Solution to HW 7

Question 2.29 (i) Consider the process

$$\tilde{Y}_t = \begin{cases} \epsilon_1 & \text{for } t = 1 \\ aY_{t-1} + \epsilon_t & \text{for } t > 1 \end{cases}$$

i.e., $\tilde{Y}_t$ equals the AR(1)-process, conditional on $Y_0 = 0$. Compute $\mathbb{E}(\tilde{Y}_t)$ $\mathbb{V}ar(\tilde{Y}_t)$ and $\mathbb{C}ov(\tilde{Y}_t, \tilde{Y}_{t+s})$. Is there something like asymptotic stationarity for $t \to \infty$?
(ii) Choose $a \in (-1, 1)$, $a \neq 0$, and compute the correlation matrix of $Y_1, \ldots, Y_{10}$.
Solution: Use the recursion

$$Y_t = aY_{t-1} + \epsilon_t = a^2 Y_{t-2} + a\epsilon_{t-1} + \epsilon_t = a^3 Y_{t-3} + a^2\epsilon_{t-2} + a\epsilon_{t-1} + \epsilon_t = \cdots$$

in order to get the representation:

$$Y_t = a^t Y_0 + a^{t-1}\epsilon_1 + a^{t-2}\epsilon_2 + \cdots + a^2\epsilon_{t-2} + a\epsilon_{t-1} + \epsilon_t$$

If $Y_0 = 0$ we get a representation for $\tilde{Y}_t$ that involves the white noise only.

All the white-noise elements have zero mean and variance σ^2 and they are uncorrelated. Consequently,

$$\mathbb{E}(\tilde{Y}_t) = 0\ , \quad \mathbb{V}ar(Y_t) = \sigma^2 \sum_{i=0}^{t-1} (a^i)^2 = \sigma^2 \frac{1 - a^{2t}}{1 - a^2}\ .$$

To get the covariance $\mathbb{C}ov(\tilde{Y}_t, \tilde{Y}_{t+s})$ notice that

$$\begin{aligned} \tilde{Y}_t &= a^{t-1}\epsilon_1 + a^{t-2}\epsilon_2 + \cdots + a^2\epsilon_{t-2} + a\epsilon_{t-1} + \epsilon_t \\ \tilde{Y}_{t+s} &= a^{t+s-1}\epsilon_1 + a^{t+s-2}\epsilon_2 + \cdots + a^2\epsilon_{t+s-2} + a\epsilon_{t+s-1} + \epsilon_{t+s} \end{aligned}$$

in order to obtain that

$$\mathbb{C}ov(\tilde{Y}_t, \tilde{Y}_{t+s}) = \sigma^2 \sum_{i=0}^{t-1} a^s (a^i)^2 = \sigma^2 a^s \frac{1 - a^{2t}}{1 - a^2}\ .$$

When $t \to \infty$ (and s is fixed) we get that the covariance and the variance converge to the variance and covariance of the stationary AR(1)-process.

For part (ii) see the attached R code.

Question 2.30 Use the IML function ARMASIM (or R) to simulate the stationary AR(2)- process

$$Y_t = -0.3Y_{t-1} + 0.3Y_{t-2} + \epsilon_t \ .$$

Estimate the parameters $a_1 = -0.3$ and $a_2 = 0.3$ by means of the Yule-Walker equations using the SAS procedure PROC ARIMA.

Solution: The stationary variance of the AR(2) process satisfies $\gamma(0) = a_1\gamma(1) + a_2\gamma(2) + \sigma^2$. From the Yule-Walker Equations we get that $\gamma(1) = a_1\gamma(0)+a_2\gamma(1)$, hence $\gamma(1) = \gamma(0)a_1/(1-a_2)$. Also, $\gamma(2) = a_1\gamma(1)+a_2\gamma(0)$. Therefore,

$$\gamma(0) = a_1\gamma(1) + a_2\gamma(2) + \sigma^2 = a_1(a_2+1)\gamma(1) + a_2^2\gamma(0) + \sigma^2$$

It follows that

$$\gamma(0) = \sigma^2/(1 - a_1^2(a_2+1)/(1-a_2) - a_2^2) \ .$$

The stationary process can be generate by simulating (Y_0, Y_1) from the stationary distribution and applying the AR(2) formula for generating subsequent observations. (See the R code.)

