\rho B(s)}{1 + s + \rho B(s)},$$

as required.

c) (Liron) From example 4 we know that the probability of all s servers being busy in a $M/M/s$ system is:

$$Q(s) = \sum_{j=s}^{\infty} \frac{\left(\frac{\lambda}{\mu}\right)^j}{s! s^{j-s} \left(\sum_{i=0}^{s-1} \frac{\left(\frac{\lambda}{\mu}\right)^i}{i!} + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!\left(1 - \frac{\lambda}{\mu s}\right)} \right)}. \tag{31}$$

In the previous items we showed that:

$$B(s) = \pi_s = \frac{\frac{\left(\frac{\lambda}{\mu}\right)^s}{s!}}{\sum_{i=0}^{s} \frac{\left(\frac{\lambda}{\mu}\right)^i}{i!}}. \tag{32}$$

Combining both of the above:

$$\begin{aligned}
Q(s) &= \sum_{j=s}^{\infty} \frac{\left(\frac{\lambda}{\mu}\right)^j}{s! s^{j-s} \left(\sum_{i=0}^{s-1} \frac{\left(\frac{\lambda}{\mu}\right)^i}{i!} + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!\left(1 - \frac{\lambda}{\mu s}\right)} \right)} \\
&= \frac{\frac{\left(\frac{\lambda}{\mu}\right)^s}{s!}}{\sum_{i=0}^{s} \frac{\left(\frac{\lambda}{\mu}\right)^i}{i!} \left(1 - B(s) + \frac{B(s)}{1 - \frac{\lambda}{\mu s}}\right)} \sum_{j=0}^{\infty} \frac{\lambda}{\mu s} \\
&= \frac{B(s)}{\left(1 - B(s) + \frac{B(s)}{1 - \frac{\lambda}{\mu s}}\right)\left(1 - \frac{\lambda}{\mu s}\right)} \\
&= \frac{B(s)}{1 - \frac{\lambda}{\mu s} + \frac{\lambda}{\mu s} B(s)} \\
&= \frac{s B(s)}{s - \frac{\lambda}{\mu}(1 - B(s))}
\end{aligned}$$

d) $\mathrm{E}(W_q) = Q(s)$ holds due to the PASTA property. As for the mean waiting time,

$$\mathrm{E}(W_q) = \mathrm{E}(W_q|W_q = 0)\mathrm{P}(W_q = 0) + \mathrm{E}(W_q|W_q > 0)\mathrm{P}(W_q > 0) = \mathrm{E}(W_q|W_q > 0)\mathrm{P}(W_q > 0).$$

The second term here is $Q(s)$ as defined in the previous item. As for the first item, note that it equals the mean waiting time (service inclusive) of a customer in an $M/M/1$ queue with λ as the arrival rate and $s\mu$ and the service rate. In other words, $\mathrm{E}(W_q|W_q > 0) = 1/(s\mu - \lambda)$ (see Equation (4.9)).

8.13 Question 13

a) In this system an idleness period follows a busy period, which follows an idleness period, etc. The mean of the idleness period equals $1/(\lambda_1 + \lambda_2)$ as this period is the minimum between exponentially distributed random variables with parameters λ_1 and λ_2. This is the numerator. The mean busy period is b_1 (respectively, b_2) if they arrival who opens the busy period is of type 1 (respectively, type 2), a probability $\lambda_1/(\lambda_1+\lambda_2)$ (respectively, $\lambda_2/(\lambda_1+\lambda_2)$) event. Hence, the denominator here is the mean cycle time. The ratio between the two is the probability of idleness which, by definition, equals π_{00}.

b) The value of b_1 is $1/(\mu - \lambda_1)$ and likewise $b_2 = 1/(\mu - \lambda_2)$. Hence, π_{00} as stated in the question equals

$$\frac{1}{1 + \frac{\lambda_1}{\mu-\lambda_1} + \frac{\lambda_2}{\mu-\lambda_2}}.$$

Minimal algebra leads to Equation (8.18).

8.14 Question 14

a) Work conservation is defined on page 53 (see Definition 4.1). In this model of retrials it is possible that jobs will be in queue (or orbit) while the server is idle. Had that been a FCFS model, the server who serve the customer(s) at this instant. Hence this is not a work-conserving. In particular, the amount of work kept in the system is larger than or equal to a similar one but under the FCFS regime.

b) We repeat below what appears in Hassin and Haviv book, pp.131-132. First, notice that using the notation of the exercise, what we are looking for is

$$L_q = \sum_{j=1}^{\infty} j\pi_{1j} + \sum_{j=1}^{\infty} j\pi_{0j}.$$

Consider the unnumbered formula after (8.21) and denote it by $G_s(x)$ as indeed it is the function of s and x. Take derivative with respect to x, in both sides and let $x = \rho$, you get,

$$\sum_{j=1}^{\infty} j\rho^{j-1}\Pi_{i=1}^{j-1}(1+\frac{s}{i}) = (1+\frac{\lambda}{\eta})(1-\rho)^{-2-\frac{\lambda}{\eta}}. \tag{33}$$

