Solution to HW 7

Question 2.29

(i) Consider the process

\[ \tilde{Y}_t = \begin{cases} 
\epsilon_1 & \text{for } t = 1 \\
aY_{t-1} + \epsilon_t & \text{for } t > 1 
\end{cases} \]

i.e., \( \tilde{Y}_t \) equals the AR(1)-process, conditional on \( Y_0 = 0 \). Compute \( \mathbb{E}(\tilde{Y}_t) \), \( \text{Var}(\tilde{Y}_t) \) and \( \text{Cov}(\tilde{Y}_t, \tilde{Y}_{t+s}) \). Is there something like asymptotic stationarity for \( t \to \infty \)?

(ii) Choose \( a \in (-1, 1) \), \( a \neq 0 \), and compute the correlation matrix of \( Y_1, \ldots, Y_{10} \).

Solution: Use the recursion

\[ Y_t = a^{t-1}Y_0 + a^{t-2}\epsilon_1 + a^{t-3}\epsilon_2 + \cdots + a^2\epsilon_{t-2} + a\epsilon_{t-1} + \epsilon_t \]

in order to get the representation:

\[ Y_t = a^tY_0 + a^{t-1}\epsilon_1 + a^{t-2}\epsilon_2 + \cdots + a^2\epsilon_{t-2} + a\epsilon_{t-1} + \epsilon_t \]

If \( Y_0 = 0 \) we get a representation for \( \tilde{Y}_t \) that involves the white noise only.

All the white-noise elements have zero mean and variance \( \sigma^2 \) and they are uncorrelated. Consequently,

\[ \mathbb{E}(\tilde{Y}_t) = 0, \quad \text{Var}(Y_t) = \sigma^2 \sum_{i=0}^{t-1} (a^i)^2 = \sigma^2 \frac{1 - a^{2t}}{1 - a^2}. \]

To get the covariance \( \text{Cov}(\tilde{Y}_t, \tilde{Y}_{t+s}) \) notice that

\[ \tilde{Y}_t = a^{t-1}\epsilon_1 + a^{t-2}\epsilon_2 + \cdots + a^2\epsilon_{t-2} + a\epsilon_{t-1} + \epsilon_t \]
\[ \tilde{Y}_{t+s} = a^{t+s-1}\epsilon_1 + a^{t+s-2}\epsilon_2 + \cdots + a^2\epsilon_{t+s-2} + a\epsilon_{t+s-1} + \epsilon_{t+s} \]

in order to obtain that

\[ \text{Cov}(\tilde{Y}_t, \tilde{Y}_{t+s}) = \sigma^2 \sum_{i=0}^{t-1} a^s(a^i)^2 = \sigma^2 a^s \frac{1 - a^{2t}}{1 - a^2}. \]

When \( t \to \infty \) (and \( s \) is fixed) we get that the covariance and the variance converge to the variance and covariance of the stationary AR(1)-process.

For part (ii) see the attached R code.
**Question 2.30** Use the IML function ARMASIM (or R) to simulate the stationary AR(2)-process

\[ Y_t = -0.3Y_{t-1} + 0.3Y_{t-2} + \epsilon_t \]

Estimate the parameters \( a_1 = -0.3 \) and \( a_2 = 0.3 \) by means of the Yule-Walker equations using the SAS procedure PROC ARIMA.

**Solution:** The stationary variance of the AR(2) process satisfies \( \gamma(0) = a_1\gamma(1) + a_2\gamma(2) + \sigma^2 \). From the Yule-Walker Equations we get that \( \gamma(1) = a_1\gamma(0) + a_2\gamma(1) \), hence \( \gamma(1) = \gamma(0)a_1/(1-a_2) \). Also, \( \gamma(2) = a_1\gamma(1) + a_2\gamma(0) \). Therefore,

\[
\gamma(0) = a_1\gamma(1) + a_2\gamma(2) + \sigma^2 = a_1(a_2 + 1)\gamma(1) + a_2^2\gamma(0) + \sigma^2
\]

It follows that

\[
\gamma(0) = \sigma^2/(1 - a_1^2(a_2 + 1)/(1 - a_2) - a_2^2)
\]

The stationary process can be generate by simulating \((Y_0, Y_1)\) from the stationary distribution and applying the AR(2) formula for generating subsequent observations. (See the R code.)

