Solution to HW 2

Question 1.18 Show that one obtains the exponential smoother also as least squares estimator of the weighted approach

$$\sum_{j=0}^{\infty}(1-\alpha)^j(Y_{t-j}-\mu)^2 = \min_{\mu}$$

with $Y_t = Y_0$, for $t < 0$.

Solution: Use the fact that for $t < 0$ one has that $Y_t = Y_0$ in order to get that

$$\begin{aligned}\sum_{j=0}^{\infty}(1-\alpha)^j(Y_{t-j}-\mu)^2 &= \sum_{j=0}^{t-1}(1-\alpha)^j(Y_{t-j}-\mu)^2 + \sum_{j=t}^{\infty}(1-\alpha)^j(Y_0-\mu)^2 \\ &= \sum_{j=0}^{t-1}(1-\alpha)^j(Y_{t-j}-\mu)^2 + [(1-\alpha)^t/\alpha](Y_0-\mu)^2 ,\end{aligned}$$

The derivative with respect to μ of the (weighted) sum of squares is equal to

$$-2\sum_{j=0}^{t-1}(1-\alpha)^j(Y_{t-j}-\mu) - 2[(1-\alpha)^t/\alpha](Y_0-\mu)$$

and the second derivative is equal to

$$2\sum_{j=0}^{t-1}(1-\alpha)^j + 2[(1-\alpha)^t/\alpha] > 0 .$$

The solution obtained by equating the first derivative to zero is indeed a minimum since the second derivative is positive. Equate the first derivative to zero to get the equation:

$$\sum_{j=0}^{t-1}(1-\alpha)^j Y_{t-j} + [(1-\alpha)^t/\alpha]Y_0 = \mu \cdot \left\{\sum_{j=0}^{t-1}(1-\alpha)^j + [(1-\alpha)^t/\alpha]\right\} .$$

However,

$$\sum_{j=0}^{t-1}(1-\alpha)^j + (1-\alpha)^t/\alpha = [1-(1-\alpha)^t]/\alpha + (1-\alpha)^t/\alpha = 1/\alpha .$$

It follows that the solution is equal to

$$\hat{\mu} = \alpha \sum_{j=0}^{t-1} (1-\alpha)^j Y_{t-j} + (1-\alpha)^t Y_0 \ ,$$

which is the exponential smother as indicated by Lemma 1.2.8.

Question 1.13 Verify that the empirical correlation $r(k)$ at lag k for the trend $y_t = t$, $t = 1, \ldots, n$ is given by

$$r(k) = 1 - 3\frac{k}{n} + 2\frac{k(k^2-1)}{n(n^2-1)} \ , \quad k = 0, \ldots, n \ .$$

Plot the correlogram for different values of n. This example shows, that the correlogram has no interpretation for non-stationary processes (see Exercise 1.20).

Solution: The empirical correlation is defined by the formula (see Page 36):

$$r(k) = \frac{c(k)}{c(0)} = \frac{\sum_{t=1}^{n-k}(y_{t+k}-\bar{y})(y_t-\bar{y})}{\sum_{t=1}^{n}(y_t-\bar{y})^2} \ .$$

We will use the formulas

$$1+2+\cdots+n = n(n+1)/2 \ , \quad 1^2+2^2+3^2+\cdots+n^2 = n(n+1)(2n+1)/6$$

to get that $\bar{t} = n(n+1)/(2n) = (n+1)/2$ and

$$\sum_{t=1}^{n}(t-\bar{t})^2 = \sum_{t=1}^{n} t^2 - \Big(\sum_{t=1}^{n} t\Big)^2/n = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)^2}{4} = \frac{n(n^2-1)}{12} \ . \tag{1}$$

Next

$$\sum_{t=1}^{n-k}(t+k-\bar{t})(t-\bar{t}) = \sum_{t=1}^{n-k}(t-\bar{t})^2 + k\sum_{t=1}^{n-k}(t-\bar{t}) \ .$$

However,

$$\sum_{t=1}^{n-k}(t-\bar{t})^2 = \sum_{t=1}^{n-k}(t-\tilde{t})^2 + (n-k)(\tilde{t}-\bar{t})^2$$

where $\tilde{t} = \sum_{t=1}^{n-k} t/(n-k) = (n-k+1)/2$. It follows that

$$\sum_{t=1}^{n-k}(t-\bar{t})^2 = \frac{(n-k)((n-k)^2-1)}{12} + (n-k)\frac{(n-k+1-(n+1))^2}{4} \ ,$$

where the sum of the sum of squares is obtained from (1) with n replaced by $n-k$. Also

$$k\sum_{t=1}^{n-k}(t-\bar{t}) = k(n-k)(\tilde{t}-\bar{t}) = k(n-k)(n-k+1-(n+1))/2 = -k^2(n-k)/2\ .$$

The summation of all the terms produces a representation to the term $\sum_{t=1}^{n-k}(t+k-\bar{t})(t-\bar{t})$:

$$\begin{aligned}
&\frac{(n-k)((n-k)^2-1)}{12} + (n-k)\frac{k^2}{4} - \frac{k^2(n-k)}{2}\\
&= \frac{(n-k)(n^2-2nk+k^2-1)}{12} - \frac{k^2(n-k)}{4}\\
&= \frac{n(n^2-1)}{12} - \frac{n(2nk-k^2)}{12} - \frac{k(n^2-2nk+k^2-1)}{12} - \frac{3(n-k)k^2}{12}\\
&= \frac{n(n^2-1)}{12} + \frac{-3n^2k+2k^3+k}{12}\\
&= \frac{n(n^2-1)}{12} - \frac{3(n^2-1)k}{12} + \frac{2k(k^2-1)}{12}\ .
\end{aligned}$$

The correlation is obtained by dividing the last line by (1).