Then, by (8.21),

$$\sum_{j=1}^{\infty} j\pi_{1j} = \sum_{j=1}^{\infty} j\rho^{j}\Pi_{i=1}^{j}(1+\frac{\lambda}{\eta i})\pi_{i0}.$$

Using the value of π_{i0} given at the bottom of page 127, coupled with (33), leads to

$$\sum_{j=1}^{\infty} j\pi_{1j} = \frac{\rho^2}{1-\rho}(1+\frac{\lambda}{\eta}). \tag{34}$$

As for the second summation, using (8.20) and (8.21)

$$\pi_{0j} = \frac{\mu}{\lambda+\eta j}\pi_{1j} = \frac{\mu}{\lambda+\eta j}\rho^{j}\Pi_{i=1}^{j}(1+\frac{\lambda}{\eta i})\pi_{i0} == \frac{\mu}{\lambda+\eta j}\rho^{j}\Pi_{i=1}^{j}(1+\frac{\lambda}{\eta i})\rho(1-\rho)^{1+\frac{\lambda}{\eta}}.$$

Multiplying by j and summing up from $j = 1$ to infinity, we get by the use of (33), that

$$\sum_{j=1}^{\infty} j\pi_{0j} = \frac{\lambda}{\eta}\rho. \tag{35}$$

All is left is to sum (34) and (35) and some minimal algebra.

c) First note that the probability that the server is busy is ρ. This can be argues by Little's law: the arrival rate to the server (and we count only those who commence service is λ since eventually all enter service (and only once). The time there is with a mean $1/\mu$ so we end up with λ/μ as the mean number of customers there. Second, we use Little's law again one when the system is defined as the orbit. What is the expect number in this system is what was just derived in the previous item. The arrival rate to this system is $\lambda\rho$ and ρ since, as was just argues, ρ is the probability of finding the server busy. Note that we count an arrival to the orbit as one (and we in fact do not consider an unsuccessful retrial as a fresh arrival to the system and look at it as 'nothing happened'). Dividing one by the other, leads to the mean time in orbit, for one who enters the orbit, which coincides with what is asked in this item.

8.15 Question 15

All one needs to do it to replace all transition probabilities in Theorem 3.8 with transition rates.

8.16 Question 16

We start with showing B_n, $n \geq 1$, obeys the recursion stated there for $n \geq 2$. Suppose a customer opens a busy period in an $M/M/1/n$ queue. The next event takes place within $1/(\lambda+\mu)$ units of time. With probability $\mu/(\lambda+\mu)$ this is due to a service completion and the busy period ends. Otherwise, and this is with probability $\lambda/(\lambda+\mu)$ a new customer arrives. The question we are after is when first the server will be idle again. In order to be ideal again, it needs first to drop the number of customer there from two to one. The reader can be convinced that this is exactly as a busy period in an M/M/1/(n-1) queue. Once this occurs, it will take in fact another busy period until the system is empty for the fist time. It is clear that $B_1 = 1/\mu$, so the recursion leads to a way for computing B_n, $n \geq 2$, one by one. This implies that it is solved by a unique solution which is the values of the mean busy periods. What is finally needs to be shown is that $\frac{1}{\mu}\Sigma_{j=0}^{n-1}\rho^j$ is s solution. Minimal algebra leads to

$$(\lambda+\mu)B_n = 1 + \lambda(B_{n-1} + B_n)$$

or

$$B_n = \frac{1}{\mu} + \rho B_{n-1}.$$

Assume the induction hypothesis that $B_n = \frac{1}{\mu}\Sigma_{j=0}^{n-2}\rho^j$ and conclude.

8.17 Question 17

Theorem 8.2 says in fact that e_i is proportional to $\pi_i q_i$, $i \in N$. In other words, up to some multiplicative constant, e_i equals $\pi_i q_i$. Note that the vector e_i, $i \in N$, obeys $e_j = \Sigma_i e_i P_{ij} = \Sigma_i e_i q_{ij}/q_i$. We need to show that $\pi_i q_i$ is such a solution. We plug it in the two sides of this equation and check for equality. Indeed, $\pi_j q_j$ is what we get in the lefthand side. On the righthand side we get $\Sigma_i \pi_i q_i q_{ij}/q_i = \Sigma_i \pi_i q_{ij}$. The two are equal for all j, as the vector π_i, $i \in N$, solves the balance equations.

8.18 Question 18

During a time of length t, the number of exits from state to state follows a Poisson distribution with parameter ct. Hence, the number of such hops is n with probability $\eta^{-ct}(ct)^n/n!$. The dynamics among states in with the transition matrix P. Hence, given n hops, the probability of being in state j, given i being the initial state, is P^n_{ij}. The rest is the use of the complete probability theorem.

