Combinatorial Agency

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August 24, 2006

Abstract

We study a combinatorial variant of the classical principal-agent model. In our setting a principal wishes to motivate a team of strategic agents to exert costly effort on his behalf, but their actions are hidden from him. Our focus is on cases where *complex combinations* of the efforts of the agents influence the outcome stochastically. The principal motivates the agents by offering to them a set of contracts, which together put the agents in an equilibrium point of the induced game. The main difficulty is that of determining the required Nash equilibrium point.

We study what sets of contracts can be obtained as optimal contracts for various production technologies, and quantify the gap between the first-best and second-best solutions. Our results highlight the qualitative and quantitative differences between production technologies that exhibit complementarities and substitutabilities between the agents' actions. We also analyze the algorithmic complexity of determining the optimal contract.

Keywords

Agency Theory, Principal-Agent Model, Incentives, Contracts, Hidden-Action, Mechanism Design, Moral Hazard

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1 Introduction

This paper considers the issue of hidden actions in multi-agent settings that exhibit complex dependencies between agents' actions. Before going into the specifics of the model itself, it would be beneficial to look at the wider context: complexity in economic settings.

With the increasing influence of the Internet, many types of economic activities are now handled with the aid of sophisticated computer systems. In such environments complex scenarios involving multiple agents and goods naturally occur and are handled by appropriately programmed "hyper-rational" software. This calls for the study of the standard issues in economic theory in new complex settings. Moreover, as the different components of the Internet are operated by autonomous companies and individuals with diverse economic interests, the administration of the Internet infrastructure itself requires economic analysis in complex settings. Examples of complex economic settings that have received much recent attention are "algorithmic mechanism design" (Nisan and Ronen 2001) and "combinatorial auctions" (see the extensive treatment in (Cramton, Shoham, and Steinberg 2006)). This paper continues this line of research by exploring the problem of *combinatorial agency*, which deals with hidden actions in complex multi-agent settings.

1.1 Problem Statement

We study settings where a team of agents works towards the welfare of a principal. The problem of incentives in production teams (as opposed to the classical principal-agent problem with a single agent) has been first introduced by (Holmstrom 1982). However, Holmstrom's model and results, as well as subsequent work on team incentives (Strausz 1996, Rasmusen 1987, Legeros and Matthews 1993), deal with a different set of research questions than the ones we consider here. With few exceptions (e.g., (Legeros and Matthews 1993)), the previous work does not address the delicate combinatorial structure of dependencies between the agents' actions (e.g., strategic substitutes or complements), which is the focus of our study.

Our model is an extension of the model studied by (Winter 2004), in which a principal wishes to induce a team of agents to exert costly effort toward his welfare, where the agents' actions are hidden from him, and he can only observe the final outcome. The outcome may be determined in a complex stochastic way by the set of actions taken by the agents. The agents' actions can exhibit, for example, complementarities, substitutability, or any combination of the two. These delicate dependencies define the *production technology function*. In the general case, each combination of efforts exerted by the n different agents may result in a different expected gain for the principal.

The general question asks which contracts (conditional payments) should the principal offer to the agents as to maximize his net utility? In contrast to (Winter 2004), who studies the optimal contract that induces *all* agents to exert effort in equilibrium, the main challenge in our setting is to determine the optimal amount of effort desired from each agent (henceforth, the optimal contract), given the value that the principal obtains from a success of the project. In addition to characterizing the second best solutions for various technologies, we quantify the loss that is incurred by the principal and the society due to the principal's inability to observe individual actions, and study whether it can be bounded.

While the hidden action problem is quite general, and exists at every level of management of any firm or organization, we find that computational settings serve us with some especially natural applications for our analysis. A candidate target application for our analysis, discussed in (Feldman, Chuang, Stoica, and Shenker 2005), is routing in complex computer networks [1]. Suppose that every intermediate link or router may exert a different amount of "effort" (e.g., allocation of bandwidth, memory, storage or CPU's processing power) when attempting to forward a packet of information within a communication network. While the final outcome of whether a packet reached its destination is clearly visible, it is rarely feasible to monitor the exact amount of effort exerted by each intermediate link. In large networks that are owned, operated and used by many autonomous individuals, firms and organizations with diverse economic interests, how can we ensure that the appropriate amount of effort is exerted by the autonomous routers?

Perhaps more speculatively, our model and results may also be applicable to future economic activities that exhibit comparable levels of complexity. Consider for example a firm that sub-contracts a collection of related tasks to many individuals (or other firms). It will often not be possible to exactly monitor the actual effort level of each sub-contractor (e.g., in cases of public-relations activities, consulting activities, or any activity that requires cooperation between different sub-contractors). When the dependencies between the different subtasks are complex, we believe that combinatorial agency models can offer a foundation for the design of contracts that create appropriate incentives.

1.2 Our Models

We start by presenting a general model: in this model each of n agents has a set of possible *actions*, the combination of actions by the players results in some *outcome*, where this happens probabilistically. The main part of the specification of a problem in this model is a function that specifies this distribution for each n-tuple of agents' actions. Additionally, the problem specifies the principal's utility for each possible outcome, and for each agent, the agent's cost for each possible action. The principal motivates the agents by offering to each of them a *contract* that specifies a payment

[1] This problem is known as Quality of Service (QoS) routing.

for each possible outcome of the whole project[2]. Key here is that the actions of the players are non-observable and thus the contract cannot make the payments directly contingent on the actions of the players, but rather only on the outcome of the whole project[3].

Given a set of contracts, the agents will each optimize his own utility: i.e. will choose the action that maximizes his expected payment minus the cost of his action. Since the outcome depends on the actions of all players together, the agents are put in a game. The principal's problem, our problem in this paper, is of designing an optimal set of contracts: i.e. contracts that maximize his expected utility from the outcome, minus his expected total payment, when the agents play a Nash equilibrium[4]. The main difficulty is that of determining the required Nash equilibrium point.

In order to focus on the main issues, the rest of the paper deals with the basic binary case [5]: each agent has only two possible actions "exert effort" and "shirk" and there are only two possible outcomes "success" and "failure". Even within this minimalistic framework, we already observe many interesting phenomena [6]. The technology gives the probability that the project will be successful given a subset of agents that exert effort. In this case, each agent's problem boils down to whether to exert effort or not, and the principal's problem boils down to which agents should be contracted to exert effort. A technology can be represented by a complete table specifying the success probability for each subset of the agents who exert effort.

We then consider a more concrete model which concerns a subclass of problem instances where this exponential size table is succinctly represented. This subclass will provide many natural types of problem instances. In this subclass every agent performs a subtask which succeeds with a low probability γ if the agent does not exert effort and with a higher probability $\delta > \gamma$, if the agent does exert effort. The whole project succeeds as a deterministic Boolean function of the success of the subtasks. This Boolean function can now be represented in various ways. Two basic examples are the "AND" function, which exhibit complementarities between the agents' actions, where the project succeeds if and only if all subtasks succeed, and the "OR" function, which exhibit substitutabilities between the agents' actions, where the project succeeds if any of the subtasks succeeds. A more complex example considers routing

[2]One could think of a different model in which the agents have intrinsic utility from the outcome and payments may not be needed, as in (Smorodinsky and Tennenholtz 2004, Smorodinsky and Tennenholtz 2005).

[3]A variant of this model, considered in (Winter 2005), allows for internal monitoring. i.e., agents can monitor the actions taken by prior agents (defined by some partial order on the agents).

[4]In this paper we focus on the "best" Nash equilibrium, which is also a strong equilibrium, as we show in section 5. One may alternatively require the existence of a unique equilibrium as in (Winter 2004), or alternatively, attempt modeling some kind of an extensive game between the agents, as in (Ronen and Wahrmann 2005, Smorodinsky and Tennenholtz 2004, Smorodinsky and Tennenholtz 2005).

[5](Winter 2004) also restricts attention to the binary case.

[6]However, some of the more advanced questions we ask for this case can be viewed as instances of the general model.

in a communication network, where each agent controls a single edge, and success of the subtask means that a message is forwarded by that edge. "Effort" by the edge increases this success probability. The complete project succeeds if there is a complete path of successful edges between a given source and sink. Complete definitions of the models appear in Section 2.

1.3 Our Results

Our first object of study is the structure of the class of sets of agents that can be contracted for a given problem instance. Let us fix a production technology function, fix the agent's costs, and let us consider the set of contracted agents for different values of the principal's associated value from success. For very low values, no agent will be contracted since even a single agent's cost is higher than the principal's value. For very high values, all agents will always be contracted since the marginal contribution of an agent multiplied by the principal's value will overtake any associated payment. What happens for intermediate principal's values?

We first observe that there is a finite number of "transitions" between different sets, as the principal's project value increases. These transitions behave very differently for different technology functions. For example, we show that for symmetric "AND" technology only a single transition occurs: for low enough values no agent will be contracted, while for higher values all agents will be contracted – there is no intermediate range for which only some of the agents are contracted. For symmetric "OR" technology, the situation is opposite: as the principal's value increases, the number of contracted agents increases one-by-one. We are able to fully characterize the symmetric technologies for which these two extreme types of transitions behavior occur. However, the structure of these transitions in general seems quite complex. While we were not able to fully analyze them, we do have several results, including a construction of a non-symmetric technology with an exponential number of transitions.

We also study what we term "the price of unaccountability": How much loss is incurred due to the inability to monitor individual actions [7]. We study both the "principal's price of unaccountability", which quantifies the loss incurred by the principal, and the "social price of unaccountability", which quantifies the loss incurred by the society. We are able to fully analyze this price for the symmetric "AND" technology, where it is shown to tend to infinity as the number of agents tends to

[7]The price of unaccountability is somewhat related to the *agency cost* coined in (Jensen and Meckling 1976), and later studies in various contexts such as (Jensen 1986, Jensen 2005, Mello and Parsons 1992). However, these costs are different, as the agency cost is the total additive loss that is incurred due to monitoring costs, bonding cost, and the residual loss to the principal, while the price of unaccountability we introduce here is the multiplicative loss incurred by the principal, assuming no monitoring or bonding.

infinity. For the symmetric "OR" technology with the additional condition that $\gamma = 1 - \delta$, we bound this loss by 5/2 (for any number of agents).

Our analysis of these questions sheds light on the difficulty of the various natural associated algorithmic problems. We find that the computational hardness of finding the optimal contract depends on the representation of the production technology and how it is being accessed. We present both possibility and hardness results concerning this issue.

In a follow-up paper (Babaioff, Feldman, and Nisan 2006b) we deal with equilibria in mixed strategies and show that the principal can gain from inducing a mixed-Nash equilibrium between the agents rather than a pure one. In an additional paper (Babaioff, Feldman, and Nisan 2006a) we also show cases where the principal can gain by asking agents to reduce their effort level, even when this effort comes for free. Both phenomena can not occur in the observable-actions setting, and we provide upper bounds to the principal's gain from these extensions in the hidden-actions case.

2 Model and Preliminaries

2.1 The General Setting

A principal employs a set of agents N of size n. Each agent $i \in N$ has a possible set of actions A_i, and a cost (effort) $c_i(a_i) \geq 0$ for each possible action $a_i \in A_i$ ($c_i : A_i \rightarrow \Re_+$). The actions of all players determine, in a probabilistic way, a "contractible" outcome $o \in O$, according to a success function $t : A_1 \times, \ldots \times A_n \rightarrow \Delta(O)$ (where $\Delta(O)$ denotes the set of probability distributions on O). A production technology is a pair, (t, c), of a success function, t, and cost functions, $c = (c_1, c_2, \ldots, c_n)$. The principal has a certain value for each possible outcome, given by the function $v : O \rightarrow \Re$. As we will only consider risk-neutral players in this paper[8], we will also treat v as a function on $\Delta(O)$, by taking simple expected value. Actions of the players are invisible, but the final outcome o is visible to him and to others (in particular the court), and he may design enforceable contracts based on the final outcome. Thus the contract for agent i is a function (payment) $p_i : O \rightarrow \Re$; again, we will also view p_i as a function on $\Delta(O)$.

Given this setting, the agents have been put in a game, where the utility of agent i under the vector of actions $a = (a_1, \ldots, a_n)$ is given by $u_i(a) = p_i(t(a)) - c_i(a_i)$. The principal's problem (which is our problem in this paper) is how to design the contracts p_i as to maximize his own expected utility $u(a) = v(t(a)) - \sum_i p_i(t(a))$, where the actions $a_1, \ldots, a_n$ are at Nash-equilibrium [9]. In the case of multiple Nash

[8]The first-best solution is not achievable here despite the fact that we assume risk-neutral agents since we impose the limited-liability constraint. The risk-averse case would obviously be a natural second step in the research of this model.

[9]This is the main difference between our work and Winter (Winter 2004), who assumes that the

equilibria, in our model we let the principal choose the desired one, and "suggest" it to the agents, thus focusing on the "best" Nash equilibrium. As we show later, in the binary-outcome model (which is the focus of our paper), the best Nash equilibrium is almost always the unique *strong equilibrium* [10].

Finally, the *social welfare* for $a \in A$ is the sum of the principal's utility and the agents' utilities, given by $u(a) + \sum_{i \in N} u_i(a) = v(t(a)) - \sum_{i \in N} c_i(a_i)$.

2.2 The Binary-Outcome Binary-Action Model

As we wish to concentrate on the complexities introduced by the combinatorial structure of the success function t, we restrict ourselves to a simpler setting that seems to focus more clearly on the structure of t. We first restrict the setting to binary action spaces, in which every agent chooses between action 0 (low effort) and 1 (high effort). The cost function of agent i is now just a scalar $c_i > 0$ denoting the cost of exerting high effort (where the low effort has cost 0). The vector of costs is $\vec{c} = (c_1, c_2, \ldots, c_n)$, and we use the notation $(t, \vec{c})$ to denote a production technology in such a binary-outcome model. We then restrict the outcome space to have only two states (binary-outcome): 0 (project failure) and 1 (project success). The principal's value for a successful project is given by a scalar $v > 0$ (where the value of project failure is 0). We assume that the principal can pay the agents but not fine them (known as the *limited liability* constraint). The contract to agent i is thus now given by a scalar value $p_i \geq 0$ that denotes the payment that i gets in case of project success. If the project fails, the agent gets 0. When the lowest cost action has zero cost (as we assume), this immediately implies that the participation constraint (also known as the individual rationality constraint) holds.

At this point the success function t becomes a function $t : \{0,1\}^n \to [0,1]$, where $t(a_1, \ldots, a_n)$ denotes the probability of project success where player i with $a_i = 0$ does not exert effort and incurs no cost, and player i with $a_i = 1$ does exert effort and incurs a cost of c_i.

As we wish to concentrate on motivating agents, rather than on the coordination between agents, we assume that more effort by an agent always leads to a better probability of success, i.e. that the success function t is strictly monotone. Formally, if we denote by $a_{-i} \in A_{-i}$ the $(n-1)$-dimensional vector of the actions of all agents excluding agent i. i.e., $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$, then a success function must satisfy:

$$\forall i \in N,\ \forall a_{-i} \in A_{-i} \quad t(1, a_{-i}) > t(0, a_{-i})$$

Additionally, we assume that $t(a) > 0$ for any $a \in A$ (or equivalently, $t(0, 0, \ldots, 0) > 0$).

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principal always wishes to induce an equilibrium in which all the agents exert their maximal effort.

[10] Except for, possibly, a profile in which all the agents in the best Nash equilibrium except for a single agent exert effort.

Definition 2.1 *The marginal contribution of agent i, given $a_{-i} \in A_{-i}$ is*

$$\Delta_i(a_{-i}) = t(1, a_{-i}) - t(0, a_{-i})$$

$\Delta_i(a_{-i})$ is the increase in success probability due to agent i moving from no effort to effort, given the effort of the others. Note that since t is monotone, Δ_i is a strictly positive function. At this point we can already make some simple observations. The best action, $a_i \in A_i$, of agent i can now be easily determined as a function of what the others do, $a_{-i} \in A_{-i}$, and his contract p_i.