As an exercise we can treat the simulated numbers as if they were real data and use them in order to estimate the parameters. For the Yule-Walker approach we compute $r(1)$ and $r(2)$ and solve the equation $\mathbf{r} = \mathbf{R}\mathbf{a}$, were $\mathbf{r} = (r(1), r(2))'$, $\mathbf{a} = (a_1, a_2)'$ and $\mathbf{R}$ is a 2×2-matrix with 1 in the main diagonal and $r(1)$ being the off-diagonal element. (See the R code.)

Question 2.31 Show that the value at lag 2 of the partial autocorrelation function of the MA(1)-process

$$Y_t = \epsilon_t + a\epsilon_{t-1} \ ,$$

is

$$\alpha(2) = -\frac{a^2}{1 + a^2 + a^4} \ .$$

Solution: The empirical autocorrelation is obtained by taking the last element in the vector $\mathbf{R}_k^{-1}\mathbf{r}_k$, where $\mathbf{R}_k$ is the autocorrelation matrix up to lag $k-1$ observations and $\mathbf{r}_k$ are the first k autocorrelations. The partial correlation function is obtained by the conducting the same operation to the theoretical counterparts.

Recall that for the MA(1)-process $\rho(1) = a_1/(1+a_1^2)$ and $\rho(2) = 0$. It follows that the matrix of autocorrelations for $k = 2$ is

$$\mathbf{P}_k = \begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix} \quad \Rightarrow \quad \mathbf{P}_k^{-1} = \frac{1}{1-\rho(1)^2} \begin{pmatrix} 1 & -\rho(1) \\ -\rho(1) & 1 \end{pmatrix}$$

The partial autocorrelation of lag 2 is obtained by taking the inner product between the second row of $\mathbf{P}_k$ and the vector $(\rho(1), 0)$:

$$\alpha(2) = \frac{-\rho(1)\times\rho(1) + 1 \times 0}{1-\rho(1)^2} = -\frac{(a_1/(1+a_1^2))^2}{1-(a_1/(1+a_1^2))^2} = -\frac{a_1^2}{(1+a_1^2)^2 - a_1^2} \ ,$$

which produces the result.

Question 2.34 Compute the autocovariance function of an ARMA(1,2)-process.

Solution: In the spirit of Examples 2.2.7 and 2.2.9, we first find the values of α_w using Theorem 2.2.6 and then apply Theorem 2.2.8 in order to obtain the autocovariances.

Indeed, $\alpha_0 = 1$. For $w = 1$ we use the second expression of the final display at (2.15) to get that $\alpha_1 = b_1 + a_1$. The third expression produces $\alpha_2 = b_2 + a_1\alpha_1 = b_2 + a_1(b_1 + a_1)$.

Using these evaluations to compute covariances we get from the first equation in (2.16) that

$$\begin{aligned} \gamma(0) - a_1\gamma(1) &= \sigma^2(\alpha_2 + b_1\alpha_1 + b_2) \\ \gamma(1) - a_1\gamma(0) &= \sigma^2(b_1\alpha_1 + b_2) \\ \gamma(2) - a_1\gamma(1) &= \sigma^2 b_2 \end{aligned}$$

and $\gamma(k) = a_1\gamma(k-1) = a_1^{k-2}\gamma(2)$, for $k > 2$. From the second equation we get that $\gamma(1) = a_1\gamma(0) + \sigma^2(b_1\alpha_1 + b_2)$. Plugging this in the first equation gives

$$\begin{aligned} \gamma(0) &= a_1[a_1\gamma(0) + \sigma^2(b_1\alpha_1 + b_2)] + \sigma^2(\alpha_2 + b_1\alpha_1 + b_2) \\ &\Rightarrow \gamma(0) = \sigma^2\frac{\alpha_2 + (a_1+1)b_1\alpha_1 + (a_1+1)b_2}{1-a_1^2} \\ \gamma(1) &= a_1\gamma(0) + \sigma^2(b_1\alpha_1 + b_2) \\ \gamma(2) &= a_1\gamma(1) + \sigma^2 b_2 \ . \end{aligned}$$