Solution to HW 5

Question 2.8 Suppose that Y_t, $t = 1, \ldots, n$, is a stationary process with mean μ. Then $\hat{\mu}_n = (1/n)\sum_{t=1}^{n} Y_t$ is an unbiased estimator of μ. Express the mean square error $\mathbb{E}(\hat{\mu}_n - \mu)^2$ in terms of the autocovariance function γ and show that $\mathbb{E}(\hat{\mu}_n - \mu)^2 \to n$ if $\gamma(n) \to 0$, $n \to \infty$

Solution: By the linearity of the expectation we get that

$$\mathbb{E}(\hat{\mu}_n) = (1/n)\sum_{t=1}^{n} \mathbb{E}(Y_t) = (1/n)\sum_{t=1}^{n} \mu = \mu.$$

Therefore, $\hat{\mu}_n$ is unbiased. The mean-square-error of an unbiased estimator is its variance. We use the formula for the variance of the sum to get that

$$\mathbb{V}ar(\hat{\mu}_n) = (1/n^2)\mathbb{V}ar\Big(\sum_{t=1}^{n} Y_t\Big) = (1/n^2)\Big(\sum_{t=1}^{n} \mathbb{V}ar(Y_t) + \sum_{t=2}^{n}\sum_{s\neq t} \mathbb{C}ov(Y_t, Y_s)\Big)$$

Use the fact that $\mathbb{C}ov(Y_t, Y_s) = \gamma(t-s) = \overline{\gamma(s-t)}$ to get that

$$\mathbb{V}ar(\hat{\mu}_n) = \gamma(0)/n + \sum_{k=1}^{n-1}[\gamma(k) + \gamma(-k)]\frac{n-k}{n^2}$$

In order to show the claim regarding convergence let us take $\tilde{\gamma}(k) = \max_{t\geq k} |\gamma(k)|$. Given Observe that

$$\mathbb{V}ar(\hat{\mu}_n) \leq \gamma(0)/n + 2\sum_{k=1}^{K-1} |\gamma(k)|\frac{n-k}{n^2} + 2\tilde{\gamma}(K)\frac{(n-K)^2}{n^2}.$$

Given $\epsilon > 0$, one may find a finite K such that $\tilde{\gamma}(K) < \epsilon$. Letting n go to infinity completes the proof of the claim.

Question 2.10 Suppose that (Y_t) is a stationary process with autocovariance function γ . Express the autocovariance function of the difference Filter of first order $\Delta Y_t = Y_t - Y_{t-1}$ in terms of γ. Find it when $\gamma(k) = \lambda^{|k|}$.

Solution: The autocovariance is given as

$$\begin{aligned}
&\mathbb{C}ov(Y_t -_{t-1}, Y_s - Y_{s-1}) = \\
&\mathbb{C}ov(Y_t, Y_s) + \mathbb{C}ov(Y_{t-1}, Y_{s-1}) - \mathbb{C}ov(Y_t, Y_{s-1}) - \mathbb{C}ov(Y_{t-1}, Y_s) \\
&= 2\gamma(t-s) - \gamma(t-s+1) - \gamma(t-s-1).
\end{aligned}$$

In the special case when $\gamma(k) = \lambda^{|k|}$ we get $\lambda^{|k|}[2 - (\lambda + 1/\lambda)]$.

Question 2.4 (i) Let (X_t), (Y_t) be stationary processes such that $\mathbb{C}ov(X_t, Y_s) = 0$ for $t, s \in \mathbb{Z}$. Show that for arbitrary $a, b \in \mathbb{C}$ the linear combinations $(aX_t + bY_t)$ yield a stationary process.
(ii) Suppose that the decomposition $Z_t = X_t + Y_t$, $t \in \mathbb{Z}$ holds. Show that stationarity of (Z_t) does not necessarily imply stationarity of (X_t).

Solution: (i) Let $Z_t = aX_t + bY_t$. Observe that $\mathbb{E}(Z_t) = a\mathbb{E}(X_t) + b\mathbb{E}(Y_t)$ is equal to a constant since the expectations of the to originating processes are constant. The covariance satisfies

$$\mathbb{C}ov(Z_t, Z_s) = a\bar{a}\mathbb{C}ov(X_t, X_s) + b\bar{b}\mathbb{C}ov(Y_t, Y_s)$$

since $\mathbb{C}ov(X_t, Y_s) = \mathbb{C}ov(Y_t, X_s) = 0$ by the assumption of lack of correlation. The statement that the covariance is a function of the difference $t - s$ results from the fact that this is the case for the covariance of the two processes that produce the sum.
(ii) Take $Y_t = -X_t$. The sum process is stationary (it is always equal to 0) but (X) can be taken to be a non-stationary process.

Question 2.13 Let (ϵ_t) be a white noise process with independent $\epsilon_t \sim N(0, 1)$. and define

$$\tilde{\epsilon}_t = \begin{cases} \epsilon_t & \text{if } t \text{ is even,} \\ (\epsilon_{t-1}^2 - 1)/\sqrt{2} & \text{if } t \text{ is odd.} \end{cases}$$

Show that $\tilde{\epsilon}_t$ is a white noise process with $\mathbb{E}(\tilde{\epsilon}_t) = 0$ and $\mathbb{V}ar(\tilde{\epsilon}_t) = 1$, where the $\tilde{\epsilon}_t$ are neither independent nor identically distributed.

Solution: The fact that the mean is zero follows from the definition for the even t and follows from the fact hat the expectation of the chi-square distribution on 1 degree of freedom is equal to 1. The variance of such random variable is 2, hence the statement regarding the variance of the new white noise.

In order to complete the proof of the fact that the process is indeed a white noise we need to show that the autocorrelation is zero for each lag larger that 0. For lag that is larger than 1 and for a lag of size two between an odd index and the following even index this claim follows from independence. For a lag of size 1 between an even and its subsequent odd this follows from the fact that for $Z \sim N(0, 1)$:

$$\mathbb{C}ov(Z, Z^2 - 1) = \mathbb{E}(Z^3) - \mathbb{E}(Z) = 0.$$

Clearly, the evens and the odds do not have the same distribution. Independence is not present, since the conditional distribution of Z^2, given $Z = z$, is determinist and not chi-square.