Claim 2.2 *Given a profile of actions $a_{-i} \in A_{-i}$, agent i's best strategy is $a_i = 1$ if $p_i \geq \frac{c_i}{\Delta_i(a_{-i})}$, and is $a_i = 0$ if $p_i \leq \frac{c_i}{\Delta_i(a_{-i})}$. (In the case of equality the agent is indifferent between the two alternatives.)*

As $p_i \geq \frac{c_i}{\Delta_i(a_{-i})}$ if and only if $u_i(1, a_{-i}) = p_i \cdot t(1, a_{-i}) - c_i \geq p_i \cdot t(0, a_{-i}) = u_i(0, a_{-i})$, i's best strategy is to choose $a_i = 1$ in this case.

This allows us to specify the contracts that are the principal's optimal, for inducing a given equilibrium.

Observation 2.3 *The best contracts (for the principal) that induce $a \in A$ as an equilibrium are $p_i = 0$ for agent i who exerts no effort ($a_i = 0$), and $p_i = \frac{c_i}{\Delta_i(a_{-i})}$ for agent i who exerts effort ($a_i = 1$).*

In this case, the expected utility of agent i who exerts effort is $c_i \cdot \frac{t(0,a_{-i})}{\Delta_i(a_{-i})}$, and 0 for an agent who shirk. The principal's expected utility is given by $u(a, v) = (v - P) \cdot t(a)$, where P is the total payment in case of success, given by $P = \sum_{i|a_i=1} \frac{c_i}{\Delta_i(a_{-i})}$.

Note that as we assume that $t(a) > 0$ for any $a \in A$, an agent that exerts effort has positive utility, and the principal does not get all the social welfare. We say that the principal *contracts with* agent i if $p_i > 0$ (and $a_i = 1$ in the equilibrium $a \in A$). The principal's goal is to maximize his utility given his value v, i.e. to determine the profile of actions $a^* \in A$, which gives the highest value of $u(a, v)$ in equilibrium. Choosing $a \in A$ corresponds to choosing a set S of agents that exert effort ($S = \{i | a_i = 1\}$). We call the set of agents $S^*(v)$ that the principal contracts with in a^* ($S^*(v) = \{i | a^*_i = 1\}$) an *optimal contract* for the principal at value v. We sometimes abuse notation and denote $t(S)$ instead of $t(a)$, when S is exactly the set of agents that exert effort in $a \in A$.

A natural yardstick by which to measure this decision is the observable-actions case, i.e. when the agents observes the agents' individual actions, thus can make the payment contingent on their actions. In this case, the principal motivates an agent to exert effort by paying the agent his cost. Therefore, the principal's utility under the profile $a \in A$ is $t(a) \cdot v - \sum_{i|a_i=1} c_i$. Recall that the social welfare under a profile a is also $t(a) \cdot v - \sum_{i|a_i=1} c_i$. Thus, in the observable-actions case, the principal's utility is equal to the social welfare, and the principal will simply choose the profile that optimizes the global efficiency ("First Best").

The worst ratio between the social welfare in this observable-actions case and the social welfare for the profile $a \in A$ chosen by the principal in the hidden-actions case, may be termed the *social price of unaccountability.*

Given a technology $(t, \vec{c})$, recall that $S^*(v)$ denotes the optimal contract in the hidden-actions case and let $S^*_{oa}(v)$ denotes an optimal contract in the observable-actions case, when the principal's value is v.

Definition 2.4 *The* social price of unaccountability $POU_S(t, \vec{c})$ *of a technology* $(t, \vec{c})$ *is defined as the worst ratio (over v) between the total social welfare in the observable-actions case and the hidden-actions case:*

$$POU_S(t, \vec{c}) = Sup_{v>0} \frac{t(S^*_{oa}(v)) \cdot v - \sum_{i \in S^*_{oa}(v)} c_i}{t(S^*(v)) \cdot v - \sum_{i \in S^*(v)} c_i}$$

In cases where several sets are optimal in the hidden-actions case, we take the worst set (i.e., the set that yields the lowest social welfare).

Similarly, the worst ratio between the *principal's utility* in the observable-actions case and in the hidden-actions case may be termed the *principal's price of unaccountability.*

Definition 2.5 *The* principal's price of unaccountability $POU_P(t, \vec{c})$ *of a technology* $(t, \vec{c})$ *is defined as the worst ratio (over v) between the principal's utility in the observable-actions case and the hidden-actions case:*

$$POU_P(t, \vec{c}) = Sup_{v>0} \frac{t(S^*_{oa}(v)) \cdot v - \sum_{i \in S^*_{oa}(v)} c_i}{t(S^*(v)) \left(v - \sum_{i \in S^*(v)} \frac{c_i}{t(S^*(v)) - t(S^*(v) \setminus i)}\right)}$$

When the technology $(t, \vec{c})$ is clear in the context we will use POU_S and POU_P to denote the respective social and principal's price of unaccountability for technology $(t, \vec{c})$. We sometime use just the term POU for arguments that hold for both to the social and the principal's POU. Note that POU_S and POU_P are at least 1 for any technology. In addition, the principal's POU always (weakly) dominates the social POU, as the following observation indicates.

Observation 2.6 *For any technology* t *is holds that* $POU_P(t) \geq POU_S(t)$.

Proof: The numerators of the two expressions are identical. Therefore, it is sufficient to show that for any v, it holds that

$$t(S^*(v)) \cdot v - \sum_{i \in S^*(v)} c_i \geq t(S^*(v)) \cdot \left(v - \sum_{i \in S^*(v)} \frac{c_i}{t(S^*(v)) - t(S^*(v) \setminus \{i\})}\right)$$

but this is true since the principal's utility in the hidden-actions case is at most the social welfare (as each agent has non-negative utility). □

As we would like to focus on results that derived from properties of the success function, in most of the paper we will deal with the case where all agents have an *identical cost* c, that is $c_i = c$ for all $i \in N$. We denote a technology $(t, \vec{c})$ with identical costs by (t, c). For the simplicity of the presentation, we sometimes use the term *technology function* to refer to the success function of the technology.

2.3 Structured Technology Functions

In order to be more concrete, we will especially focus on technology functions whose structure can be described easily as being derived from independent agent tasks – we call these *structured technology functions*. This subclass will first give us some natural examples of technology function, and will also provide a succinct and natural way to represent the technology functions.

In a structured technology function, each individual succeeds or fails in his own "task" independently. The project's success or failure depends, possibly in a complex way, on the set of successful sub-tasks. Thus we will assume a monotone Boolean function $f : \{0,1\}^n \to \{0,1\}$ which denotes whether the project succeeds as a function of the success of the n agents' tasks (and is not determined by any set of $n-1$ agents). Additionally there are constants $0 < \gamma_i < \delta_i < 1$, where γ_i denotes the probability of success for agent i if he does not exert effort, and δ_i $(> \gamma_i)$ denotes the probability of success if he does exert effort. A natural restriction with much less parameters is the case that all agents have the same parameters for their tasks, that is $\gamma_1 = \ldots = \gamma_n = \gamma$ and $\delta_1 = \ldots = \delta_n = \delta$ thus leaving ourselves with only two parameters $0 < \gamma < \delta < 1$. We sometimes farther assume that $\delta = 1 - \gamma$ s.t. $0 < \gamma < \frac{1}{2}$, and when doing so we state it explicitly.

Under this structure, the technology function t is defined by $t(a_1, \ldots, a_n)$ being the probability that $f(x_1, \ldots, x_n) = 1$ where the bits $x_1, \ldots, x_n$ are chosen according to the following distribution: if $a_i = 0$ then $x_i = 1$ with probability γ and $x_i = 0$ with probability $1 - \gamma$; otherwise, i.e. if $a_i = 1$, then $x_i = 1$ with probability δ and $x_i = 0$ with probability $1 - \delta$. We denote $x = (x_1, \ldots, x_n)$.

The question of the representation of the technology function is now reduced to that of representing the underlying monotone Boolean function f. In the most general case, the function f can be given by a general monotone Boolean circuit. An especially natural sub-class of functions in the structured technologies setting would be functions that can be represented as a *network technology* – a graph with a given source and sink, where every edge is labeled by a different agent. The project succeeds if the edges that belong to players whose task succeeded form a path between the source and the sink[11].

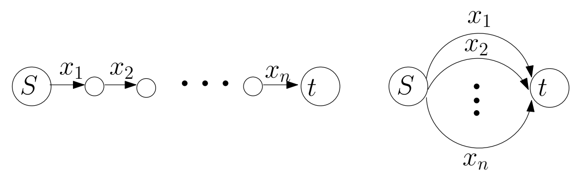
[11] One may view this representation as directly corresponding to the project of delivering a message from the source to the sink in a real network of computers, with the edges being controlled by selfish

A few simple examples should be in order here:

1. The "AND" technology [12]: $f(x_1, \ldots, x_n)$ is the logical conjunction of x_i ($f(x) = \bigwedge_{i \in N} x_i$). Thus the project succeeds only if all agents succeed in their tasks. This technology exhibits only complementarities between the agents. This is shown graphically as a network in Figure 1(a). If m agents exert effort ($\sum_i a_i = m$), then $t(a) = t_m = \delta^m \cdot \gamma^{n-m}$. E.g. for two players, the technology function $t(a_1 a_2) = t_{a_1+a_2}$ is given by $t_0 = t(00) = \gamma^2$, $t_1 = t(01) = t(10) = \delta \cdot \gamma$, and $t_2 = t(11) = \delta^2$.
2. The "OR" technology: $f(x_1, \ldots, x_n)$ is the logical disjunction of x_i ($f(x) = \bigvee_{i \in N} x_i$). Thus the project succeeds if at least one of the agents succeed in their tasks. This technology exhibits only substitutabilities between the agents. This is shown graphically as a network in Figure 1(b). If m agents exert effort, then $t_m = 1 - (1-\delta)^m (1-\gamma)^{n-m}$. E.g. for two players, the technology function is given by $t(00) = 1 - (1-\gamma)^2$, $t(01) = t(10) = 1 - (1-\delta)(1-\gamma)$, and $t(11) = 1 - (1-\delta)^2$.
3. The "Or-of-Ands" (OOA) technology: $f(x)$ is the logical disjunction of conjunctions. In the simplest case of equal-length clauses (denote by n_c the number of clauses and by n_l their length), $f(x) = \bigvee_{j=1}^{n_c} (\bigwedge_{k=1}^{n_l} x_k^j)$. Thus the project succeeds if in at least one clause all agents succeed in their tasks. This is shown graphically as a network in Figure 2(a). If m_i agents on path i exert effort, then $t(m_1, ..., m_{n_c}) = 1 - \prod_i (1 - \delta^{m_i} \cdot \gamma^{n_l - m_i})$. E.g. for four players, the technology function $t(a_1^1 \ a_2^1, a_1^2 \ a_2^2)$ is given by $t(00,00) = 1 - (1-\gamma^2)^2$, $t(01,00) = t(10,00) = t(00,01) = t(00,10) = 1 - (1-\delta\gamma)(1-\gamma^2)$, and so on.
4. The "And-of-Ors" (AOO) technology: $f(x)$ is the logical conjunction of disjunctions. In the simplest case of equal-length clauses (denote by n_l the number of clauses and by n_c their length), $f(x) = \bigwedge_{j=1}^{n_l} (\bigvee_{k=1}^{n_c} x_k^j)$. Thus the project succeeds if at least one agent from each disjunctive-form-clause succeeds in his tasks. This is shown graphically as a network in Figure 2(b). If m_i agents on clause i exert effort, then $t(m_1, ..., m_{n_c}) = \prod_i (1 - (1-\delta)^{m_i} (1-\gamma)^{n_c - m_i})$. E.g. for four players, the technology function $t(a_1^1 \ a_2^1, a_1^2 \ a_2^2)$ is given by $t(00,00) = (1 - (1-\gamma)^2)^2$, $t(01,00) = t(10,00) = t(00,01) = t(00,10) = (1 - (1-\delta)(1-\gamma))(1 - (1-\gamma)^2)$, and so on.
5. The "Majority" technology: $f(x)$ is "1" if a majority of the values x_i are 1. Thus the project succeeds if most players succeed. The majority function, even on 3 inputs, can not be represented by a network in which every edge is labeled by a different player, therefore it is not a *network technology*, but is easily represented by a monotone Boolean formula $maj(x,y,z) = xy + yz + xz$,

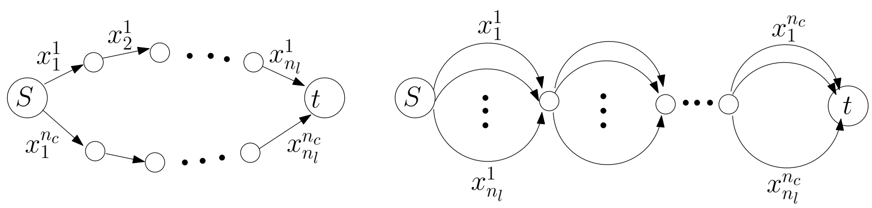
agents. Recall that this example was presented in the introduction.

[12] The AND technology is the benchmark model studied in (Winter 2004). However, the results obtained here are different since the two models use different solution concepts. See Section 5 for additional discussion.



(a) AND technology (b) OR technology

Figure 1: Graphical representations of (a) AND and (b) OR technologies.



(a) OR-of-ANDs technology (b) AND-of-ORs technology

Figure 2: Graphical representations of (a) OOA and (b) AOO technologies.

thus it is a structured technology. In this case the technology function for $\gamma = 1 - \delta$ is given by $t(000) = 3\gamma^2(1-\gamma) + \gamma^3$, $t(001) = t(010) = t(100) = \gamma^3 + 2(1-\gamma)^2\gamma + \gamma^2(1-\gamma)$, etc.

3 Analysis of Some Anonymous Technologies

A success function t is called *anonymous* if it is symmetric with respect to the players. I.e. $t(a_1, \ldots, a_n)$ depends only on $\sum_{i \in N} a_i$ (the number of agents that exert effort). A technology (t, c) is *anonymous* if t is anonymous and the cost c is identical to all agents ($\exists c$ s.t. $c_i = c \ \forall i$). Of the examples presented above, the AND, OR, and majority technologies were anonymous (but not AOO and OOA). As for an anonymous t only the number of agents that exert effort is important, we can shorten the notations and denote $t_m = t(1^m, 0^{n-m})$, $\Delta_m = t_{m+1} - t_m$, $p_m = \frac{c}{\Delta_{m-1}}$ and $u_m = t_m \cdot (v - m \cdot p_m)$, for the case of identical cost c.

3.1 *AND* and *OR* Technologies

Let us start with a direct and full analysis of the AND and OR technologies for two players for the case $\gamma = 1 - \delta = 1/4$ and $c = 1$.

Example 3.1 AND **technology with two agents,** $c = 1$**,** $\gamma = 1 - \delta = 1/4$**:** *we have* $t_0 = \gamma^2 = 1/16$*,* $t_1 = \gamma(1-\gamma) = 3/16$*, and* $t_2 = (1-\gamma)^2 = 9/16$ *thus* $\Delta_0 = 1/8$

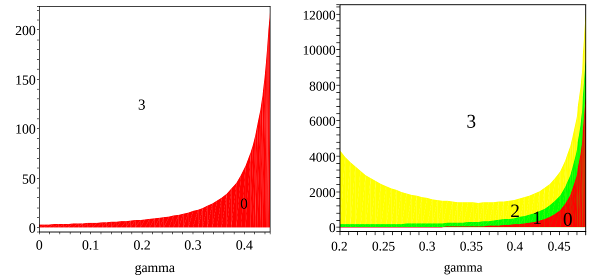


Figure 3: Number of agents in the optimal contract of the *AND* (left) and *OR* (right) technologies with 3 players, as a function of $\gamma = 1 - \delta$ and v. *AND* technology: either 0 or 3 agents are contracted, and the transition value is monotonic in γ. *OR* technology: for any γ we can see all transitions.

and $\Delta_1 = 3/8$. *The principal has 3 possibilities: contracting with 0, 1, or 2 agents. Let us write down the expressions for his utility in these 3 cases:*

- **0 Agents:** *No agent is paid thus and the principal's utility is* $u_0 = t_0 \cdot v = v/16$.
- **1 Agent:** *This agent is paid* $p_1 = c/\Delta_0 = 8$ *on success and the principal's utility is* $u_1 = t_1(v - p_1) = 3v/16 - 3/2$.
- **2 Agents:** *each agent is paid* $p_2 = c/\Delta_1 = 8/3$ *on success, and the principal's utility is* $u_2 = t_2(v - 2p_2) = 9v/16 - 3$.