As an exercise we can treat the simulated numbers as if they were real data and use them in order to estimate the parameters. For the Yule-Walker approach we compute \( r(1) \) and \( r(2) \) and solve the equation \( r = Ra \), were \( r = (r(1), r(2))' \), \( a = (a_1, a_2)' \) and \( R \) is a 2 × 2-matrix with 1 in the main diagonal and \( r(1) \) being the off-diagonal element. (See the R code.)

**Question 2.31** Show that the value at lag 2 of the partial autocorrelation function of the MA(1)-process

\[ Y_t = \epsilon_t + a\epsilon_{t-1} \]

is

\[
a(2) = -\frac{a^2}{1 + a^2 + a^4}.
\]

**Solution:** The empirical autocorrelation is obtained by taking the last element in the vector \( R_k^{-1}r_k \), where \( R_k \) is the autocorrelation matrix up to lag \( k - 1 \) observations and \( r_k \) are the first \( k \) autocorrelations. The partial correlation function is obtained by the conducting the same operation to the theoretical counterparts.
Recall that for the MA(1)-process $\rho(1) = a_1 / (1 + a_1^2)$ and $\rho(2) = 0$. It follows that the matrix of autocorrelations for $k = 2$ is

$$
P_k = \begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix} \Rightarrow P_k^{-1} = \frac{1}{1 - \rho(1)^2} \begin{pmatrix} 1 & -\rho(1) \\ -\rho(1) & 1 \end{pmatrix}
$$

The partial autocorrelation of lag 2 is obtained by taking the inner product between the second row of $P_k$ and the vector $(\rho(1), 0)$:

$$\alpha(2) = -\rho(1) \times \rho(1) + 1 \times 0 \frac{(a_1/ (1 + a_1^2))^2}{1 - (a_1/ (1 + a_1^2))^2} = -\frac{a_1^2}{1 + a_1^2 - a_1^2},$$

which produces the result.

**Question 2.34** Compute the autocovariance function of an ARMA(1,2)-process.

**Solution:** In the spirit of Examples 2.2.7 and 2.2.9, we first find the values of $\alpha_w$ using Theorem 2.2.6 and then apply Theorem 2.2.8 in order to obtain the autocovariances.

Indeed, $\alpha_0 = 1$. For $w = 1$ we use the second expression of the final display at (2.15) to get that $\alpha_1 = b_1 + a_1$. The third expression produces $\alpha_2 = b_2 + a_1 \alpha_1 = b_2 + a_1 (b_1 + a_1)$.

Using these evaluations to compute covariances we get from the first equation in (2.16) that

$$
\gamma(0) - a_1 \gamma(1) = \sigma^2(\alpha_2 + b_1 \alpha_1 + b_2) \\
\gamma(1) - a_1 \gamma(0) = \sigma^2(b_1 \alpha_1 + b_2) \\
\gamma(2) - a_1 \gamma(1) = \sigma^2 b_2
$$

and $\gamma(k) = a_1 \gamma(k - 1) = a_1^{k-2} \gamma(2)$, for $k > 2$. From the second equation we get that $\gamma(1) = a_1 \gamma(0) + \sigma^2(b_1 \alpha_1 + b_2)$. Plugging this in the first equation gives

$$
\gamma(0) = a_1 [a_1 \gamma(0) + \sigma^2(b_1 \alpha_1 + b_2)] + \sigma^2(\alpha_2 + b_1 \alpha_1 + b_2) \\
\Rightarrow \gamma(0) = \sigma^2 \frac{a_2 + b_1 \alpha_1 + (a_1 + 1) b_2}{1 - a_1^2} \\
\gamma(1) = a_1 \gamma(0) + \sigma^2(b_1 \alpha_1 + b_2) \\
\gamma(2) = a_1 \gamma(1) + \sigma^2 b_2.
$$