8.19 Question 19

a) The answer is Poisson with parameter λW. Firth, note that this is easily the answer had the question been referring to the time instant of departure. Yet, by considered the time reversed process (which has the same dynamics), the instants of arrivals and departure swap and hence have the same statistics.

b) Denote by S the service time. First note that given $W = w$, S gets only values between 0 and w. Moreover, there is an atom at w due to the fact that 'no waiting' is a non-zero probability event. Indeed, we start with the atom. Since regardless of service time, it coincides with the waiting time with due to finding an empty system upon arrival, a probability $1 - \rho$ event,

$$\mathrm{P}(S = w|W = w) = \frac{f_S(s)\mathrm{P}(W = w|S = w)}{f_W(w)} = \frac{\mu e^{-\mu w}(1-\rho)}{(\mu - \lambda)e^{-(\mu-\lambda)w}},$$

which by minimal algebra is seen to equals $e^{-\lambda w}$. Looking from the time reserved angle, the event that $W = S$ is equivalent to no arrivals during a waiting time, in this case, time of length w. This probability equals $e^{-\lambda w}$.

Next for the case $f_{S|W=w}(s)$ for $0 \le s \le w$:

$$f_{S|W=w} = \frac{f_S(s) f_{W|S=s}(w)}{f_W(w)} = \frac{\mu e^{-\mu s} f_{W_q}(w-s)}{(\mu - \lambda)e^{-(\mu-\lambda)w}},$$

where W_q is the random variable of the time in queue (exclusive of service. This equals,

$$\frac{\mu e^{-\mu s}\rho(\mu-\lambda)e^{(\mu-\lambda)(w-s)}}{(\mu-\lambda)e^{-(\mu-\lambda)w}} = \lambda e^{-\lambda s}.$$

Looking for a time-reservable argument, time of departure corresponds to time of arrival and the event $S = s$, means that at this instant, which is the last departure in the last w units of time, is in fact the first arrival in the reversed process. This is with density $\lambda e^{-\lambda s}$.

9 Chapter 9

9.1 Question 1

a)

$$\sum_j P^*_{ij} = \sum_j \frac{\gamma_j}{\gamma_i} P_{ji} = \frac{1}{\gamma_i} \sum_j \gamma_j P_{ji},$$

which by (9.1) equals

$$\frac{1}{\gamma_i}(\gamma_i - \lambda_i) = 1 - \frac{\lambda_i}{\gamma_i} < 1.$$

b) By what was just shown

$$\lambda_i + \gamma_i \sum_j P^* ij = \lambda_i + \gamma_i (1 - \frac{\lambda_i}{\gamma_i}) = \gamma_i,$$

as required.

c) Consider the departure rate from station i in the time-reversed process. The external rate equals λ_i as this is the external arrival rate in the original process. The internal departure rate in the time-reversed process equals $\gamma_i \Sigma_j P^*_{ij}$. Sum the two an get the departure rate. This should of course equal the throughput of this station which is γ_i.

d) The lefthand side equals

$$\lambda_i \pi(\underline{n}) \frac{1}{\rho_i} + \sum_{j \neq i} \pi(\underline{n}) \frac{\rho_j}{\rho_i} \mu_j P_{ji} = \pi(\underline{n})[\lambda_i \frac{\mu_i}{\gamma_i} + \sum_{j \neq i} \frac{\gamma_j}{\gamma_i} \mu_i P_{ji}]$$

$$\pi(\underline{n}) \frac{\mu_i}{\gamma_i} [\lambda_i + \sum_{j \neq i} \gamma_j P_{ji}] = \pi(\underline{n}) \mu_i \frac{\gamma_i}{\gamma_i} = \pi(\underline{n}) \mu_i,$$

where the one before last equality is due to (9.1).

9.2 Question 2

(Liron)

a) The Markov process of am open network of exponential queues is time reversible if and only if $\gamma_i P_{ij} = \gamma_j P_{ji}$ for every pair $1 \leq i, j \leq M$.
Proof: By Theorem (8.4), a Markov process is time reversible if and only if for every pair of states $i \in \mathcal{N},\ j \in \mathcal{N}$: $g^*_{ij} = q_{ij}$, where $g^*_{ij} =$

$\frac{\pi_j}{\pi_i} q_{ji}$.
Recall that the state space is the vectors $\underline{n} \in \mathbb{N}^M$. We apply Theorem (9.1) in order to compute q^* for the different possible states:

$$\begin{aligned} q^*(\underline{n}, \underline{n}+e_i) &= \frac{\pi(\underline{n}+e_i)}{\pi(\underline{n})} q(\underline{n}+e_i, \underline{n}) = \rho_i \mu_i P_i^{'} \ , \ 1 \le i \le M, \\ q^*(\underline{n}, \underline{n}-e_i) &= \frac{\pi(\underline{n}-e_i)}{\pi(\underline{n})} q(\underline{n}-e_i, \underline{n}) = \frac{\lambda_i}{\rho_i} \ , \ 1 \le i \le M \ , \ n_i \ge 1, \\ q^*(\underline{n}, \underline{n}-e_i+e_j) &= \frac{\pi(\underline{n}-e_i+e_j)}{\pi(\underline{n})} q(\underline{n}-e_i+e_j, \underline{n}) = \frac{\rho_j}{\rho_i} \mu_i P_{ji} \ , \ 1 \le i,j \le M \ , \ n_i \ge 1. \end{aligned}$$