Notice that the option of contracting with one agent is always inferior to either contracting with both or with none, and will never be taken by the principal. The principal will contract with no agent when $v < 6$, *with both agents whenever* $v > 6$, *and with either non or both for* $v = 6$.

This should be contrasted with the observable-actions case in which the principal observes the individual actions. In this case, the principal can pay each agent exactly his cost in order to induce him to exert effort. In the observable-actions case, therefore, the principal simply optimizes globally the social welfare. In the example above, the principal will make both agents exert effort whenever $v \geq 4$. *Thus for example, for* $v = 6$ *the globally optimal decision (observable-actions case) would give a global utility of* $6 \cdot 9/16 - 2 = 11/8$ *while the principal's decision (in the hidden-actions case) would give a global utility of* $3/8$, *giving a ratio of* $11/3$.

It turns out that this is the worst social price of unaccountability in this example, and it is obtained exactly at the transition point of the hidden-actions case. This point also yields the worst principal's price of unaccountability, which is equal to the social POU in this example.

Example 3.2 ***OR*** **technology with two agents,** $c = 1$, $\gamma = 1 - \delta = 1/4$**:** *we have* $t_0 = 1 - (1-\gamma)^2 = 7/16$, $t_1 = 1 - \gamma(1-\gamma) = 13/16$, *and* $t_2 = 1 - \gamma^2 = 15/16$ *thus*

$\Delta_0 = 3/8$ *and* $\Delta_1 = 1/8$. *Let us write down the expressions for the principal's utility in these three cases:*

- **0 Agents:** *No agent is paid and the principal's utility is* $u_0 = t_0 \cdot v = 7v/16$.
- **1 Agent:** *This agent is paid* $p_1 = c/\Delta_0 = 8/3$ *on success and the principal's utility is* $u_1 = t_1(v - p_1) = 13v/16 - 13/6$.
- **2 Agents:** *each agent is paid* $p_2 = c/\Delta_1 = 8$ *on success, and the principal's utility is* $u_2 = t_2(v - 2p_2) = 15v/16 - 15$.

Now contracting with one agent is better than contracting with none whenever $v > 52/9$ *(and is equivalent for* $v = 52/9$*), and contracting with both agents is better than contracting with one agent whenever* $v > 308/3$ *(and is equivalent for* $v = 308/3$*), thus the principal will contract with no agent for* $0 \leq v \leq 52/9$*, with one agent for* $52/9 \leq v \leq 308/3$*, and with both agents for* $v \geq 308/3$.

In the observable-actions case, in comparison, the principal will make a single agent exert effort for $v > 8/3$*, and the second one exert effort as well when* $v > 8$.

It turns out that the social price of unaccountability here is 19/13, and is achieved at $v = 52/9$, which is exactly the transition point from 0 to 1 contracted agents in the hidden-actions case. The same is true for the principal's POU. As Lemma 3.4 below shows, it is not a coincidence that for both the AND and OR technologies the social and the principal's POU are obtained for v that is a transition point.

We first present a fundamental observation, which is used in the proof of Lemma 3.4 below, as well as in many additional proofs throughout the paper.

Lemma 3.3 *(***Monotonicity lemma***) For any technology* $(t, \vec{c})$*, in both the hidden-actions and the observable-actions cases, the expected utility of the principal at the optimal contracts, the success probability of the optimal contracts, and the expected payment of the optimal contract, are all monotonically non-decreasing with the value.*

Proof: Suppose the sets of agents S_1 and S_2 are optimal in v_1 and $v_2 < v_1$, respectively. Let $Q(S)$ denote the expected total payment to all agents in S in the case that the principal contracts with the set S and the project succeeds (for the hidden-actions case, $Q(S) = t(S) \cdot \sum_{i \in S} \frac{c_i}{t(S) - t(S \setminus i)}$, while for the observable-actions case $Q(S) = \sum_{i \in S} c_i$). The principal's utility is a linear function of the value, $u(S, v) = t(S) \cdot v - Q(S)$. As S_1 is optimal at v_1, $u(S_1, v_1) \geq u(S_2, v_1)$, and as $t(S_2) \geq 0$ and $v_1 > v_2$, $u(S_2, v_1) \geq u(S_2, v_2)$. We conclude that $u(S_1, v_1) \geq u(S_2, v_2)$, thus the utility is monotonic non-decreasing in the value.

Next we show that the success probability is monotonic non-decreasing in the value. S_1 is optimal at v_1, thus:

$$t(S_1) \cdot v_1 - Q(S_1) \geq t(S_2) \cdot v_1 - Q(S_2)$$

S_2 is optimal at v_2, thus:

$$t(S_2) \cdot v_2 - Q(S_2) \geq t(S_1) \cdot v_2 - Q(S_1)$$

Summing these two equations, we get that $(t(S_1)-t(S_2)) \cdot (v_1 - v_2) \geq 0$, which implies that if $v_1 > v_2$ than $t(S_1) \geq t(S_2)$.

Finally we show that the expected payment is monotonic non-decreasing in the value. As S_2 is optimal at v_2 and $t(S_1) \geq t(S_2)$, we observe that:

$$t(S_2) \cdot v_2 - Q(S_2) \geq t(S_1) \cdot v_2 - Q(S_1) \geq t(S_2) \cdot v_2 - Q(S_1)$$

or equivalently, $Q(S_2) \leq Q(S_1)$, which is what we wanted to show. □

This lemma implies that for anonymous technologies (as the anonymous AND and OR technologies considered above) the number of contracted agents is non-decreasing with the value. With this, we are ready to prove that the price of unaccountability is always obtained at a transition point.

Lemma 3.4 *For any given technology $(t, \vec{c})$ the social price of unaccountability is obtained at some value v which is a transition point, of either the hidden-actions or the observable-actions cases. The same holds for the principal's price of unaccountability.*

Proof: For technology $(t, \vec{c})$ let $f(v)$ be the social welfare ratio at v, that is $f(v) = \frac{t(S^*_{oa}) \cdot v - \sum_{i \in S^*_{oa}} c_i}{t(S^*) \cdot v - \sum_{i \in S^*} c_i}$. By definition the social POU is $POU_S(t, \vec{c}) = Sup_{v>0} f(v)$. By the monotonicity lemma (Lemma 3.3), in both the hidden-actions and observable-actions cases, contracting with no agent can be optimal up to some value, and never optimal for larger values. Additionally, once contracting with all agents is optimal, no other contract be optimal for any larger value. Let $\underline{v}$ be the lowest value transition point, from contracting with no agent to contracting with some agents, of either the hidden-actions or observable-actions case. Let $\overline{v}$ be the highest value transition point, from contracting with some agents to contracting with all agents, of either the hidden-actions or observable-actions case. Note that for value $v \leq \underline{v}$ no agent is contracted in both the hidden-actions and observable-actions cases, and for value $v \geq \overline{v}$ all agents are contracted in both cases. Thus for $v \leq \underline{v}$ and for $v \geq \overline{v}$, $f(v) = 1$. We conclude that if the social POU is obtained, it happens for a finite positive value $v \in [\underline{v}, \overline{v}]$.

We can assume w.l.o.g. that ties between optimal sets are broken in a consistent way (as we only care about the welfare of the principal). By Lemma 3.3, we can partition the $[\underline{v}, \overline{v}]$ interval to at most 2^n segments, in each the optimal contract in the hidden-actions case is fixed. Similarly, each of these segments can be partitioned to at most 2^n segments, in each the optimal contract for the observable-actions case is fixed. We conclude that there exists a finite partition of the $[\underline{v}, \overline{v}]$ interval to segments, such that the end points of the segments are transition points, and on each segment there is one contract that is optimal for the entire segment for both the hidden-actions and observable-actions cases.

To complete the proof, we use the following lemma (see proof in Appendix A.1) which shows that for each segment, the supremum of $f(v)$ is obtained at an end point of the segment.

Lemma 3.5 *Let $f(x) = \frac{a \cdot x - b}{c \cdot x - d}$ be a function of x, and assume that $c > 0$. Let $\overline{x} \geq \underline{x} > 0$ be two points for which $c\underline{x} - d > 0$. Then the supremum of f on the range $[\underline{x}, \overline{x}]$ is obtained at either $\underline{x}$ or $\overline{x}$. Additionally, if $a = c > 0$, $d > b$ and for some $\overline{x} > 0$ it holds that $a\overline{x} - d > 0$ then the supremum of f on $[\overline{x}, \infty)$ is obtained at $\overline{x}$.*

On each of the segments mentioned above, $f(v)$ satisfies the conditions of the first part of Lemma 3.5, thus its supremum is obtained at an end point of the segment. The global suprimum (over all segments) is obtained as the maximum of finitely many maximal numbers obtained, one in each segment. Therefore, the lemma holds for the social POU.

Let $g(v) = \frac{t(S^*_{oa}) \cdot v - \sum_{i \in S^*_{oa}} c_i}{t(S^*) \cdot v - \sum_{i \in S^*} \frac{c_i}{t(S^*) - t(S^* \setminus \{i\})}}$ be the principal's utility ratio at v. By definition the principal's POU is $POU_P(t, \vec{c}) = Sup_{v>0} g(v)$. For any $v \leq \underline{v}$, $g(v) = 1$. Additionally, $g(v)$ satisfies the conditions of the second part of Lemma 3.5, thus we need not care about values $v \geq \overline{v}$. Finally, for values $[\underline{v}, \overline{v}]$, the same arguments that we use to prove that the POU_S is obtained at a transition point also hold for $g(v)$, and therefore the lemma holds for the principal's POU as well. □

We already see a qualitative difference between the AND and OR technologies (even with 2 agents): in the first case either all agents are contracted or none, while in the second case, for some intermediate range of values v, exactly one agent is contracted. Figure 3 shows the same phenomena for AND and OR technologies with 3 players, and we next show that it holds for any number of players.

Theorem 3.6 *For any anonymous AND technology*[13]*:*

- *there exists a value*[14] *$v_* < \infty$ such that for any $v < v_*$ it is optimal to contract with no agent, for $v > v_*$ it is optimal to contract with all n agents, and for $v = v_*$, both contracts $(0, n)$ are optimal.*
- *the social price of unaccountability and the principal's price of unaccountability are obtained at the transition point of the hidden-actions case, and satisfy*

$$POU_P = POU_S = \left(\frac{\delta}{\gamma}\right)^{n-1} + 1 - \frac{\gamma}{\delta}$$

In Section 3.2 we present a general characterization of technologies with a single transition in the hidden-actions and the observable-actions cases. The above theorem

[13] AND technology with any number of agents n and any γ, any $\delta > \gamma$ and any identical cost c.
[14] v_* is a function of n, γ, δ and c.

is a special case of the general characterization. Its first part is proved in Lemma 3.15, and its second part is proven in Corollary 3.18.

The property of a single transition in the AND technology occurs in both the hidden-actions and the observable-actions cases, where the transition occurs at a smaller value of v in the observable-actions case. Notice that the social and the principal's POU are not bounded across the AND family of technologies (for various n, γ) as they approach ∞ either if $\gamma \to 0$ (for any given $n \geq 2$ and δ bounded from 0 - in particular when $\delta = 1 - \gamma > 1/2$) or $n \to \infty$ (for any fixed $\delta > \gamma > 0$).

Next we consider the OR technology and show that it exhibits all n transitions.

Theorem 3.7 *For any anonymous OR technology*[15]*, there exist finite positive values* $v_1 < v_2 < \ldots < v_n$ *such that for any* v *s.t.* $v_k < v < v_{k+1}$*, contracting with exactly* k *agents is optimal (for* $v < v_1$*, no agent is contracted, and for* $v > v_n$*, all* n *agents are contracted). For* $v = v_k$*, the principal is indifferent between contracting with* $k-1$ *or* k *agents.*

This characterization is a direct corollary of a more general characterization given in Section 3.2, and is proven in Lemma 3.16 in that section.

While in the AND technology we were able to fully determine the principal's and social POU analytically, for the OR technology we cannot fully characterize it. Nevertheless, we show that in any anonymous OR technology with n agents, the principal's and the social POU are bounded by a small constant.

Theorem 3.8 *For any anonymous OR technology with* n *agents with* $\gamma = 1-\delta < 1/2$*, it holds that*

$$POU_S \leq POU_P \leq \frac{5}{2}$$

Proof: By Observation 2.6 for any technology it holds that $POU_S \leq POU_P$, thus it is sufficient to show that $POU_P \leq \frac{5}{2}$. Let v^*_{oa} be the value for which the principal is independent between contracting with 0 or 1 agents in the observable-actions case. If $v \leq v^*_{oa}$, the utility ratio is 1 as the optimal contract in both cases is with 0 agents.

We next look at the value v^* for which the principal is independent between contracting with 0 or 1 agents in the hidden-actions case. At $v = v^*$ it holds that $t_1 \cdot (v - \frac{1}{\Delta_1}) = t_0 \cdot v$ thus $v^* = \frac{t_1}{\Delta_1^2}$. For a value $v^*_{oa} \leq v \leq v^*$ it is enough to bound $\frac{t_n \cdot v - 1}{t_0 \cdot v}$, which is monotonicity increasing with v, thus it is enough to bound the ratio at v^*. For $v > v^*$ it is enough to bound $\frac{t_n \cdot v - 1}{t_1 \cdot (v - \frac{1}{\Delta_1})} = \frac{t_n}{t_1} \cdot (1 + \frac{\frac{1}{\Delta_1} - \frac{1}{t_n}}{v - \frac{1}{\Delta_1}})$, which is monotonicity decreasing with v, thus it is enough to bound the ratio at v^*. We conclude that we only need to bound the ratio $\frac{t_n \cdot v^* - 1}{t_0 \cdot v^*}$.

$$\frac{t_n \cdot v^* - 1}{t_0 \cdot v^*} = \frac{t_n}{t_0} - \frac{1}{t_0 \cdot v^*} = \frac{t_n}{t_0} - \frac{\Delta_1^2}{t_0 \cdot t_1} = \frac{t_n}{t_0} - \frac{t_1^2 - 2t_0 t_1 + t_0^2}{t_0 \cdot t_1} = \frac{t_n}{t_0} - \frac{t_1}{t_0} + 2 - \frac{t_0}{t_1} = 2 + \frac{t_n - t_1}{t_0} - \frac{t_0}{t_1}$$

[15] OR technology with any number of agents n and any γ, any $\delta > \gamma$ and any identical cost c.

$t_n \leq 1$ and $1 \geq t_1 \geq \delta$. Additionally, as $n \geq 2$, $t_0 = 1-(1-\gamma)^n \geq 1-(1-\gamma)^2 = \gamma(2-\gamma)$. Thus

$$\frac{t_n \cdot v^* - 1}{t_0 \cdot v^*} = 2 + \frac{t_n - t_1}{t_0} - \frac{t_0}{t_1} \leq 2 + \frac{1-\delta}{\gamma(2-\gamma)} - \gamma(2-\gamma)$$

This will be bounded by a constant if $\frac{1-\delta}{\gamma}$ is bounded by a constant. For the special case that $\delta = 1-\gamma$ we get

$$2 + \frac{1-\delta}{\gamma(2-\gamma)} - \gamma(2-\gamma) = 2 + \frac{1}{2-\gamma} - \gamma(2-\gamma) \leq 2 + \frac{1}{2} = \frac{5}{2}$$

The function $\frac{1}{2-\gamma} - \gamma(2-\gamma)$ is monotonically decreasing in $[0, 0.5]$. To see this, note that the derivative is $\frac{1}{(2-\gamma)^2} - 2 + 2\gamma$, which is negative for $\gamma \in [0, 0.5]$. This is true since the second derivative is $\frac{2}{(2-\gamma)^3} + 2 > 0$ and therefore it is sufficient to show that the first derivative is negative at $\gamma = 0.5$. But this holds since $\frac{2}{(2-0.5)^2} - 2 + 2 \cdot 0.5 = -1/9 < 0$. The maximum when $\gamma = 0$ is $\frac{1}{2}$. □

We already observe a qualitative difference between the POU in the *AND* and *OR* technologies.

