Solution to HW 2

Question 1.8 Give an update equation for a simple moving average of (even) order 2s.

Solution: Notice that the even moving average gives half the weight to the most extreme observations. In the update observations with half weight are added or deleted. An observation on the left that perviously had a full weight should be changed to half weight and the one one the right that had only half a weight should be given full weight. The update formula is:

$$Y^*_{t+1} = Y^*_t + \frac{1}{4s}\Big(Y_{t+s} + Y_{t+s+1} - Y_{t-s} - Y_{t-s+1}\Big)$$

Question 1.12 Show that the rank of a matrix A equals the rank of $A^T A$.

Solution: Let x be a vector of coefficients of length equal to the column dimension of A. The two linear transformations have the same null space if $Ax = 0 \Leftrightarrow A^T Ax = 0$. (Notice, the length of the first 0 is equal to the row dimension of A and the length of the second is equal to the column dimension.)

The direction "$\Rightarrow$" of the proof is trivial. Let us prove the direction "$\Leftarrow$." Multiply the second equation from the left by x^T. This produces the equation $x^T A^T Ax = (Ax)^T(Ax) = 0$. It follows that the length of the vector Ax is equal to zero. However, the only vector of length 0 in $\mathbb{R}^d$ is the zero vector. Consequently, $Ax = 0$ and the proof is complete.

Question 1.13 The $p + 1$ columns of the design matrix X in (1.17) are linearly independent.

Solution: Let β be a vector of coefficients of length $p + 1$. This vector defines the polynom $\beta_0 + \beta_1 x + \cdots + \beta_p x^p$ of degree p. The equation $X\beta = 0$ corresponds to the statement that the numbers $-k, -k+1, \ldots, k$ are all roots of the polynom. A non-trivial polynom of degree p has at most p distinct roots. Consequently, once $2k + 1 > p$ we must have that the polynom is trivial, namely all the components of β are equal to 0.