Applying simple algebra and plugging in the rates into the time-reversibility condition ($q^* = q$) we obtain the following conditions:

$$\gamma_i \left(1 - \sum_{j=1}^{M} P_{ij} \right) = \lambda_i \ , \ 1 \le i \le M \tag{36}$$

$$\gamma_i P_{ij} = \gamma_j P_{ji} \ , \ 1 \le i,j \le M \tag{37}$$

If we plug (37) into (36) we get:

$$\gamma_i = \lambda_i + \sum_{j=1}^{M} \gamma_j P_{ji},$$

which is met by definition in this model: $\gamma = \lambda(I-P)^{-1}$. We can therefore conclude that the second condition is a necessary and sufficient condition for time reversibility. ■

b) If the process is time-reversible we show that the external arrival rate equals the external departure rate.
The external arrival rate to server $1 \le i \le M$ is $\sum_{j=1}^{M} \gamma_j P_{ji}$. From the condition of (a): $\gamma_j P_{ji} = \gamma_i P_{ij} \ \forall 1 \le i,j \le M$ we get that the arrival rate equals $\sum_{j=1}^{M} \gamma_i P_{ij}$, which is exactly the external departure rate.

9.3 Question 3

a) CANCELED

b) (i) Establishing the fact that the waiting times in the first two queues are independent is easy: No queueing time prior to the second service and this service time is drawn independently of the waiting time in queue 1. By looking at the time-reversed process and using the time-reversibility of this system, we get, by the same reasoning used to show the independence between waiting times at the first two queues, that the waiting time in the second queue (i.e., the corresponding service time) and the waiting time at the third queue are independent.

(ii) Let us look first at the unconditional probability. The probability that the one in the back of a customer overtake hims, is 1/4. The reason behind that is the one in the back should first complete service before this customer finishes his second service. This is a probability 1/2 event. Due to the memoryless property, that he will also finishes service at the second server first is also 1/2, leading to 1/4. However, if the customer under consideration is served for a very long time, he is very likely to be overtaken. In fact, this probability can become large as one wants, by taking this service time to be long enough. In fact, for any such service time x, we look for $\mathrm{P}(X \leq x)$ where X follows an Erlang distribution with parameter 2 (for two stages) and 1 (the rate of service at each stage). Clearly, $\lim_{x\to\infty} \mathrm{P}(X \leq x) = 1$.

9.4 Question 4

a) The departure rate from station i equals $\gamma_i P'_i$, $1 \leq i \leq M$. Hence, in case of a departure it is from station i with probability

$$\frac{\gamma_i P'_i}{\sum_{j=1}^{M} \gamma_i P'_i}, \quad 1 \leq i \leq M.$$

b) Denote by W_i^d the mean time spent in the system by a customer who depart from station i, $1 \leq i \leq M$. Then, in a similar way to (9.6) but for the time-reversed process, we get that

$$W_i^d = \frac{1}{\mu(1-\rho_i)} + \sum_{j=1}^{M} P^*_{ij} W_j^d, \quad 1 \leq i \leq M.$$

Then, as in (9.7),

$$W_i^d = \sum_{j=1}^{M} (I - P^*)^{-1} \frac{1}{\mu_j(1-\rho_j)}, \quad 1 \leq i \leq M.$$

Using the previous item, we conclude that the mean time in the system of one who just departed is

$$W^d \equiv \sum_{i=1}^{M} \frac{\gamma_i P_i'}{\sum_{j=1}^{M} \gamma_j P_j'} W_i^d.$$

c) Of course, $W = W^d$. Hence, this is an alternative way for computing the mean time in the system for a random customer.

10 Chapter 10

10.1 Question 1

Although in the model dealt with in this chapter, there are M state variables, we in fact have only $M-1$. This is the case since one degree of freedom is lost due to the fact that the sum across the values of the state variables, is fixed to the value of N. In particular, in the case where $M=2$ we in fact have only one state variable that of how many are in station 1. The number there is a random random between 0 and N. The birth rate is m_2 as long as the number there is below N: This is due to a service completion as the other server (recall that we assume $P_{ii}=0$ and hence customers hop from one server to the other). The death rate is μ_1 from the same reason and this is the case as long as this station is not empty. Since the $M/M/1N$ is also a birth and death process with the same state space, our proof is completed.

10.2 Question 2

Since $e(\underline{n}-e_i+e_j)=e(\underline{n})\rho_j/\rho_i$ for any pair of i and j, $1\leq i,j\leq M$ (as long as $n_i\geq 1$), we get that the lefthand side equals

$$e(\underline{n})\frac{1}{\rho_i}\sum_{j\neq i}\rho_j\mu_jP_{ji}=e(\underline{n})\frac{1}{\rho_i}\sum_{j\neq i}\gamma_jP_{ji}$$

which equals, by the definition of the vector γ and the assumption that $P_{ii}=0$, to

$$e(\underline{n})\frac{1}{\rho_i}\gamma_i=e(\underline{n})\mu_i,$$

as required.