Observation 3.9 *While in the AND family of technologies the POU_P and POU_S are not bounded from above, in the OR family of technologies with $\gamma = 1-\delta$, both are always bounded by 5/2.*

3.2 What Determines the Transitions?

In this section we characterize anonymous technologies with a single transition, and with all n transitions for both the hidden-actions and the observable-actions cases. The proofs for the claims presented below appear in Appendix A.1.

We begin by an example that shows that transitions are not necessarily the same in the hidden-actions and the observable-actions cases.

Example 3.10 *Consider the following anonymous technology with two agents: $t_0 = 0$, $t_1 = 0.3$, and $t_2 = 0.61$. One can verify that in the hidden-actions case, the principal will contract with no agent for $0 \leq v \leq 3.33..$, with one agent for $3.33.. \leq v \leq 9.47..$, and with both agents for $v \geq 9.47..$, thus either 0, 1, or 2 agents can be obtained as the optimal contract. In contrast, in the observable-actions case, the principal contracts with no agent for $0 \leq v \leq 3.33..$ and with both agents for $v \geq 3.33...$ Contracting with a single agent is never optimal.*

A technology (t, c) has *all transitions* if for each $k \in \{0, 1, \ldots, n\}$, there exists v for which $\forall k' \neq k, u(k) > u(k')$. [16]

[16] A technology (t, c) has *all transitions* in the *weak* sense if the inequality holds only weakly.

What determines the number of transitions in the hidden-actions and observable-actions cases? It turns out that in the observable-actions case, the conditions are simple.

We denote $a_{-i} < b_{-i}$ if for any agent $j \neq i$ it holds that $a_j \leq b_j$, and for some $j \neq i$ it holds that $a_j < b_j$.

Definition 3.11 *A technology success function*[17] t *exhibits*

- *(strictly)* **increasing returns to scale (IRS)** *if for every* i *and every* $a_{-i} < b_{-i}$ *it holds that* $\Delta_i(a_{-i}) < \Delta_i(b_{-i})$
- *(strictly)* **decreasing returns to scale (DRS)** *if for every* i *and every* $a_{-i} < b_{-i}$ *it holds that* $\Delta_i(a_{-i}) > \Delta_i(b_{-i})$
- *(strictly)* **under-proportional contribution (UPC)** *if the technology is anonymous, and for every* k *it holds that*

$$\frac{k}{n} > \frac{t_k - t_0}{t_n - t_0}$$

Intuitively, IRS means that more effort by the other agents increases the influence of an agent on the success of the project. On the other hand, DRS means that more effort by the other agents decreases the influence of an agent on the success of the project. Loosely speaking, UPC means that for any number of agents k ($0 < k < n$), the average contribution of each of the k agents is smaller than the average contribution of each agent when all n agents exert effort. This is a weaker condition than IRS, as any anonymous technology that exhibits IRS also exhibits UPC, but not vice versa (see Lemma A.8 in Appendix A.1). The following theorem provides a complete characterization of the observable-actions case. For additional details refer to A.1.

Theorem 3.12 *In the observable-actions model, an anonymous technology* (t, c) *has*

- *all* n *transitions if and only if it exhibits DRS.*
- *a single transition if and only if it exhibits UPC.*

The analysis of the hidden-actions case is more complex and involves less intuitive conditions.

Definition 3.13 *For an anonymous technology* (t, c) *let* $Q_k = \frac{c \cdot k \cdot t_k}{t_k - t_{k-1}}$ *be the total expected payment in the best contract for which there exist an equilibrium with* k *agents exerting effort (the minimal cost of implementing a contract with* k *agents).*

An anonymous technology[18] (t, c) *exhibits*

[17] If the success function of the technology with identical costs (t, c) exhibits some property, we also say that the technology exhibits the same property.

[18] Note that these are conditions on the success function, independent of the identical cost c.

- *(strictly)* **over-payment (OP)** *if for any k, it holds that*

$$\frac{Q_k}{Q_n} > \frac{t_k - t_0}{t_n - t_0}$$

- *(strictly)* **increasing relative marginal payment (IRMP)** *if for any k, it holds that*

$$\frac{Q_{k+1} - Q_k}{t_{k+1} - t_k} > \frac{Q_k - Q_{k-1}}{t_k - t_{k-1}}$$

Intuitively, the over-payment condition compares the proportional increase in success probability when moving from k agents to n agents ($\frac{t_k - t_0}{t_n - t_0}$), to the proportional expected payment to k agents with respect to the expected payment to n agents ($\frac{Q_k}{Q_n}$). If any set of k agents ($0 < k < n$) needs to be paid "over proportionally", the principal will contract either with 0 agents or with all n agents.

The IRMP condition looks at the proportional increase in payment ($Q_k - Q_{k-1}$) with respect to the increase in success probability ($t_k - t_{k-1}$), when increasing the number of contracting agents by one, from $k-1$ to k. The IRMP condition requires that any additional agent has a larger effect than its predecessor.

Notice that if we adjust Q_k to the observable-actions case (for which $Q_k = c \cdot k$), OP is equivalent to UPC, and IRMP is equivalent to DRS (see Observation A.6 in Appendix A.1).

Theorem 3.14 *In the hidden-actions model, an anonymous technology (t, c) has*

- *all n transitions if and only if it exhibits IRMP.*
- *a single transition if and only if it exhibits OP.*

Theorem 3.14 can be derived from a more general analysis, which can be found in Appendix A.1.

The UPC condition is not equivalent to over-payment and the DRS condition is not equivalent to IRMP (see Observation A.9 in Appendix A.1). Thus, it is not true in general that the transitions are the same in the hidden-actions and the observable-actions cases (also recall Example 3.10). Yet, it is true for anonymous AND and OR functions, as the following lemmas prove.

Lemma 3.15 *Anonymous AND technology exhibits both UPC and Over-Payment, thus has a single transition in both the hidden-actions and observable-actions cases.*

Lemma 3.16 *Anonymous OR technology exhibits DRS and IRMP, thus has all n transitions in both the hidden-actions and observable-actions cases.*

We next show that for anonymous technologies that exhibit both UPC and Over-Payment, we can fully characterize the social and the principal's POU. The proof for the next lemma appears in the appendix.

Lemma 3.17 *For any anonymous technology (t, c) that exhibits both UPC and Over-Payment, it holds that*

$$POU_S = POU_P = 1 + \frac{t_{n-1}}{t_0} - \frac{t_{n-1}}{t_n}$$

and both are obtained at a transition point of the hidden-actions case.

The AND technology is an example of such a technology. Thus, we derive the following corollary.

Corollary 3.18 *The social and principal's POU for AND technology (with any c and any $\delta > \gamma > 0$) are obtained at the transition point of the hidden-actions case, and satisfy*

$$POU_P = POU_S = \left(\frac{\delta}{\gamma}\right)^{n-1} + 1 - \frac{\gamma}{\delta}$$

Proof: By the above Lemma, AND technology exhibits both UPC and Over-Payment. Now, the proof is a direct result of Lemma 3.17 and the fact that for AND technology with $\delta > \gamma > 0$, $\frac{t_{n-1}}{t_0} = \frac{\delta^{n-1}\cdot\gamma}{\gamma^n} = \left(\frac{\delta}{\gamma}\right)^{n-1}$ and $\frac{t_{n-1}}{t_n} = \frac{\delta^{n-1}\cdot\gamma}{\delta^n} = \frac{\gamma}{\delta}$. □

3.3 The MAJORITY Technology

The project under the MAJORITY function succeeds if the majority of the agents succeed in their tasks (see Section 2.3). Figure 4 presents the optimal number of contracted agents as a function of v and γ, for $\delta = 1 - \gamma$ and $n = 5$. While we cannot analyze the MAJORITY technology, the phenomena that we observe in this example (and others that we looked at) leads us to the following conjecture.

Conjecture 3.19 *For any Majority technology (any $n, \gamma = 1 - \delta$ and c), there exists l, $1 \leq l \leq \lceil n/2 \rceil$ such that the first transition is from 0 to l agents, and then all the remaining $n - l$ transitions exist.*

Moreover, for any fixed c, n, $l = 1$ when γ is close enough to $\frac{1}{2}$, l is a non-decreasing function of γ (with image $\{1, \ldots, \lceil n/2 \rceil\}$), and $l = \lceil n/2 \rceil$ when γ is close enough to 0.

4 Non-Anonymous Technologies

In non-anonymous technologies (even with identical costs), we need to talk about the contracted *set* of agents and not only about the number of contracted agents. In this section, we identify the sets of agents that can be obtained as the optimal contract for some v. These sets construct the *orbit* of a technology, which corresponds to the principal's "Pareto Frontier".

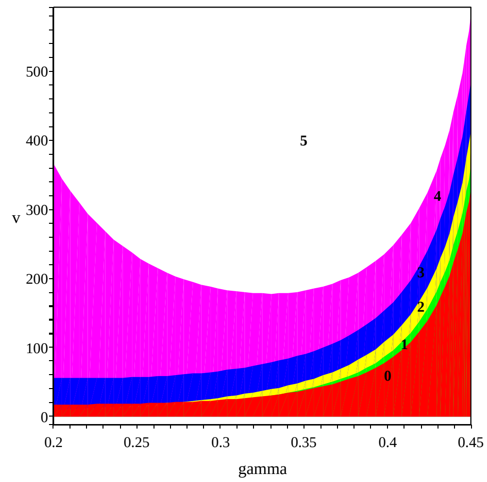


Figure 4: Simulations results showing the number of agents in the optimal contract of the $MAJORITY$ technology with 5 players, as a function of γ and v, when $\delta = 1 - \gamma$. As γ decreases the first transition is at a lower value and to a higher number of agents. For any sufficiently small γ, the first transition is to $3 = \lceil 5/2 \rceil$ agents, and for any sufficiently large γ, the first transition is to 1 agents. For any γ, the first transition is never to more than 3 agents, and after the first transition we see all following possible transitions.

Definition 4.1 *For a technology t, a set of agents S is in the* orbit of t *if for some value v, the optimal contract is exactly with the set S of agents (where ties between different S's are broken according to a lexicographic order*[19]*). The* k-orbit *of t is the collection of sets of size exactly k in the orbit.*

4.1 AOO and OOA Technologies

We begin our discussion of non-anonymous technologies with two examples; the And-of-Ors (AOO) and Or-of-Ands (OOA) technologies.

The AOO technology (see figure 2) is composed of multiple *OR-components* that are "And"ed together. The next theorem portrays the orbit structure of AOO technologies. Part of the proof of the theorem is based on such AOO technology being a special case of a more general family of technologies, in which disjoint anonymous technologies are "And"-ed together, as explained in the next section.

Theorem 4.2 *Let h be an anonymous OR technology, and let $f = \bigwedge_{j=1}^{n_c} h$ be the AOO technology that is obtained by a conjunction of n_c of these OR-components on disjoint inputs. Then for any value v, an optimal contract contracts with the same number of agents in each OR-component.*

[19]This implies that there are no two sets with the same success probability in the orbit.

Proof: We prove that for any AOO technology with n_c OR-components, each with n_l symmetric agents, any optimal contract has equal number of agents contracted in each OR-component (for any v, c, γ, and any n_c, n_l). We prove the claim by induction on n_c. The base case of $n_c = 2$ is proven first. Let h denote the OR technology for a single component (by symmetry h is the same for all components).

Claim 4.3 *For any AOO technology with two OR-components, each with n_l symmetric agents, any optimal contract has the same number of agents contracted in each OR-component.*

Proof: Assume that for some v the optimal contract has k_1 agents in the first OR-component, and k_2 in the second OR-component. Assume in contradiction (wlog) that $k_1 > k_2$, thus $h(k_1) > h(k_2)$. By Lemma 4.12, k_1 is optimal for h on $v \cdot h(k_2)$, and k_2 is optimal for h on $v \cdot h(k_1) > v \cdot h(k_2)$. This contradicts Observation A.2 which shows that if k_2 is optimal for a larger value than k_1, then $k_2 \geq k_1$. □

By induction, assume that for any number of OR-components that is smaller than n_c ($n_c > 2$), the optimal contract has the same number of agents in each component. We show that the optimal contract has the same number of agents in each component if there are n_c components. Assume that the optimal contract has k_1 agents on the first OR-component. Let g be the conjunction of the rest $n_c - 1$ components. By Lemma 4.12, the contract on g is an optimal contract at the value $v \cdot h(k_1)$, thus by our induction hypothesis has the same number of agents k_2 contracted at each OR-component. To conclude the proof we need to show that $k_1 = k_2$.

Let h_2 be the conjunction of the first two OR-components. Again by Lemma 4.12 the contract on h_2 is an optimal contract for some value, and by the induction hypothesis has the same number of agents contracted in each of the two components, k_3. Since in the first component k_1 agents are contracted then $k_1 = k_3$. Since in the second component k_2 agents are contracted then $k_2 = k_3$. Thus $k_1 = k_2$, and we conclude the proof. □

Note that the properties of the OR function are not used in the proof, and therefore, the proof above holds also when we replace OR with any anonymous technology.

Observation 4.4 *Let h be an anonymous technology, and let $f = \bigwedge_{j=1}^{n_c} h$ be a technology that is obtained by a conjunction of n_c copies of h, with pairwise disjoint inputs. Then for any value v, an optimal contract contracts with the same number of agents in each h component.*

We conjecture that a similar result holds for the OOA technology. We are unable to prove it in general, but can prove it for the case of two parallel paths of length two, as the following claim reveals. (see proof in Appendix B)

Claim 4.5 *In the OOA technology with two parallel paths of length two, for any values of c and $\gamma = 1 - \delta$, there exist values $v_1 < v_2$ in the optimal contract such that:*

- *for any $v \le v_1$, no agent is contracted.*
- *for any $v \in [v_1, v_2]$, two agents on the same path are contracted.*
- *for any $v \ge v_2$, all four agents are contracted.*

4.2 Orbit Characterization

Given a non-anonymous technology, we wish to investigate whether there is a structure on the collection of optimal contracts of the technology. That is, how the optimal contract changes as the principal's value from the project (v) increases. A basic observation is that the orbit of a technology is actually an ordered list of sets of agents, where the order is by increasing success probability of the contracted set, as determined by the success monotonicity lemma 3.3.

For an anonymous technology, this means that the number of agents that are contracted is monotonically non decreasing with the value. The above lemma also implies that there are at most $2^n - 1$ changes to the optimal contract; but does there actually exist a technology that has such a large number of transitions? Clearly, the k-orbit of any technology with identical costs in the observable-actions case is of size at most one, since among all the sets of size k, only the one with the highest probability can be in the orbit. For the hidden-actions case, however, the orbit might include more than a single set of each size. Yet, the following observation indicates that in the case of identical costs, it is impossible for *all* subsets of agents to be on the orbit.

Observation 4.6 *The 1-orbit of any technology (even non-anonymous) with identical costs is of size at most one.*

Proof: Only the agent that gives the highest success probability (when only he exerts effort) can be on the orbit since his effort yields the highest probability and the lowest payment (over all contracts with a single agent). □

Nevertheless, we next show that in the hidden-actions case the orbit can have exponential size (in n).

A collection of sets of k elements (out of n) is "admissible", if every two sets in the collection differ by at least 2 elements (e.g. for k=3, 123 and 234 can not be together in the collection, but 123 and 345 can be).

Theorem 4.7 *Every admissible collection can be obtained as the $k - orbit$ of some t.*

We first present the high-level arguments of the proof, and then present the full proof. The proof is constructive. Let $\mathcal{S}$ be some admissible collection of k-size sets. For each set $S \in \mathcal{S}$ in the collection we pick ϵ_S, such that for any two admissible sets $S_i \neq S_j$, $\epsilon_{S_i} \neq \epsilon_{S_j}$. We then define the technology function t as follows: for any $S \in \mathcal{S}$, $t(S) = 1/2 - \epsilon_S$ and $\forall i \in S$, $t(S \setminus i) = 1/2 - 2\epsilon_S$. [20] Thus, the marginal contribution of every $i \in S$ is ϵ_S. Note that since $\mathcal{S}$ is admissible, t is well defined, as for any two sets $S, S' \in \mathcal{S}$ and any two agents i, j, $S \setminus i \neq S' \setminus j$. For any other set Z, we define $t(Z)$ in a way that ensures that the marginal contribution of each agent in Z is a very small ϵ (the rest of the details appear below). This completes the definition of t.

We show that each admissible set $S \in \mathcal{S}$ is optimal at the value $v_S = \frac{ck}{2\epsilon_S^2}$. We first show that it is better than any other $S' \in \mathcal{S}$. At the value $v_S = \frac{ck}{2\epsilon_S^2}$, the set S that corresponds to ϵ_S maximizes the utility of the principal (This result is obtained by checking the first-order condition). Therefore S yields a higher utility than any other $S' \in \mathcal{S}$. We also pick the range of ϵ_S to ensure that at v_S, S is better than any other set $S' \setminus i$ s.t. $S' \in \mathcal{S}$. Now we are left to show that at v_S, the set S yields a higher utility than any other set Z ($Z \neq S$ and $Z \neq S \setminus i$ for any $S \in \mathcal{S}$ and any agent i). The construction of $t(Z)$ ensures this since the marginal contribution of each agent in Z is such a small ϵ, that the payment is too high for the set to be optimal.

The full proof follows.

Proof: Let $\mathcal{S}$ be some admissible collection of k size sets. For each set $S \in \mathcal{S}$ in the collection we pick $\epsilon_S \in (0.17, 0.2]$, such that for any two admissible sets $S_i \neq S_j$, $\epsilon_{S_i} \neq \epsilon_{S_j}$, and for some $S \in \mathcal{S}$, $\epsilon_S = 0.2$. We also pick a small enough $\epsilon > 0$.

Next, we define the technology t:

- For any set $S \in \mathcal{S}$ let $t(S) = 1/2 - \epsilon_S$.
- For any set $S \in \mathcal{S}$ and every $i \in S$, $t(S \setminus i) = 1/2 - 2\epsilon_S$.
- Let $\mathcal{Z}$ be the family of all sets for which the above defined t for ($\mathcal{Z} = \mathcal{S} \cup \bigcup_{S \in \mathcal{S}, i \in S} \{S \setminus i\}$). For any set T that is not in $\mathcal{Z}$, we define: $t(T) = max_{Z: Z \subset T, Z \in \mathcal{Z}} \{t(Z) + (|T| - |Z|) \cdot \epsilon\}$ (if there is no $Z \in \mathcal{Z}$ such that $T \supset Z$, then we define $t(T) = \epsilon \cdot |T|$).

Note that t is well defined, as $\mathcal{S}$ is an admissible set, thus for each set T, $t(T)$ was only defined once (that is, for any two sets $S, S' \in \mathcal{S}$ and any two agents i, j, $S \setminus i \neq S' \setminus j$).

We show that each admissible set S is the optimal at the value $v_S = \frac{c \cdot k}{2{\epsilon_S}^2}$. We first show that it is better than contract with a different $S' \in \mathcal{S}$, and than we show that it is better than any contract for other sets (in particular $k-1$ size sets).

The utility of the principal from a contract with a set $S \in \mathcal{S}$ is $u(S, v) = t(S) \cdot (v - \sum_{i \in S} \frac{c}{t(S) - t(S \setminus i)}) = (\frac{1}{2} - \epsilon_S)(v - \frac{c \cdot k}{\epsilon_S}) = \frac{v}{2} + k \cdot c - (\frac{c \cdot k}{2\epsilon_S} + v \cdot \epsilon_S)$. The utility is maximized when $(\frac{c \cdot k}{2\epsilon_S} + v \cdot \epsilon_S)$ is minimized, which happens when $\frac{c \cdot k}{2\epsilon_S} = v \cdot \epsilon_S$, or when $v = \frac{c \cdot k}{2{\epsilon_S}^2}$. (The derivative of $u(S, v)$ by ϵ_s is $\frac{c \cdot k}{2{\epsilon_S}^2} - v$). We denote by v_S the value for which ϵ_S maximizes the utility of the principal ($v_S = \frac{c \cdot k}{2{\epsilon_S}^2}$). Note that at v_S, any set $S' \neq S$ has a lower utility for the principal.

[20] With some abuse of notation, for a set S and agent i we use $S \setminus i$ to denote the set $S \setminus \{i\}$.

Additionally, note that for any admissible set S, and any set T of size $k-1$, $t(S) > t(T)$. This is true as $t(S) \geq \frac{1}{2} - \epsilon_S > 0.3$, while even if $T = S' \setminus i$ for some S', i, then $t(T) \leq \frac{1}{2} - 2\epsilon_{S'} < 0.16$ (otherwise $t(T) = \epsilon \cdot |T|$, and we can pick ϵ as small as we want). This implies that in order to show that any $S \in \mathcal{S}$ has higher utility than any T of size $k-1$ at the value v_S, it is sufficient to show that this hold for the smallest v_S, which is achieved for the largest ϵ_S, which is 0.2 by our construction. We denote the largest ϵ_S by $\bar{\epsilon} = 0.2 = 1/5$, its corresponding set by $\bar{S}$, and its corresponding value by $\bar{v} = \frac{k \cdot c}{2 \cdot \bar{\epsilon}^2}$.

The utility of the principal from a set $T = S \setminus i$ for some admissible set S and agent i, at the value $\bar{v}$ is $u(T, \bar{v}) = (\frac{1}{2} - 2\epsilon_S) \cdot (\bar{v} - \frac{(k-1) \cdot c}{(\frac{1}{2} - 2\epsilon_S) - (k-2)\epsilon})$. As we can take ϵ to be as small as we like, we can neglect the $(k-2)\epsilon$ term. Thus, $u(T, \bar{v}) = \frac{\bar{v}}{2} - 2\epsilon_S \cdot \bar{v} - (k-1) \cdot c$.

We need to verify that $u(T, \bar{v}) = \frac{\bar{v}}{2} - 2\epsilon_S \cdot \bar{v} - (k-1) \cdot c < \frac{\bar{v}}{2} + k \cdot c - \frac{c \cdot k}{\bar{\epsilon}} = u(\bar{S}, \bar{v})$. As $\bar{v} = \frac{k \cdot c}{2 \cdot \bar{\epsilon}^2}$, this is equivalent to $c(1 - 2k + \frac{k}{\bar{\epsilon}}) < 2\epsilon_S \cdot \frac{k \cdot c}{2 \cdot \bar{\epsilon}^2}$. Equivalently, $\epsilon_S > \bar{\epsilon} - \bar{\epsilon}^2 \cdot (2 - \frac{1}{k})$. As $k = 1$ maximizes the right side, we need to verify that $\epsilon_S > \bar{\epsilon} - \bar{\epsilon}^2 = 1/5 - 1/25 = 4/25 = 0.16$, and this holds as $\epsilon_S > 0.17$.

Finally, note that for any other contract, there is at least one agent that is paid $\frac{1}{\epsilon}$. As we can choose ϵ arbitrarily small, we can make sure that for any set $S \in \mathcal{S}$, at the value $v_S = \frac{c \cdot k}{2\epsilon_S{}^2}$ the payment to at least one agent in any other contract is at least as the value v_S (recall that $\epsilon_S > 0.17$ thus v_S is bounded from above by $\frac{c \cdot k}{2 \cdot 0.17^2}$), and the utility is negative. This implies that for any v_S, the optimal contract is not any set other that $S \in \mathcal{S}$ or $T = S \setminus i$ for some $S \in \mathcal{S}$ and agent $i \in S$.

We conclude that at any v_S the optimal contract is the admissible set $S \in \mathcal{S}$. □

We next show that there exist very large admissible collections.

Lemma 4.8 *For any $n \geq k$, there exists an admissible collection of k-size sets of size $\Omega(\frac{1}{n} \cdot \binom{n}{k})$* [21].

Proof: Take an error correcting code that corrects a single error.[22] The Hamming distance (number of differ bits) between any two code words in such a code is at least 3. Therefore, the set of code words is an admissible collection. Now, it is known that there exist such codes with $\Omega(2^n/n)$ code words. The only thing left is to show that an appropriate fraction of these code words have weight k (i.e., k 1's out of n bits). This can be easily achieved by renaming at random for each coordinate 0 and 1. I.e. choose a random n-bit vector r and XOR each codeword (bitwise) with r. It is easy to see that the Hamming distance between any two code words remains the same after being Xor-ed with r. Therefore, the obtained code is still admissible. Now each single codeword is uniformly mapped to the whole cube, and thus its probability of having weight k (after the Xor-ing) is exactly $\binom{n}{k}/2^n$. Thus the expected number

[21]We say that $f(n) = \Omega(g(n))$ if and only if $\exists c > 0 \; \exists n_0 \; \forall n > n_0 \; c \cdot f(n) \leq g(n)$. Thus, the Ω notation is used to denote an asymptotic lower bound.

[22]This is a set of binary strings such that any two strings differ at least at 3 locations. Given a string from this family with at most one error (bit flip), we can tell what is the original string -and correct this single error - as there is only one string in the set that can be the original string).

of weight k words (in the code after Xor-ing) is the product of this probability and the number of code words, which gives $\Omega(\binom{n}{k}/n)$, and for some r this expectation is achieved or exceeded. □

For $k = n/2$ we can construct an exponential size admissible collection, which by Theorem 4.7 can be used to build a technology with exponential size orbit.

Corollary 4.9 *There exists a technology (t, c) with orbit of size $\Omega(\frac{2^n}{n\sqrt{n}})$.*

Proof: By Stirling approximation [23] $\binom{2k}{k} \approx \frac{2^{2k}}{\sqrt{\pi k}}$, thus for n=2k we derive that there exists an orbit of size $\Omega\left(\binom{n}{n/2} \cdot \frac{1}{n}\right) = \Omega\left(\frac{2^n}{\sqrt{\pi \cdot n/2}} \cdot \frac{1}{n}\right) = \Omega(\frac{2^n}{n\sqrt{n}})$ □

Thus, we are able to construct a technology with exponential orbit, but this technology is not a network technology or a structured technology.

Open Question 4.10 *Is there a network technology with exponential orbit? Is there a structured technology with exponential orbit?*

So far, we have not seen examples of series-parallel networks [24] whose orbit size is larger than $n + 1$.

Open Question 4.11 *How big can the orbit size of a series-parallel network be?*

We make the first step toward a solution of this question by showing that the size of the orbit of a conjunction of two disjoint networks (taking the two in a serial) is at most the sum of the two networks' orbit sizes.

Let h and g be two Boolean functions on disjoint inputs H and G, respectively, and let $f = h \bigwedge g$ (i.e., take their networks in series). The optimal contract for f for some v, denoted by S, is composed of some agents from the h-part and some from the g-part, call them $T = H \cap S$ and $R = G \cap S$ respectively.

Lemma 4.12 *Let S be an optimal contract for $f = h \bigwedge g$ on v. Then, $T = H \cap S$ is an optimal contract for h on $v \cdot t_g(R)$, and $R = G \cap S$ is an optimal contract for g on $v \cdot t_h(T)$.*

Proof: We will abuse the notation and use the functions to denote the technology as well ($f(S)$ will denote the probability of success with the function f and the contract S).

The utility of the principal with value v from S when using technology f is

$$\begin{aligned} U(S,v) &= f(S)\left(v - \sum_{i \in S} \frac{c_i}{\Delta_i^f(S \setminus i)}\right) \\ &= h(T) \cdot g(R) \cdot \left(v - \sum_{i \in T} \frac{c_i}{\Delta_i^f(S \setminus i)} + \sum_{i \in R} \frac{c_i}{\Delta_i^f(S \setminus i)}\right) \end{aligned}$$

[23]Stirling approximation for $n!$ is $n! \approx \sqrt{2\pi n} \cdot (n/e)^n$.

[24]The formal inductive definition of series-parallel networks appears in appendix B.

For any $i \in T$, $\Delta_i^f(S \setminus i) = h(1, T \setminus i) \cdot g(R) - h(0, T \setminus i) \cdot g(R) = g(R) \cdot \Delta_i^h(T \setminus i)$. Similarly, for any $i \in R$, $\Delta_i^f(S \setminus i) = h(T) \cdot \Delta_i^g(R \setminus i)$.

We derive that

$$\begin{aligned} U(S, v) &= h(T) \cdot \left(g(R) \cdot v - \sum_{i \in T} \frac{g(R) \cdot c_i}{g(R) \cdot \Delta_i^h(T \setminus i)} \right) + h(T) \cdot g(R) \cdot \sum_{i \in R} \frac{c_i}{h(T) \cdot \Delta_i^g(R \setminus i)} \\ &= h(T) \left(g(R) \cdot v - \sum_{i \in T} \frac{c_i}{\Delta_i^h(T \setminus i)} \right) + g(R) \cdot \sum_{i \in R} \frac{c_i}{\Delta_i^g(R \setminus i)} \end{aligned}$$

the first term is maximized exactly at a set T that is optimal for h on the value $g(R) \cdot v$, while the second term is independent of T and h. We conclude that S is an optimal contract for f on v if and only if T is an optimal contract for h on $v \cdot t_g(R)$. The proof that R is an optimal contract for g on $v \cdot t_h(T)$ is similar and is omitted. □

Lemma 4.13 *The real function $v \rightarrow t_h(T)$, where T is the $h-$part of an optimal contract for f on v, is monotone non-decreasing (and similarly for the function $v \rightarrow t_g(R)$).*

Proof: Let $S_1 = T_1 \cup R_1$ be the optimal contract for f on v_1, and let $S_2 = T_2 \cup R_2$ be the optimal contract for f on $v_2 < v_1$. By Lemma 3.3 $f(S_1) \geq f(S_2)$, and since $f = g \cdot h$, $f(S_1) = h(T_1) \cdot g(R_1) \geq h(T_2) \cdot g(R_2) = f(S_2)$. Assume in contradiction that $h(T_1) < h(T_2)$, then since $h(T_1) \cdot g(R_1) \geq h(T_2) \cdot g(R_2)$ this implies that $g(R_1) > g(R_2)$. By Lemma 4.12, T_1 is optimal for h on $v_1 \cdot g(R_1)$, and T_2 is optimal for h on $v_2 \cdot g(R_2)$. As $v_1 > v_2$ and $g(R_1) > g(R_2)$, T_1 is optimal for h on a larger value than T_2, thus by Lemma 3.3, $h(T_1) \geq h(T_2)$, a contradiction. □

Based on Lemma 4.12 and Lemma 4.13, we obtain the following Lemma.

Lemma 4.14 *Let g and h be two Boolean functions on disjoint inputs and let $f = g \bigwedge h$ (i.e., take their networks in series). Suppose x and y are the respective orbit sizes of g and h; then, the size of the orbit of f is at most $x + y - 1$.*

Proof: By Lemma 4.12 an optimal contract for f is constructed from optimal contracts for h and g. By Lemma 4.13 the orbit of h consists of sets $T_1, T_2, \ldots, T_y$ with increasing success probabilities (because of consistent tie breaking). Similarly, the orbit of g consists of sets $R_1, R_2, \ldots, R_x$ with increasing success probabilities.

The orbit of f consists of contracts of the form $T_i \cup R_j$. If we order the orbit of f by increasing success probabilities: $S_1, S_2, \ldots S_z$, where $S_l = T_{i(l)} \cup R_{j(l)}$, then By Lemma 4.13 both $i(l)$ and $j(l)$ are monotonically non decreasing, and at least one of them must increase when we move from l to $l + 1$. As for any l , $x \geq i(l) \geq 1$ and $y \geq j(l) \geq 1$, the orbit size of f is of size at most $x + y - 1$. □

By induction we get the following corollary.