10.3 Question 3

Suppose μ_i is changed to $\mu_i t$, $1\leq i\leq M$. Then, $G(N)$ as defined in (10.4) is multiplied by $1/t^N$. Of course, $G(N-1)$ is multiplied by $1/t^{N-1}$. Inserting this in (10.11), leads to the fact that $C(N)$ is multiplied by t. The same is the case with the throughputs $X_i(N)$, $1\leq i\leq N$ (see (10.10)). Indeed if μ_i measures the number of service completions per minute, $60\mu_i$ is the number of service completions per hours. Of course, the throughputs measured in hours are 60 times larger than the throughput measured per minutes.

10.4 Question 4

(It is recommended to read first the first remark in page 154.) Note that all parameters of the model depend on the matrix P only throught the vector of γ. In particular, two transition matrices having the same γ, end up with the same values computed throughout this chapter. Next, for a transition matrix specified in the question, the corresponding g has uniform entries, which can be assumed to equal one. Replacing μ_i by γ_i/m_i (both g_i and μ_i for some given model), led to a new ρ_i which equals $1/(\mu_i/g_i) = \gamma_i/m_i$ which coincides with the original ρ_i, $1 \le i \le M$. Since $G(N)$ (see (10.4)), $e(\underline{n})$ (see 10.4), and then the throughputs as defined in (10.10) are only function of (the unchanged) ρ_i, $1 \le i \le M$, the proof is completed.

10.5 Question 5

The first thing to do is to compute $\Sigma_{i=1}^{M}\rho_i^n$, for $n = 0, \ldots, N$. This is an $O(NM)$ task. Next initiate with $G(0) = 1$ which is fixed and with $G(n) = 0$ for $1, \ldots, N$ which are only temprory values. Then,

$$\text{For } n = 1, \ldots, N$$

$$\text{For } k = 1, \ldots, n$$

$$G(n) \leftarrow G(n) + G(n-k)\sum_{j=1}^{M}\rho_j^k$$

$$G(n) \rightarrow \frac{G(n)}{n}$$

Since we have a double look each of which with N steps, the complexity of this part is $O(N^2)$. Thus, the complexity of the algorith is $O(\max\{NM, N^2\})$ which is inferior to the $O(MN)$ complexity of both the convolution algorithm and the MVA algorithm which were described in Sections 10.2 and 10.5.1, respectively.

10.6 Question 6

The Markov process underlying the model is time-reversible if and only if

$$e(\underline{n}\mu_i P_{ij} = e(\underline{n} + e_j - e_i)\mu_j P_{ji}$$

for the case where $n_i \ge 1$. Since $e(\underline{n} + e_j - e_i) = e(\underline{n})\rho_j/\rho_i$, this condition is met if and only if

$$\mu_i P_{ij} = \frac{\rho_j}{\rho_i}\mu_j P_{ji}$$

or equivalently, after some algebra, if and only if

$$\gamma_i P_{ij} \gamma_j P_{ji}, \quad 1 \le i, j \le M.$$

The the case where $P_{ij} = 1/(M-1)$, $1 \le i \ne j \le M$, all enties in the vector g are identical which immediately led to this conclusion.

11 Chapter 11

11.1 Question 1

a) For the M/G/1/1 case:

(i) This is not a renewal process: from past loss instants one can learn differently (and statistically) the age of service upon the loss, which in turn effects the time of the next loss.

(ii) This is a renewal process. The time between two renewal follows the independent sum of service time (distributed G) and an arrival time (exponentially distributed). Note the need for the Poisson arrival rate here: As soon as service is completed, the next service commencement is within an exponential service time, regardless the service length.

(iii) This is also a renewal processes. In fact, the underlying distribution is as for the enter to service process.

(iv) This is a renewal process. In fact, this is also a Poisson process with a rate which equals the arrival rate. The reason behind that is this is a symmetric queue (see the bottom of page 166) for which Theorem 11.3 holds. Note the departure process here is the super-positioned process for loss arrival and service completions (and not only the latter process).

b) For the M/M/1/1 case:

(i) Now the loss process is a renewal process as the age of service upon the current loss is irrelevant. As for the underlying distribution, we claim that its LST equals

$$\frac{\lambda(\lambda+s)}{s^2+(2\lambda+\mu)s+\lambda^2}.$$

This is proved as follows: Denote by $F_X^*(s)$ the LST we are after. The consider the first event after a loss, which can be either arrival or departure. It has an exponential distribution with parameter $\lambda+\mu$. Hence, its LST equals $(\lambda+\mu)/(\lambda+\mu+s)$. Independently, the next loss will take no time (with probability of $\lambda/(\lambda+\mu)$ which corresponds to arrival taken place first) or the next loss will require the sum of two independent random variables: an arrival plus an independent replica of X (with the complementary probability of $\mu/(\lambda+\mu)$). In summary,

$$F_X^*(s) = \frac{\lambda+\mu}{\lambda+\mu+s}\left(\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}\frac{\lambda}{\lambda+s}F_X^*(s)\right).$$

The rest follows some minimal algebra.

(ii) The entrance to service is a renewal process as in any M/G/1/1 above. Here we can be specific and say that the underlying distribution is the sum of two independent exponentially distributed random variables, one with λ, the other with μ (the arrival rate and service rate respectively).

(iii) Looking at the general case of M/G/1/1, we again reach the same conclusion as in the previous item (that of the loss processes).