Corollary 4.15 *Assume that* $\{(g_j, c_j)\}_{j=1}^m$ *is a set of anonymous technologies on disjoint inputs, each with identical agent cost (all agents of technology* g_j *has the same cost* c_j*). Then the orbit of* $f = \bigwedge_{j=1}^m g_j$ *is of size at most* $(\sum_{j=1}^m n_j) - 1$*, where* n_j *is the number of agents in technology* g_j *(the orbit is linear in the number of agents).*

Proof: As the technology (g_j, c_j) is anonymous the size of its orbit is at most n_j+1. By induction we get that he size of f is at most $\sum_{j=1}^m (n_j+1)-(m-1) = (\sum_{j=1}^m n_j)-1$. □

In particular, this holds for *AOO* technology where each OR-component is anonymous.

It would also be interesting to consider a disjunction of two Boolean functions.

Open Question 4.16 *Does Lemma 4.14 hold also for the Boolean function* $f = g \bigvee h$ *(i.e., when the networks* g, h *are taken in parallel)?*

We conjecture that this is indeed the case, and that the corresponding Lemmas 4.12 and 4.14 exist for the OR case as well. If this is true, this will show that series-parallel networks have polynomial size orbit.

5 Robustness to Coordinated Deviations

A solution concept defines acceptable solutions for games that are played by rational agents. The choice of the solution concept greatly affects the analysis and the results of the question in hand. For example, Winter (Winter 2004) showed that the optimal contract that induces a *unique* Nash equilibrium in IRS technologies (see definition 3.11) must have discriminatory payments. In contrast, our weaker requirement for the existence of a Nash equilibrium trivially results in a symmetric optimal contract (i.e., a contract in which all the payments to investing agents are equal).

A *strong equilibrium*, due to Aumann (Aumann 1959), is a refinement of Nash equilibrium that requires that no subgroup of players (henceforth *coalition*) can coordinate a joint deviation such that every member of the coalition strictly improves his utility. This is a much stronger solution concept than Nash, since it is robust against deviations in groups of any size, which a Nash equilibrium is robust only against unilateral deviations.

We observe that the payments that induce the strategy profile S^* as the best Nash equilibrium (i.e., the Nash equilibrium that maximizes the principal's utility) actually induce S^* as a *strong equilibrium*. Moreover, we show that under these payments S^* becomes the *unique* strong equilibrium almost always[25]. This is a generalization of

[25]except for, possibly, a profile in which a single agent in S^* does not exert effort. However, assuming that the principal breaks ties in favor of smaller groups, for an arbitrary small increase in the rewards, S^* is the unique strong equilibrium. In addition, note that the agent that does not exert effort is indifferent between exerting effort and shirking and the agents that exert effort are worse-off.

Winter's result (Winter 2004), which shows that for any monotone technology, the equilibrium in which all agents exert effort is the unique strong equilibrium. The definition of a strong equilibrium follows.

Definition 5.1 *A strategy profile $a \in A$ is a* strong equilibrium (SE) *if there does not exist any coalition $\Gamma \subseteq N$ and a strategy profile $a'_\Gamma \in \times_{i \in \Gamma} A_i$ such that for any $i \in \Gamma$, $u_i(a'_{-\Gamma}, a_\Gamma) > u_i(a)$.*

Theorem 5.2 *Under the optimal payments that induce the optimal contract S^* (i.e., $p_i = \frac{c_i}{\Delta_i(a_{-i})}$ for any $i \in S^*$, and $p_i = 0$ for any $i \in N \setminus S^*$), S^* is a strong equilibrium. Moreover, S^* is the unique strong equilibrium, except for, possibly, an action profile $S^* \setminus i$ for some agent $i \in S^*$.*

Proof: It is clear that any agent for which $p_i = 0$ cannot improve his utility by exerting effort. Therefore, we can restrict attention to deviations in which some of the agents in S^* do not exert effort. Let $T \subset S^*$ denote the set of agents that do not deviate (i.e., agents that continue to exert effort.) $S^* \setminus T$ are the agents that deviate. For any agent $i \in S^* \setminus T$ (i.e., an agent that exerts effort in S^* but not in T), i's utility when the set S^* exerts effort is given by

$$u_i(S^*) = p_i(S^*) \cdot t(S^*) - c_i = p_i(S^*) \cdot t(S^* \setminus i)$$

As $p_i(S^*)$ is chosen to make i indifferent between exerting effort and shirking. This should be compared to his utility when the set T exerts effort, which is given by $u_i(T) = p_i(S^*) \cdot t(T)$. But since $T \subset S^* \setminus i$, it follows that $t(T) \leq t(S^* \setminus i)$. Therefore, i cannot improve his utility by the deviation, thus S^* is a strong equilibrium.

To prove the uniqueness of S^*, consider an action profile $T \subseteq S^*$ such that $|T| \leq |S^*| - 2$ (i.e., at least two agents in S^* deviate and do not exert effort). The utility of each agent $i \in S^* \setminus T$ is $u_i(T) = p_i(S^*) \cdot t(T)$. We will show that if all the agents in $S^* \setminus T$ deviate to $a_i = 1$, each member of the coalition strictly improves his utility. Under this deviation, the utility of each agent $i \in S^* \setminus T$ is given by $u_i(S^*) = p_i(S^*) \cdot t(S^* \setminus i)$, which is greater than $p_i(S^*) \cdot t(T)$ since $|T| \leq |S^*| - 2$. Therefore, all the agents in the coalition strictly improve their utilities, thus T is not a strong equilibrium. □

To emphasize the uniqueness of the equilibrium, we observe that the strategy profile S^* is unique not only with respect to the strong equilibrium solution concept, but also to the related, but weaker, solution concept of *coalition-proof* equilibrium. The latter requires resilience against deviations that are, themselves, resilient against further deviations (the formal definition is recursive and appears in (Bernheim, Peleg, and Whinston 1987))[26].

[26] Winter (Winter 2004) demonstrates the same result for an equilibrium in which all agents exert effort

Observation 5.3 *Under the optimal payments that induce the optimal contract* $S^* = \{i|a_i = 1\}$ *(i.e.,* $p_i = \frac{c_i}{\Delta_i(a_{-i})}$ *for any* $i \in S^*$*, and* $p_i = 0$ *for any* $i \in N \setminus S^*$*),* S^* *is the unique coalition-proof equilibrium, except for, possibly, an action profile* $S^* \setminus i$ *for some agent* $i \in S^*$*.*

Proof: Theorem 5.2 shows that for any $T \subseteq S^*$ s.t. $|T| \leq |S^*| - 2$, if all the agents in $S^* \setminus T$ deviate to $a_i = 1$, each member of the coalition strictly improves his utility. Thus, in order to show that T is not a coalition-proof equilibrium, we only need to show that the deviation above is itself resilient to further deviations. But this follows from Theorem 5.2, which assures that S^* is a strong equilibrium, thus resilient to deviations of any coalition. □

6 Algorithmic Aspects

Throughout this paper we have been analyzing the structure of the optimal contract for various production technology functions. In this section, we are interested in a different, yet related issue, which is the algorithmic aspects of computing the best contract. While the two questions are different in their nature, our analysis throughout the paper sheds some light on the latter.

The computational hardness of finding the optimal contract depends on the representation of the technology and how it is being accessed.

We first consider the general model where the technology function is given by an arbitrary monotone function t (with rational values), and we then consider the case of structured technologies given by a network representation of the underlying Boolean function.

6.1 Binary-Outcome Binary-Action Technologies

Here we assume that we are given a technology and value v as the input, and our output should be the optimal contract, i.e. the set $S^*(v)$ of agents to be contracted and the contract p_i for each $i \in S^*(v)$. In the general case, the success function t is of size exponential in n, the number of agents (as for each of 2^n subsets we need to specify its success probability), and we will need to deal with that. In the special case of anonymous technologies, the description of t is only the $n + 1$ numbers $t_0, \ldots, t_n$, and in this case our analysis in section 3 completely suffices for computing the optimal contract.

Proposition 6.1 *Given as input the full description of a technology (the values* $t_0, \ldots, t_n$ *and the identical cost* c *for an anonymous technology, or the value* $t(S)$ *for all the* 2^n *possible subsets* $S \subseteq N$ *of the players, and a vector of costs* $\vec{c}$ *for non-anonymous technologies), the following can all be computed in polynomial time:*

- *The orbit of the technology in both the hidden-actions and the observable-actions cases.*
- *An optimal contract for any given value v, for both the hidden-actions and the observable-actions cases.*
- *The social and principal's price of unaccountability.*

Proof: We prove the claims for the non-anonymous case, the proof for the anonymous case is similar.

We first show how to construct the orbit of the technology. To construct the orbit we find all transition points and the sets that are in the orbit. The empty contract is always optimal for $v = 0$. Assume that we have calculated the optimal contracts and the transition points up to some transition point v for which S is an optimal contract with the highest success probability. We show how to calculate the next transition point and the next optimal contract.

By Lemma 3.3 the next contract on the orbit (for higher values) has a higher success probability (there are no two sets with the same success probability on the orbit). We calculate the next optimal contract by the following procedure. We go over all sets T such that $t(T) > t(S)$, and calculate the value for which the principal is indifferent between contracting with T and contracting with S. The minimal indifference value is the next transition point and the contract that has the minimal indifference value is the next optimal contract. Linearity of the utility in the value and monotonicity of the success probability of the optimal contracts ensure that the above works. Clearly the above calculation is polynomial in the input size.

Once we have the orbit, it is clear that an optimal contract for any given value v can be calculated. We find the largest transition point that is not larger than the value v, and the optimal contract at v is the set with the higher success probability at this transition point.

Finally, as we can calculate the orbit of the technology in both the hidden-actions and the observable-actions cases in polynomial time, we can find the social and principal's price of unaccountability in polynomial time. By Lemma 3.4 the social and principal's price of unaccountability are obtained at some transition points, so we only need to go over all transition points, and find the one with the maximal social welfare ratio and the one with the maximal principal's utility ratio. □

A more interesting question is whether if given the function t as a black box (i.e., a device which outputs $t(a)$ given the profile a as an input), we can compute the optimal contract in time that is polynomial in n. We can show that, in general, this is not the case:

Theorem 6.2 *Given as input a black box for a success function t (when the costs are identical), and a value v, the number of queries that is needed, in the worst case, to find the optimal contract is exponential in n.*

Proof: Consider the following family of technologies. For some small $\epsilon > 0$ and $k = \lceil n/2 \rceil$ we define the success probability for a given set T as follows. If $|T| < k$, then $t(T) = |T| \cdot \epsilon$. If $|T| > k$, then $t(T) = 1 - (n - |T|) \cdot \epsilon$. For each set of agents $\hat{T}$ of size k, the technology $t_{\hat{T}}$ is defined by $t(\hat{T}) = 1 - (n - |\hat{T}|) \cdot \epsilon$ and $t(T) = |T| \cdot \epsilon$ for any $T \neq \hat{T}$ of size k.

For the value $v = c \cdot (k + 1/2)$, the optimal contract for $t_{\hat{T}}$ is $\hat{T}$ (for the contract $\hat{T}$ the utility of the principal is about $v - c \cdot k = 1/2 \cdot c > 0$, while for any other contract the utility is negative).

If the algorithm queries about at most $\binom{n}{\lceil n/2 \rceil} - 2$ sets of size k, then it cannot always determine the optimal contract (as any of the sets that it has not queried about might be the optimal one). We conclude that $\binom{n}{\lceil n/2 \rceil} - 1$ queries are needed to determine the optimal contract, and this is exponential in n (it is about $\frac{2^n}{\sqrt{\pi n/2}}$ by Stirling approximation). □

6.2 Structured Technologies

In this section we will consider the natural representation of networks for the underlying Boolean function. Thus the problem we address will be:

The Optimal Contract Problem for Network Technologies:

Input: A network $G = (V, E)$, with two specific vertices s, t; rational values γ_e, δ_e for each player $e \in E$ (and $c_e = 1$), and a rational value v.

Output: A set $S^*(v)$ of agents who should be contracted in an optimal contract for value v.

Let $t(E)$ denote the probability of success when each edge succeeds with probability δ_e. We first notice that even computing the value $t(E)$ is a hard problem: it is called the network reliability problem and is known to be $\#P - hard$ (Provan and Ball 1983)[27]. Just a little effort will reveal that our problem is not easier:

Theorem 6.3 *The Optimal Contract Problem for network technologies is #P-hard (under Turing reductions).*

In the proof (presented in Appendix C), we use a technique called reduction, which proves the hardness of our problem based on the well-known hardness of another problem - the network reliability problem in our case.

We conclude that computing the optimal contract is hard even for the subclass of network technologies. This result suggest two natural research directions. The first

[27] $\#P$ is a complexity class in computational complexity theory, which is the set of counting problems associated with the decision problems in the set NP. NP problems are decision problems that can be solved in polynomial time on non-deterministic Turing machine. The corresponding $\#P$ problem, which is harder, counts the number of solutions.

avenue is to study families of technologies whose optimal contracts can be computed in polynomial time. A possible candidate for the first direction is the family of series-parallel networks, for which the network reliability problem (computing the value of t) is polynomial. The second avenue is to explore approximation algorithms for the optimal contract problem. That is, algorithms that run in polynomial time and find the optimal contract up to a small constant factor.

Open Question 6.4 *Can the optimal contract problem for series-parallel network technologies be solved in polynomial time?*

We can only show that if we have two technologies for which we can calculate the orbit, we can also calculate the orbit of the technology constructed from the two technology in series (conjunction of the two). In particular, this shows that we can handle *AOO* networks (when the components are any anonymous OR technologies, not necessarily identical), by induction and by the fact that we can calculate the orbit of an anonymous *OR* technology in polynomial time.

Lemma 6.5 *Given two technology g and h for which we can calculate the orbit in polynomial time, one can also calculate the orbit for $h \bigwedge g$ in polynomial time.*

Proof: Give the orbits of h and g, we calculate all the transition points and the optimal contract at each interval between any two consecutive points (the orbit) of $h \bigwedge g$.

Let $X = (X_1, X_2, \ldots, X_x)$ be the set of optimal contracts for h, sorted in increasing success probabilities ($h(X_i) < h(X_{i+1})$). Similarly, let $Y = (Y_1, Y_2, \ldots, Y_y)$ be the set of optimal contracts for g sorted by increasing probabilities.

Using Lemma 4.12 we show how to combine the the two orbits to create the orbit of $h \bigwedge g$. By the Lemma, the only candidates for optimal contracts are $(X_i, Y_j) \in (X, Y)$. We denote the indexes of the $l-th$ optimal contract for h by $(i(l), j(l))$. The first optimal contract for $h \bigwedge g$ is $(X_{i(1)}, Y_{j(1)}) = (X_1, Y_1)$, which is the $(0, 0)$ contract. Assume that we calculated the optimal contracts for $h \bigwedge g$ up to the l-th contract. We calculate the $l+1$-th optimal contract by the following procedure. We go over all i, j pairs such that $i \geq i(l)$ and $j \geq j(l)$, and calculate the value for which the principal is indifferent between the contract $(X_i(l), Y_j(l))$ and the contract (X_i, Y_j). The contract that has the minimal indifference value is the next optimal contract, and we continue to find the next optimal contract by the same procedure.

Clearly, finding each additional optimal contract can be done in polynomial time, as the number of optimal contracts in $h \bigwedge g$ grows at most linearly in the size of the orbits of h and g. □

7 Conclusions

In cases where a principal should motivate a team of agents to exert costly effort when he can only observe the final outcome, the production technology greatly affects the structure of the optimal contract. In this paper, we focus on the change in the optimal contract as the principal's value from the project increases (the "orbit"). we compute the "orbit" for specific technologies such as AND, OR and AND-of-ORs, and demonstrate that different technologies may result in very different "orbit" structures. While we are able to analyze the structure of the transitions for simple technologies, we show that computing the optimal contract in general is hard. In addition, we quantify the worse multiplicative loss that is incurred due to the inability of the principal to observe individual actions. We show that this loss may be unbounded for AND technologies (which exhibit complementarities between agents), while it can be bounded by a small constant for OR technologies (which exhibit substitutabilities between agents).

Acknowledgments. This work is supported by the Israel Science Foundation, the USA-Israel Binational Science Foundation, the Lady Davis Fellowship Trust, and by a National Science Foundation grant number ANI-0331659.