(iv) This is just a special case of an M/G/1/1 model dealt above. In particular, the answer is that this is a Poisson process with rate λ.

11.2 Question 2

For the M/G/s/s model:

a) The loss process is not a renewal process as it is not in the M/G/1/1 case. Yet, it is for the M/M/2/2 case.

b) The entrance to service is not a renewal process and this is also the case in the M/M/2/2 model.

c) The previous conclusion holds here too.

d) The situation here is as in the M/G/1/1 case due to Theorem 11.3. See Question 1 above. In particular, this is a Poisson process.

11.3 Question 3

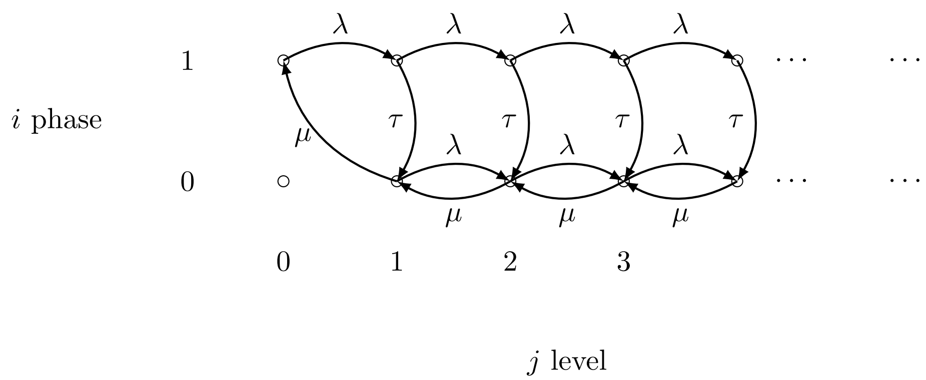
What Theorem 11.5 basically says that as long as a new arrival commences service as soon as it arrives (and with the full dedication of the server) in order to get a product from for the limit probabilities (and other phenomena associated with it as insensitivity, Poisson departure process, etc.) the order of return to service of those previously preempted is not important, as long as it is done without anticipation, namely without reference to past or future service times of those on line from which one needs to be selected to resume service. Thus, the variation suggested here does not change anything. Now more formally, in the presentation of the model, the fourth bullet from the top of page 176, needs to be that the transition rate from $(d_1, d_2, \ldots, d_n)$ to $(d_{\pi_{n-1}^{-1}(1)+1}, d_{\pi_{n-1}^{-1}(2)+1}, \ldots, d_{\pi_{n-1}^{-1}(n-1)+1})$ equals $\mu \frac{q_{d_1+1}}{q_{d_1}} s_{\pi(n-1)}$. Next, when we update the proof, we need to consider only Case 2 as the rest stays unchanged. Specifically, $x = (d_1, d_2, \ldots, d_n)$ and $y = (d_{\pi_{n-1}^{-1}(1)+1}, d_{\pi_{n-1}^{-1}(2)+1}, \ldots, d_{\pi_{n-1}^{-1}(n-1)+1})$ and the transition rate from x to y equals $\mu \frac{q_{d_1+1}}{q_{d_1}} s_{\pi(n-1)}$. The transition rate from y to x in the time-reversed process equals $\mu s_{\pi(n-1)}$. The rest of the proof is the same: An extra term which equals $s_{\pi(n-1)}$ appears now in both (11.17) and (11.18).

12 Chapter 12

12.1 Question 1

(Benny)

a) (i)



(ii)

$$Q_0(st) = \begin{cases} \lambda & s = t \\ 0 & \text{otherwise} \end{cases}$$

$$Q_2(st) = \begin{cases} \mu & s = t = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$Q_1(st) = \begin{cases} \tau & s = 1, t = 0 \\ 0 & s = 0, t = 1 \\ -(\lambda + \mu) & s = t = 0 \\ -(\lambda + \tau) & s = t = 1 \end{cases}$$

(iii)

$$Q_0 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$Q_1 = \begin{pmatrix} -(\lambda + \mu) & 0 \\ \tau & -(\lambda + \tau) \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix}$$

(iv) For the case where $j \geq 1$, the balance equations, in matrix notation are

$$\underline{\pi}_j Q_0 + \underline{\pi}_{j+1} Q_1 + \underline{\pi}_{j+2} Q_2 = \underline{0}$$

or, in detail,

$$(\lambda + \mu)\pi_{0,j+1} = \lambda \pi_{0,j} + \tau \pi_{1,j+1} + \mu \pi_{0,j+2}$$