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A Analysis of Some Anonymous Technologies

A.1 Transitions: Proofs

Lemma 3.5 *Let* $f(x) = \frac{a \cdot x - b}{c \cdot x - d}$ *be a function of* x*, and assume that* $c > 0$*. Let*

$\overline{x} \geq \underline{x} > 0$ *be two points for which* $c\underline{x} - d > 0$. *Then the supremum of* f *on the range* $[\underline{x}, \overline{x}]$ *is obtained at either* $\underline{x}$ *or* $\overline{x}$. *Additionally, if* $a = c > 0$, $d > b$ *and for some* $\overline{x} > 0$ *it holds that* $a\overline{x} - d > 0$ *then the supremum of* f *on* $[\overline{x}, \infty)$ *is obtained at* $\overline{x}$.

Proof: As f is a continuous function (recall that $cx - d > 0$ on the range as $c > 0$ and $c\underline{x} - d > 0$) on a compact range, its supremum is obtained.

In order to find the maximum of f, we take the first derivative and equate to zero:

$$\frac{\partial f}{\partial x} = \frac{a(cx-d) - c(ax-b)}{(cx-d)^2} = \frac{bc - ad}{(cx-d)^2} = 0$$

which holds if and only if $bc = ad$. As this equality is independent of x, it either hold for any x (and in particular for $\overline{x}$ and $\underline{x}$), or for no x. If it holds for no x, then the maximum must be obtained at either $\underline{x}$ or $\overline{x}$.

If if $a = c > 0$, $d > b$ and for $\overline{x}$, $a\overline{x} - d > 0$ then f is continuous on $[\overline{x}, \infty)$. Additionally, $\frac{\partial f}{\partial x} = \frac{bc-ad}{(cx-d)^2} = \frac{a\cdot(b-d)}{(ax-d)^2} < 0$ for any $x \geq \overline{x}$. Thus the function f monotonically decreases on $[\overline{x}, \infty)$ and supremum of f is obtained at $\overline{x}$. □

The following characterization holds for both the hidden-actions and the observable-actions cases. Let Q_k be the total expected payment to all agents in the best (minimal payment) contract in which k agents exert effort (for the observable-actions case $Q_k = c \cdot k$ and for the hidden-actions case $Q_k = \frac{c \cdot k \cdot t_k}{t_k - t_{k-1}}$). Note that Q_k is only a function of the technology t.

Theorem A.1 *An anonymous technology* (t, c) *has*

- *a single transition if and only if it exhibits Over-Payment, that is for any* $k \in \{1, \ldots, n\}$ *it holds that*
$$\frac{Q_k}{Q_n} > \frac{t_k - t_0}{t_n - t_0}$$
- *all* n *transitions at different values if and only if it exhibits IRMP, that is for any* $k \in \{1, 2, \ldots, n-1\}$ *it holds that*
$$\frac{Q_{k+1} - Q_k}{t_{k+1} - t_k} > \frac{Q_k - Q_{k-1}}{t_k - t_{k-1}}$$

Proof: We begin by a lemma that characterize the two cases by some properties of the principal utilities, and than show that these properties are equivalent to the properties presented by the theorem.

We use the following corollary from Lemma 3.3.

Observation A.2 *For any anonymous technology* (t, c), *assume that contracting with* k_1 *agents is optimal for* v_1, *and contracting with* k_2 *agents is optimal for* v_2. *If* $v_1 > v_2$ *then* $k_1 \geq k_2$.

Given an anonymous technology (t, c), let $u(k, v)$ be the utility at value v, when optimally contracting with k agents[28], and let $v_{i,j}$ be the value v in which the principal is indifferent between contracting with either i agents or with j agents (by the definition of $v_{i,j}$, $u(j, v_{i,j}) = u(i, v_{i,j})$. That is $t_i \cdot v_{i,j} - Q_i = t_j \cdot v_{i,j} - Q_j$, or equivalently $v_{i,j} = \frac{Q_j - Q_i}{t_j - t_i}$).

Lemma A.3 *An anonymous technology (t, c) has*

1. *all n transitions at different values if and only if $u(k, v_{k-1,k}) > u(k+1, v_{k-1,k})$ for all $k \in \{1, 2, \ldots, n-1\}$, and*
2. *a single transition (from 0 agents to n agents) if and only if $u(n, v_{0,n}) > u(k, v_{0,n})$ for all $k \in \{1, 2, \ldots, n-1\}$.*

Proof: First we show that a technology t has all n transitions at different values if and only if for all $k \in \{1, 2, \ldots, n-1\}$ it holds that $u(k, v_{k-1,k}) > u(k+1, v_{k-1,k})$.

case if: Assume that $u(k, v_{k-1,k}) > u(k+1, v_{k-1,k})$ for all $k \in \{1, 2, \ldots, n-1\}$. By Lemma A.5 this condition is equivalently to the IRMP condition, that is $\frac{Q_{k+1} - Q_k}{t_{k+1} - t_k} > \frac{Q_k - Q_{k-1}}{t_k - t_{k-1}}$ for any $k \in \{1, 2, \ldots, n-1\}$. As $v_{i,j} = \frac{Q_j - Q_i}{t_j - t_i}$, IRMP is equivalent to $v_{k,k+1} > v_{k-1,k}$ for any $k \in \{1, 2, \ldots, n-1\}$. We next show that at any value $v \in (v_{k-1,k}, v_{k,k+1})$ for some $k \in \{1, 2, \ldots, n-1\}$, contracting with k agents is optimal for the principal, and this is the only optimal contract for him. Together with the fact that 0 is optimal for value of 0, and n is optimal for values larger that $v_{n-1,n}$, we conclude that all transitions occur.

We first show by induction that at any value $v \in (v_{k-1,k}, v_{k,k+1})$ for some $k \in \{1, 2, \ldots, n-1\}$, contracting with k agents has higher utility than contracting with $j < k$ agents. Clearly the claim holds for $k = 0$. Assume that we have proven the claim of to $k-1$, which means that at $v_{k-1,k}$ contracting with $k-1$ agents has higher utility than contracting with $j < k-1$ agents. By definition of $v_{k-1,k}$, contracting with k agents is better than contracting with $k-1$ agents for any value larger than $v_{k-1,k}$, thus contracting with k agents is better than contracting with any $j < k$ agents.

A similar argument shows by induction that at any value $v \in (v_{k-1,k}, v_{k,k+1})$ for some $k \in \{1, 2, \ldots, n-1\}$, contracting with k agents has higher utility than contracting with $j > k$ agents. This is proven by starting from $k = n$ agents and going backwards. Combining the two claims we derive that contracting with k agents at a value $v \in (v_{k-1,k}, v_{k,k+1})$ achieves higher utility for the principal than contracting with any other number of agents.

case only if: Assume that t has all n transitions, and at different values. For all $k \in \{1, 2, \ldots, n-1\}$, by Observation A.2 k is not optimal for $v < v_{k-1,k}$ and is an

[28]Note that in the hidden-actions case, $u(k, v)$ denotes the utility of the principal, and in the observable-actions case, the principal's utility coincides with the social welfare.

optimal contract at $v_{k-1,k}$, thus $u(k, v_{k-1,k}) \geq u(k+1, v_{k-1,k})$. If $u(k, v_{k-1,k}) = u(k+1, v_{k-1,k})$, then (again by the same Observation), k is not optimal for any $v > v_{k-1,k}$, contradicting the transitions in different values. We conclude that $u(k, v_{k-1,k}) > u(k+1, v_{k-1,k})$ for all $k \in \{1, 2, \ldots, n-1\}$.

Next we show that a technology t has a single transition (from 0 agents to n agents) if and only if $u(n, v_{0,n}) > u(k, v_{0,n})$ for all $k \in \{1, 2, \ldots, n-1\}$.

case if: Assume that $u(n, v_{0,n}) > u(k, v_{0,n})$ for all $k \in \{1, 2, \ldots, n-1\}$. By Observation A.2, since n is optimal contract at $v_{0,n}$, for any $v > v_{0,n}$, n is the only optimal contract. On the other hand, as 0 is optimal at $v_{0,n}$, by Observation A.2, if $k > 0$ was optimal for any $v < v_{0,n}$ then 0 was not optimal for $v_{0,n}$. Thus for any $v < v_{0,n}$, 0 is the only optimal contract. As at $v_{0,n}$ the only optimal contracts are 0 and n, this implies that any $k \in \{1, 2, \ldots, n-1\}$ is never optimal, thus t has a single transition.

case only if: Assume that t has a single transition from 0 to n at $v_{0,n}$. This implies that for any $v \leq v_{0,n}$, 0 is the optimal contract, thus $u(0, v) > u(k, v)$ for all $k \in \{1, 2, \ldots, n-1\}$. for any $v \geq v_{0,n}$, n is the optimal contract, thus $u(n, v) > u(k, v)$ for all $k \in \{1, 2, \ldots, n-1\}$. We conclude that at $v = v_{0,n}$, $u(n, v_{0,n}) = u(0, v_{0,n}) > u(k, v_{0,n})$ for all $k \in \{1, 2, \ldots, n-1\}$. □

Next we show that Condition 1 in Lemma A.3 is equivalent to the Over-Payment condition.

Lemma A.4 *$u(n, v_{0,n}) > u(k, v_{0,n})$ for any $k \in \{1, 2, \ldots, n-1\}$ if and only if $\frac{Q_k}{Q_n} > \frac{t_k - t_0}{t_n - t_0}$ for any $k \in \{1, 2, \ldots, n-1\}$.*

Proof: For all $k \in \{1, 2, \ldots, n-1\}$

$u(n, v_{0,n}) > u(k, v_{0,n}) \;\Leftrightarrow\; t_n \cdot v_{0,n} - Q_n > t_k \cdot v_{0,n} - Q_k \;\Leftrightarrow\; (t_n - t_k) \cdot v_{0,n} > Q_n - Q_k$

As $v_{0,n} = \frac{Q_n}{t_n - t_0}$, the above happens if and only if

$$(t_n - t_k) \cdot \frac{Q_n}{t_n - t_0} > Q_n - Q_k \quad \Leftrightarrow \quad \frac{Q_k}{Q_n} > \frac{t_k - t_0}{t_n - t_0}$$

which is what we wanted to prove. □

Next we show that Condition 2 in Lemma A.3 is equivalent to the IRMP condition.

Lemma A.5 *$u(k, v_{k-1,k}) > u(k+1, v_{k-1,k})$ for any $k \in \{1, 2, \ldots, n-1\}$ if and only if $\frac{Q_{k+1} - Q_k}{t_{k+1} - t_k} > \frac{Q_k - Q_{k-1}}{t_k - t_{k-1}}$ for any $k \in \{1, 2, \ldots, n-1\}$.*

Proof: For all $k \in \{1, 2, \ldots, n-1\}$

$$u(k, v_{k-1,k}) > u(k+1, v_{k-1,k}) \;\Leftrightarrow\; t_k \cdot v_{k-1,k} - Q_k > t_{k+1} \cdot v_{k-1,k} - Q_{k+1}$$

As $v_{k-1,k} = \frac{Q_k - Q_{k-1}}{t_k - t_{k-1}}$, the above happens if and only if

$$Q_{k+1} - Q_k > (t_{k+1} - t_k) \cdot v_{k-1,k} = (t_{k+1} - t_k) \cdot \frac{Q_k - Q_{k-1}}{t_k - t_{k-1}}$$

$$\Updownarrow$$

$$\frac{Q_{k+1}-Q_k}{t_{k+1}-t_k} > \frac{Q_k-Q_{k-1}}{t_k-t_{k-1}}$$

which is what we wanted to prove. $\Box$

the theorem is now a direct result from the claims above. $\Box$

We now turn to the observable-actions case and show that the Over-Payment and IRMP conditions are equivalent to DRS and UPC, respectively.

Observation A.6 *For the observable-actions case, an anonymous technology (t, c):*

1. *exhibits Over-Payment if and only if it exhibits UPC.*
2. *exhibits IRMP if and only if it exhibits DRS.*

Proof: As for the observable-actions case the total expected payment to all agents in the best (minimal payment) contract in which k agents exert effort is $c \cdot k$, this means that $Q_k = c \cdot k$ for any k. Thus, for the observable-actions case $\frac{Q_k}{Q_n} = \frac{k}{n}$, which implies the first claim, and $Q_{k+1} - Q_k = Q_k - Q_{k-1} = c$ which implies the second claim. $\Box$

Observation A.7 *For the hidden-actions case, an anonymous technology (t, c) it holds that $Q_k = \frac{c \cdot k \cdot t_k}{t_k - t_{k-1}}$. Thus technology (t, c)*

1. *has all n transitions if and only if for any $k \in \{1, \ldots, n\}$ it holds that $\frac{Q_k}{Q_n} > \frac{t_k - t_0}{t_n - t_0}$ (exhibits Over-Payment), for Q_k as defined above.*
2. *has a single transition if and only if for any $k \in \{1, 2, \ldots, n-1\}$ it holds that $\frac{Q_{k+1}-Q_k}{t_{k+1}-t_k} > \frac{Q_k - Q_{k-1}}{t_k - t_{k-1}}$ (exhibits IRMP) for Q_k as defined above.*

Proof: As for the hidden-actions case the total expected payment to all agents in the best (minimal payment) contract in which k agents exert effort is $\frac{c \cdot k \cdot t_k}{t_k - t_{k-1}}$, this observation is derived directly from Theorem A.1. $\Box$

Lemma A.8 *An anonymous technology that exhibits IRS also exhibits UPC, but not vise versa.*

Proof: Assume in contradiction that the technology exhibits IRS and not UPC. Then there is a $k \in \{1, \ldots, n\}$ s.t. $\frac{k}{n} \leq \frac{t_k - t_0}{t_n - t_0}$. As the technology exhibits IRS, it holds that for any $i \in \{2, \ldots, n\}$, $t_i - t_{i-1} > t_{i-1} - t_{i-2}$, thus for any $i \in \{1, \ldots k-1\}$ it holds that $t_k - t_{k-1} > t_i - t_{i-1}$. Therefore by summation

$$(k-1)(t_k - t_{k-1}) = \sum_{i=1}^{k-1}(t_k - t_{k-1}) > \sum_{i=1}^{k-1}(t_i - t_{i-1}) = t_{k-1} - t_0$$

Equivalently $t_k - t_{k-1} > \frac{t_k - t_0}{k}$. As for k we assumed that $\frac{k}{n} \leq \frac{t_k - t_0}{t_n - t_0}$, it implies that $t_k - t_{k-1} > \frac{t_n - t_0}{n}$. As for any $i \in \{(k+1), \ldots, n\}$, by IRS $t_k - t_{k-1} < t_i - t_{i-1}$,

$$(n-k)(t_k - t_{k-1}) = \sum_{i=k+1}^{n} (t_k - t_{k-1}) < \sum_{i=k+1}^{n} (t_i - t_{i-1}) = t_n - t_k$$

which implies that $t_n - t_k > \frac{n-k}{n}(t_n - t_0)$. With the assumption that $\frac{k}{n}(t_n - t_0) \leq t_k - t_0$ we observe that $t_n - t_0 = (t_n - t_k) + (t_k - t_0) > t_n - t_0$, a contradiction.