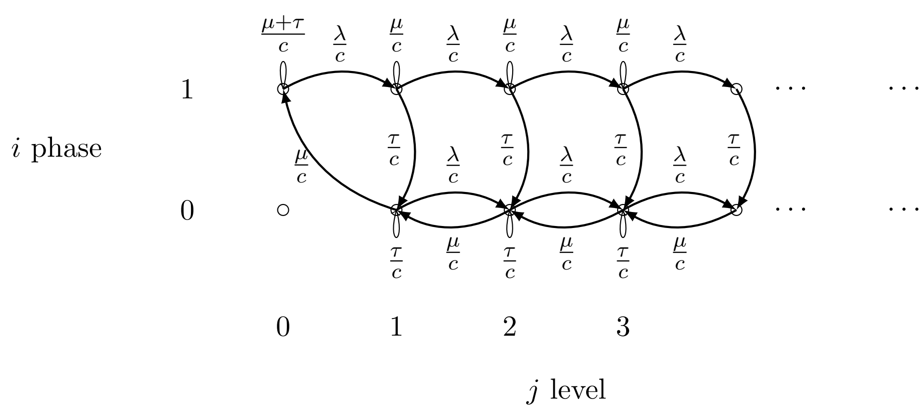
and

$$(\lambda + \tau)\pi_{1,j+1} = \lambda \pi_{1,j}$$

b) We use a probabilistic argument (as stated at the end of Theorem 12.1) to show that the rate matrix equals

$$\begin{pmatrix} \frac{\lambda}{\mu} & 0 \\ \frac{\lambda}{\mu} & \frac{\lambda}{\lambda+\tau} \end{pmatrix}.$$

First, we look at the corresponding discrete-time process which we get by uniformization (we take the value of c to equal $\lambda + \mu + \tau$).



The following observation will be needed in the sequel. Once the process is in state $(0, j+1)$, unless the level goes down, a probability $\frac{\mu}{c}$ event, an additional visit in state $(0, j+1)$ is guaranteed. Thus, the number of visits to state $(0, j+1)$ prior to reaching level j follows a geometric distribution with mean $\frac{c}{\mu}$.

For process that commences at $(0, j)$, in order to reach state $(0, j+1)$ prior to returning to level j, it must move immediately to state $(0, j+1)$, which is a probability $\frac{\lambda}{c}$ event. By the argument above, the number of future visits there (inclusive of the first one), is with a mean of $\frac{c}{\mu}$. Thus, R_{00} equals λ/μ. For process that commences at $(1, j)$, in order to reach state $(0, j+1)$ prior to returning to level j, it must move immediately to state $(1, j+1)$, which is a probability $\frac{\lambda}{c}$ event. Once in state $(1, j+1)$, it will reach state $(0, j+1)$ for sure, sooner or later. By the argument above, the number of future visits there (inclusive of the first one), is with a mean of $\frac{c}{\mu}$. Thus, R_{10} equals $\frac{\lambda}{\mu}$. On the other hand, once in state $(1, j+1)$, the process will revisit there with probability $\frac{\mu}{c}$ or leave this state with no possibility to revisit prior to returning to level j implying that the number of future visits there (inclusive of the first one), follows a geometric distribution with mean $\frac{1}{1-\frac{\mu}{c}}$. Thus, R_{11} equals $\frac{\lambda}{c}\frac{1}{1-\frac{\mu}{c}} = \frac{\lambda}{\lambda+\theta}$.

Finally, process that commences at $(0, j)$ will never reach state $(1, j+1)$ prior to returning to level j and hence $R_{01} = 0$.

An alternative way is to verify that R is a solution to (12.23)

$$R^2Q_2 + RQ_1 + Q_0 =$$
$$\begin{pmatrix} (\frac{\lambda}{\mu})^2 & 0 \\ (\frac{\lambda}{\mu})^2 + \frac{\lambda^2}{(\lambda+\tau)\mu} & (\frac{\lambda}{\lambda+\tau})^2 \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix} +$$
$$\begin{pmatrix} \frac{\lambda}{\mu} & 0 \\ \frac{\lambda}{\mu} & \frac{\lambda}{\lambda+\tau} \end{pmatrix} \begin{pmatrix} -(\lambda+\mu) & 0 \\ \tau & -(\lambda+\tau) \end{pmatrix} + \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} =$$
$$\begin{pmatrix} \frac{\lambda^2}{\mu} & 0 \\ \frac{\lambda^2}{\mu} + \frac{\lambda^2}{\lambda+\tau} & 0 \end{pmatrix} + \begin{pmatrix} -\frac{\lambda(\lambda+\mu)}{\mu} & 0 \\ -\frac{\lambda(\lambda+\mu)}{\mu} + \frac{\lambda\tau}{\lambda+\tau} & -\lambda \end{pmatrix} + \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = 0$$

c) The characteristic polynomial of R is

$$(\frac{\lambda}{\mu} - w)(\frac{\lambda}{\lambda+\tau} - w)$$

and the two solutions, i.e., the two eigen values are $w_1 = \frac{\lambda}{\mu}$ and $w_2 = \frac{\lambda}{\lambda+\tau}$.

(i) A necessary and sufficient condition for stability is $|w_i| < 1$, $1 \leq i \leq 2$ (see 12.5). Since $0 < w_2 < 1$, it is left to guarantee that $w_1 < 1$, i.e., $\lambda < \mu$.

(ii) From

$$\underline{\pi}_{j+1} = \underline{\pi}_j R = (\frac{\lambda}{\mu}(\pi_{0j} + \pi_{1j}), \frac{\lambda}{\lambda+\tau}\pi_{1j})$$

it is possible to see that the probability of phase 1 decay with geometric factor of $\frac{\lambda}{\lambda+\tau}$. If τ is small enough, more specifically $\tau < \lambda - \mu$, then $w_2 > w_1$ and w_2 is the largest eigenvalue and its corresponding left eigenvector v_2, that by Perron-Frobinius theorem is known to be positive, real, and unique, satifies

$$(v_2)_i = \lim_{j\to\infty} \frac{\pi_{ij}}{\pi_{0j} + \pi_{1j}} \ .$$

In that case, it is easy to see that $v_2 = (0, 1)$ and hence, for large j, $\pi_{.j} \approx \pi_{1j}$, that, as said, decay with geometric factor of $\frac{\lambda}{\lambda+\tau}$.

12.2 Question 2

CANCELED

12.3 Question 3

(Benny)

a)

$$Q_0(st) = \begin{cases} \lambda & s = r,\ t = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$Q_2(st) = \begin{cases} \mu & 1 \le s = t \le r \\ 0 & \text{otherwise} \end{cases}$$

and

$$Q_1(st) = \begin{cases} -(\lambda+\mu) & s = t \\ \lambda & 0 \le s \le r-1,\ t = s+1 \\ 0 & \text{otherwise} \end{cases}$$

Or, equivalently,

$$Q_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \lambda & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mu & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mu \end{pmatrix} = \mu I$$

and

$$Q_1 = \begin{pmatrix} -(\lambda+\mu) & \lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & -(\lambda+\mu) & \lambda & \cdots & 0 & 0 & 0 \\ 0 & 0 & -(\lambda+\mu) & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -(\lambda+\mu) & \lambda \\ 0 & 0 & 0 & 0 & \cdots & 0 & -(\lambda+\mu) \end{pmatrix}$$

b) The process can be looked as the phases are ordered in a cycle where the move to the next phase occurs upon any completion of stage in the arrival process. These transitions are not affected by changes in the level. The phase process is with generator matrix

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\lambda & \lambda & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\lambda & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\lambda & \lambda \\ \lambda & 0 & 0 & 0 & \cdots & 0 & -\lambda \end{pmatrix}$$

and clearly, $\underline{\pi} = (1/r, \ldots, 1/r)$ solve the balance equation $\underline{0} = \underline{\pi}Q$ and hence the limit distribution is uniform in the integers 1 through r.

c) (i) When one approximates the solution of (12.23) via the entry-wise monotonic matrix sequence $\{X(k)|k \geq 0\}$ defined through the recursion stated in (12.24) while initializing with $X(0) = 0$, one gets that $X(1) = A_0$. Due to the shape of Q_0, all the entries of $X(2)$ but those in the last row equal zero too. Moreover, as the iterative procedure continues, the same is the case with all matrices $X(k)$, $k \geq 0$. As $\{X(k)\}_{k=0}^{\infty}$ converges, as k goes to infinity, to a solution of (12.23), R itself possesses the same shape. In summary, $R_{ij} = 0$ for $1 \leq i \leq r-1$ and $1 \leq j \leq r$.

Thus, for some row vector $\underline{w} \in \mathbf{R}^r$

$$R = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ w_1 & \cdots & w_r \end{pmatrix}$$

(ii) Due to the structure of R, we get that

$$R^j = w_r^{j-1} R, \quad j \geq 1$$

and as $\underline{\pi}_j = \underline{\pi}_0 R^j$, $j \geq 0$, we get that

$$\underline{\pi}_j = \pi_{r0} w_r^{j-1} \underline{w}, \quad j \geq 1.$$

In other words,

$$\pi_{ij} = \pi_{r0} w_i w_r^{j-1}, \;\; 0 \leq i \leq n, \; j \geq 1. \tag{38}$$

Denote the number of customers in the system by L and the number of completed stages in the arrival process by S.

$$P(S = i | L = j) = \frac{\pi_{ij}}{\sum_{k=1}^{r} \pi_{ij}} = \frac{w_i}{\sum_{k=1}^{r} w_k}, \quad 1 \leq i \leq r, \quad j \geq 1$$

and as long as $j \geq 1$, this probability is not a function of j. Hence, given that $L \geq 1$, i.e., the server is busy, S and L are independent.

(iii) Using (38) and the fact that the limit distribution of the phase is uniform we get

$$\frac{1}{r} = \pi_{r.} = \pi_{r0} + \pi_{r0} \sum_{j=1}^{\infty} w_r w_r^{j-1} = \frac{\pi_{r0}}{1 - w_r}$$

implying $\pi_{r0} = \frac{1-w_r}{r}$
and

$$\frac{1}{r} = \pi_{i.} = \pi_{i0} + \pi_{r0} \sum_{j=1}^{\infty} w_i w_r^{j-1} = \pi_{i0} + \frac{1 - w_r}{r} \frac{w_i}{1 - w_r}\ , \quad 1 \leq i \leq r$$

implying $\pi_{i0} = \frac{1-w_i}{r}$.
Finally, from (38) we get

$$\pi_{ij} = \frac{1 - w_r}{r} w_i w_r^{j-1}, \;\; 0 \leq i \leq n, \; j \geq 1.$$