Finally, we observe that the technology with 3 agents defined as $t_0 = 1/10, t_1 = 3/10, t_2 = 45/100, t_3 = 1$, exhibits UPC but not IRS. It exhibits UPC as $1/3 > (3/10 - 1/10)/(1 - 1/10) = 2/9$ and $2/3 > (45/100 - 1/10)/(1 - 1/10) = 7/18$. It does not exhibits IRS as $3/20 = 45/100 - 3/10 < 3/10 - 1/10 = 1/5$. $\square$

Observation A.9 *There exists a technology that exhibits UPC but does not exhibit over-payment (OP). Additionally, there exists a technology that exhibits IRMP but does not exhibit DRS.*

Proof: The technology with 2 agents and $t_0 = 1/10$, $t_1 = 4/10$ and $t_2 = 9/10$ exhibits UPC as $1/2 > (4/10 - 1/10)/(9/10 - 1/10) = 3/8$. The technology does not exhibit OP as

$$\frac{Q_1}{Q_2} = \frac{\frac{t_1}{t_1 - t_0}}{\frac{2 \cdot t_2}{t_2 - t_1}} = \frac{\frac{4/10}{4/10 - 1/10}}{\frac{2 \cdot 9/10}{9/10 - 4/10}} = \frac{4/3}{18/5} = 10/27 < 3/8 = \frac{4/10 - 1/10}{9/10 - 1/10} = \frac{t_1 - t_0}{t_2 - t_0}$$

This technology does not exhibit DRS as $t_2 - t_1 = 9/10 - 4/10 = 1/2 > 3/10 = 4/10 - 1/10 = t_1 - t_0$. However, it exhibits IRMP as $Q_2 = 18/5$, $Q_1 = 4/3$ and $Q_0 = 0$ thus $\frac{Q_2 - Q_1}{t_2 - t_1} > \frac{Q_1 - Q_0}{t_1 - t_0}$ as

$$\frac{Q_2 - Q_1}{t_2 - t_1} = \frac{18/5 - 4/3}{9/10 - 4/10} = \frac{68}{15} > \frac{40}{9} = \frac{4/3 - 0}{4/10 - 1/10} = \frac{Q_1 - Q_0}{t_1 - t_0}$$

Which concludes the proof. $\square$

A.1.1 Anonymous *AND* and *OR* technologies

Observation A.10 *The AND technology exhibits IRS.*

Proof: Let $r^a, r^b \in [0,1]^n$ be two profiles of actions, such that $r^b \geq r^a$ (for any i, $r_i^b \geq r_i^a$). We need to show that for every i, $t_i(r_i^b, r_{-i}^b) - t_i(r_i^a, r_{-i}^b) \geq t_i(r_i^b, r_{-i}^a) - t_i(r_i^a, r_{-i}^a)$. Indeed, $t_i(r_i^b, r_{-i}^b) - t_i(r_i^a, r_{-i}^b) = (r_i^b - r_i^a) \cdot \prod_{j \neq i} r_j^b \geq (r_i^b - r_i^a) \cdot \prod_{j \neq i} r_j^a = t_i(r_i^b, r_{-i}^a) - t_i(r_i^a, r_{-i}^a)$. $\square$

Lemma 3.15 *Anonymous AND technology exhibits both UPC and Over-Payment, thus has a single transition in both the hidden-actions and observable-actions cases.*

Proof: First we observe that AND exhibits IRS (Observation A.10) thus exhibits UPC (Lemma A.8). Next we show that it exhibits Over-Payment. As the technology is anonymous, for any $i \in N$, $c_i = c, \gamma_i = \gamma$ and $\delta_i = \delta$ for some $\gamma < \delta$. As for any k, $\frac{t_k}{t_k - t_{k-1}} = \frac{\delta}{\delta - \gamma}$, it implies that $\frac{Q_k}{Q_n} = \frac{\left(\frac{c \cdot k \cdot t_k}{t_k - t_{k-1}}\right)}{\left(\frac{c \cdot n \cdot t_n}{t_n - t_{n-1}}\right)} = \frac{k}{n}$. Thus, Over-Payment is equivalent to UPC for AND technology. □

The following holds for the symmetric case (in γ and c). Let $AND(n, \gamma, c)$ be the AND technology with n symmetric agents, each with cost c. Fixing n and c, let $v(\gamma)$ be the transition value of the optimal contract from the 0 contract to the n contract, when we use the parameter γ. We next show that the function $v(\gamma)$ is a monotonic function of γ. This means that if the success probability in case that an agent exert effort increases, the principal will move to the n contract earlier.

Lemma A.11 *For $AND(n, \gamma_1, c)$ and $AND(n, \gamma_2, c)$, where $\gamma_1 < \gamma_2$ it holds that $v(\gamma_1) < v(\gamma_2)$.*

Proof: Recall that at $v(\gamma)$ the utility of the agent is the same for the 0 and n contracts. Thus $v(\gamma) = \frac{c \cdot n \cdot t_n}{(t_n - t_{n-1})(t_n - t_0)}$. For AND technology $\frac{t_n}{t_n - t_{n-1}} = \frac{(1-\gamma)^n}{(1-\gamma)^n - \gamma \cdot (1-\gamma)^{n-1}} = \frac{1-\gamma}{1-2\gamma}$. For $0 < \gamma < \frac{1}{2}$ this is a monotonic function of γ. Additionally, $\frac{1}{t_n - t_0} = \frac{1}{(1-\gamma)^n - \gamma^n}$ is also a monotonic function of γ, for $0 < \gamma < \frac{1}{2}$. □

Lemma 3.16 *Any anonymous OR technology exhibits DRS and IRMP, thus has all n transitions in both the hidden-actions and observable-actions cases.*

Proof: As the technology is anonymous, for any $i \in N$, $c_i = c, \gamma_i = \gamma$ and $\delta_i = \delta$ for some $\gamma < \delta$. Let t_k denote the success probability when k agents, out of $n \geq k$ participants, exert effort. $t_k = 1 - (1-\delta)^k (1-\gamma)^{n-k} = 1 - r^k (1-\gamma)^n$ for $r = \frac{1-\delta}{1-\gamma}$. and $\Delta_k = t_k - t_{k-1} = \left(1 - r^k(1-\gamma)^n\right) - \left(1 - r^{k-1}(1-\gamma)^n\right) = r^{k-1}(1-\gamma)^n(\delta - \gamma)$ This implies that $\Delta_{k+1} = r\Delta_k$, and as $r < 1$ we conclude that OR exhibits DRS. Next we show that it also exhibits $IRMP$.

To show that the technology exhibits IRMP we need to show that for any k,

$$\frac{Q_{k+1} - Q_k}{\Delta_{k+1}} > \frac{Q_k - Q_{k-1}}{\Delta_k}$$

where $Q_k = \frac{c \cdot k \cdot t_k}{\Delta_k}$ is the total expected payment in the best contract for which there exist an equilibrium with k agents exerting effort.

$$\frac{Q_{k+1} - Q_k}{\Delta_{k+1}} = \frac{c}{\Delta_{k+1}} \cdot \left(\frac{(k+1) \cdot t_{k+1}}{\Delta_{k+1}} - \frac{k \cdot t_k}{\Delta_k}\right) =$$

$$\frac{c\left((k+1) \cdot t_{k+1} - r \cdot k \cdot t_k\right)}{(\Delta_{k+1})^2} = \frac{c\left(r \cdot t_k + (1-r) \cdot (k+1)\right)}{(\Delta_{k+1})^2}$$

Where the last equality is derived from $t_{k+1} = r \cdot t_k + 1 - r$.

We use the facts that $\Delta_{k+1} = r \cdot \Delta_k$, and that $r \cdot t_{k-1} = t_k - (1-r)$ to conclude:

$$\frac{Q_{k+1} - Q_k}{\Delta_{k+1}} > \frac{Q_k - Q_{k-1}}{\Delta_k}$$

$$\Updownarrow$$

$$\frac{c\left(r \cdot t_k + (1-r) \cdot (k+1)\right)}{(\Delta_{k+1})^2} > \frac{c\left(r \cdot t_{k-1} + (1-r) \cdot k\right)}{(\Delta_k)^2}$$

$$\Updownarrow$$

$$r \cdot t_k + (1-r) \cdot (k+1) > r^2 \cdot (r \cdot t_{k-1} + (1-r) \cdot k)$$

$$\Updownarrow$$

$$r \cdot t_k + (1-r) \cdot (k+1) > r^2 \cdot (t_k + (1-r) \cdot (k-1))$$

$$\Updownarrow$$

$$r(1-r)t_k > (1-r)\left(r^2(k-1) - (k+1)\right)$$

$$\Updownarrow$$

$$r \cdot t_k > r^2(k-1) - (k+1)$$

And this holds as $0 < r < 1$ thus $r \cdot t_k > 0$ while $r^2(k-1) - (k+1) < 0$. $\square$

A.2 The Price of Unaccountability

Lemma 3.17 *For any anonymous technology (t, c) that exhibits both UPC and Over-Payment and $t_0 > 0$, the POU is*

$$POU_S = POU_P = 1 + \frac{t_{n-1}}{t_0} - \frac{t_{n-1}}{t_n}$$

and it is obtained at the transition point of the hidden-actions case.

Proof: By Theorem 3.12 and Theorem 3.14 the technology has a single transition in both the hidden-actions and observable-actions cases. Let v_{ha} be the value at which the transition occurs in the hidden-actions case, and let v_{oa} be the value at which the transition occurs in the observable-actions case. The transition value is the value in which the principal is indifferent between contracting with 0 agents and contracting with n agents. Thus v_{oa} solves the equation $v_{oa} \cdot t_n - c \cdot n = v_{oa} \cdot t_0$, so $v_{oa} = \frac{c \cdot n}{t_n - t_0}$. Additionally, v_{ha} solves the equation $t_n \cdot (v_{ha} - \frac{c \cdot n}{t_n - t_{n-1}}) = v_{ha} \cdot t_0$, so $v_{ha} = \frac{c \cdot n}{t_n - t_0} \cdot \frac{t_n}{t_n - t_{n-1}} = v_{oa} \cdot \frac{t_n}{t_n - t_{n-1}}$.

As we assumed that $t_0 > 0$ then $t_{n-1} > 0$. Thus, $\frac{t_n}{t_n - t_{n-1}} > 1$, and therefore $v_{ha} > v_{oa}$ (i.e., the transition in the hidden-actions case occurs at a larger value than in the observable-actions case). By Lemma 3.4, both the social and the principal's POU are obtained at one of the two transition points (v_{oa} and v_{ha}). As at v_{oa} no agent is contracted in the hidden-actions case, both the social welfare ratio and the principal's utility ratio are 1 at v_{oa}. Thus, we only need to check the ratios at v_{ha}.

As v_{ha} is a transition point of the hidden-actions case, between 0 and n, the worst social welfare at this point is obtained with 0 agents exerting effort, and it is $v_{ha} \cdot t_0$. This is also the principal's utility at v_{ha}. Thus, the the social and principal's POU are $POU_S = POU_P = \frac{v_{ha} \cdot t_n - c \cdot n}{v_{ha} \cdot t_0} = \frac{t_n}{t_0} - \frac{c \cdot n}{v_{ha} \cdot t_0}$. As $\frac{c \cdot n}{v_{ha} \cdot t_0} = \frac{c \cdot n}{\frac{c \cdot n}{t_n - t_0} \cdot \frac{t_n}{t_n - t_{n-1}} \cdot t_0} = \left(\frac{t_n}{t_0} - 1\right) \cdot \left(1 - \frac{t_{n-1}}{t_n}\right)$, we derive that

$$POU_S = POU_P = \frac{t_n}{t_0} - \frac{c \cdot n}{v_{ha} \cdot t_0} = \frac{t_n}{t_0} - \left(\frac{t_n}{t_0} - 1\right) \cdot \left(1 - \frac{t_{n-1}}{t_n}\right) =$$

$$\frac{t_{n-1}}{t_n} \cdot \frac{t_n}{t_0} + 1 - \frac{t_{n-1}}{t_n} = 1 + \frac{t_{n-1}}{t_0} - \frac{t_{n-1}}{t_n}$$

which concludes the proof. □

B Non-Anonymous Technologies

Theorem 4.5 *In the OOA technology with two parallel paths of length two, for any values of c and $\gamma = 1 - \delta$, there exist values $v_1 < v_2$ in the optimal contract such that:*

- *for any $v \leq v_1$, no agent is contracted.*
- *for any $v \in [v_1, v_2]$, two agents on the same path are contracted.*
- *for any $v \geq v_2$, all four agents are contracted.*

Proof: Let (k_1, k_2) be the profile with k_1 agents exerting effort on the first path, and k_2 agents exerting effort on the second path. Let v_1 be the value for which the principal is indifferent between $(0, 0)$ and $(2, 0)$, and let v_2 be the value for which the principal is indifferent between $(2, 0)$ and $(2, 2)$. One can show that $v_1 < v_2$, which implies that out of the three contracts $(0, 0), (2, 0)$ and $(2, 2)$, up to v_1 the contract $(0, 0)$ is optimal, for value between v_1 and v_2 the contract $(2, 0)$ is optimal, and for any value larger than v_2, the contract $(2, 2)$ is optimal. To conclude the proof we also need 3 observations, showing that the profiles $(1, 1), (1, 0)$ and $(2, 1)$ are never optimal.

Let $u((k_1, k_2), v)$ be the utility of the principal with value v, if the contract is (k_1, k_2). First, for any value v one can show that the $u((1, 1), v) < u((2, 0), v)$ (as when moving to $(2, 0)$ the success probability improves and the payment decreases). Secondly, one can show that up to the value v_1 the $(0, 0)$ contract gives higher utility than the $(1,0)$ contract, and for any higher value the $(2,0)$ contract gives higher utility. Finally, one can show that up to the value v_2 the $(2,0)$ contract gives higher utility than the $(2,1)$ contract, and for any higher value the $(2,2)$ contract gives higher utility. We omit the details of the proof due to lack of space. □

Definition B.1 *A series-parallel network (SPN) is a network (V, E, s, t) such that one of the following conditions hold:*

- *Base case: a single edge (s,t), i.e., a network of the form $(\{s,t\}, \{(s,t)\}, s, t)$ is an SPN.*
- *Series: Suppose that $\Gamma_1 = (V_1, E_1, s_1, t_1)$ and $\Gamma_2 = (V_2, E_2, s_2, t_2)$ are SPN such that $V_1 \cap V_2 = \emptyset$. Set $V = V_1 \cup V_2$, $E = E_1 \cup E_2$, and merge t_1 with s_2. Then $(V, E, s = s_1, t = t_2)$ is an SPN.*
- *Parallel: Suppose that $\Gamma_1 = (V_1, E_1, s_1, t_1)$ and $\Gamma_2 = (V_2, E_2, s_2, t_2)$ are SPN such that $V_1 \cap V_2 = \emptyset$. Set $V = V_1 \cup V_2$, $E = E_1 \cup E_2$, and merge s_1 with s_2 and t_1 with t_2. Then $(V, E, s = s_1 = s_2, t = t_1 = t_2)$ is an SPN.*

C Algorithmic Aspects

Theorem 6.3 *The Optimal Contract Problem for Read Once Networks is #P-hard (under Turing reductions).*

Proof: We will show that an algorithm for this problem can be used to solve the network reliability problem. Given an instance of a network reliability problem $< G, \{\zeta_e\}_{e \in E} >$ (where ζ_e denotes e's probability of success), we define an instance of the optimal contract problem as follows: first define a new graph G' which is obtained by "And"ing G with a new player x, with γ_x very close to $\frac{1}{2}$ and $\delta_x = 1 - \gamma_x$. For the other edges, we let $\delta_e = \zeta_e$ and $\gamma_e = \zeta_e/2$. By choosing γ_x close enough to $\frac{1}{2}$, we can make sure that player x will enter the optimal contract only for very large values of v, after all other agents are contracted (if we can find the optimal contract for any value, it is easy to find a value for which in the original network the optimal contract is E, by keep doubling the value and asking for the optimal contract. Once we find such a value, we choose γ_x s.t. $\frac{c}{1-2\gamma_x}$ is larger than that value). Let us denote $\beta_x = 1 - 2\gamma_x$.

The critical value of v where player x enters the optimal contract of G', can be found using binary search over the algorithm that supposedly finds the optimal contract for any network and any value. Note that at this critical value v, the principal is indifferent between the set E and $E \cup \{x\}$. Now when we write the expression for this indifference, in terms of $t(E)$ and $\Delta_i^t(E)$, we observe the following.

$$t(E)\cdot\gamma_x\cdot\left(v-\sum_{i\in E}\frac{c}{\gamma_x\cdot\Delta_i^t(E\setminus i)}\right)=t(E)(1-\gamma_x)\left(v-\sum_{i\in E}\frac{c}{(1-\gamma_x)\cdot\Delta_i^t(E\setminus i)}-\frac{c}{t(E)\cdot\beta_x}\right)$$

if and only if

$$t(E)=\frac{(1-\gamma_x)\cdot c}{(\beta_x)^2\cdot v}$$

thus, if we can always find the optimal contract we are also able to compute the value of $t(E)$. $\square$