On stochastic orders of absolute value of order statistics in symmetric distributions

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Abstract

Let $Y_1, \ldots, Y_n$ be the order statistics of a simple random sample from a finite or infinite population, having median $= M$. We compare the variables $|Y_j - M|$ and $|Y_m - M|$, where Y_m is the sample median, that is, $m = \frac{n+1}{2}$ for odd n. The comparison is in terms of the likelihood ratio order, which implies stochastic order as well as other orders. The results were motivated by the study of best invariant and minimax estimators for the k/N quantile of a finite population of size N, with a natural loss function of the type $g(\left|F_N(t) - \frac{k}{N}\right|)$, where F_N is the population distribution function, t is an estimate, and g is an increasing function.

Key words:
likelihood ratio order, median, finite population, minimax strategy, sampling without replacement

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1. Introduction

1.1. Motivation

The main results of this paper are stated in Theorems 1 and 2 of Section 1.2 below and their corollaries. The results concern stochastic orders for variables like $|Y_j - M|$, where $Y_1, \ldots, Y_n$ are the order statistics of a simple random sample from a finite or infinite population with median M. These results on stochastic ordering and order statistics are motivated by the study of minimax strategies for estimating quantiles of a finite population, where a strategy consists of a sampling design and an estimator. One way of finding such strategies is to start with

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minimax invariant estimators. For related problems in the context of infinite populations, see, e.g., Zieliński (1999), Yu and Chow (1991), Yu and Phadia (1992), Stępień-Baran (2009), and Ferguson (1967).

Let $x = (x_1, x_2, ..., x_N)$ be an N-dimensional vector of finite population values of some measurement, where each x_i is a real number associated with the population unit labeled i. We assume that $x \in \Upsilon$, a (known) parameter space, where $\Upsilon = \{(x_1, x_2, ..., x_N) : x_i \in I,\ \ x_i \text{ distinct}\}$, and I is a finite or infinite interval in $\mathbb{R}$. Define the population distribution function by $F_N(t) = \frac{1}{N}\sum_{j=1}^{N} \mathbb{I}_{(-\infty,\, t]}(x_j)$, where $\mathbb{I}_A(x)$ stands for the indicator function of the event $x \in A$. A k-th quantile of F_N is any value θ such that $F_N(\theta) = k/N$. We consider estimation of the quantiles with a loss function of the form $L(x,a) = \left|F_N(a) - \frac{k}{N}\right|^r$ for some $r > 0$, or more generally $L(x,a) = g(\left|F_N(a) - \frac{k}{N}\right|)$ for $k = 1, ..., N$, where a is the estimate, and g increasing (see, e.g. Ferguson (1967)). For odd N, the median is obtained when $k = (N+1)/2$.

A sampling design $\mathcal{P}$ is a probability function on the space of all subsets S of $\{1, \ldots, N\}$. Simple random sampling without replacement of size n is denoted by $\mathcal{P}_s$ and satisfies $\mathcal{P}_s(S) = 1/\binom{N}{n}$ for subsets S of size n. The class of sampling designs having sample size n is denoted by $\mathbb{P}_n$.

The data consist of the set of pairs $D = \{(i, x_i) : i \in S\}$, that is , the x-values in the sample S and their corresponding labels. An estimator t is a function $t(D)$ of the data.

Given a strictly increasing function $\varphi : \mathbb{R} \to \mathbb{R}$, we extend its operation also to vectors in the parameter space, by $\varphi(x) = (\varphi(x_1), ..., \varphi(x_N))$. Let Φ denote the group of all strictly increasing (hence one-to-one) functions such that the extension to vectors satisfies $\varphi : \Upsilon \to \Upsilon$ and onto (bijections). Define $\varphi(D) = \{(i, \varphi(x_i)\,) : i \in S\}$. A nonrandomized estimator $t(D)$ is said to be invariant if for all D and all $\varphi \in \Phi$, $t(\varphi(D)) = \varphi(t(D))$. The class of nonrandomized invariant estimators is denoted by T_I.

Let $Y_1, \ldots, Y_n$ be the order statistics of a sample of size n using $\mathcal{P}_s$, and set $j^* = \arg\min_{1\le j\le n} E_{\mathcal{P}_s}\left|F_N(Y_j) - \frac{k}{N}\right|^r$. The following minimax result is given in Malinovsky and Rinott (2009).

Theorem. *The strategy $(\mathcal{P}_s, Y_{j^*})$ is minimax among all strategies $(\mathcal{P}, t)$ consisting of a sampling design $\mathcal{P}$ having a fixed sample size n, and a nonrandomized invariant estimator t, that is,*

$$\inf_{t\in T_I, \mathcal{P}\in\mathbb{P}_n}\ \sup_{x\in\Upsilon} E_{\mathcal{P}}\left|F_N(t(D)) - \frac{k}{N}\right|^r = \sup_{x\in\Upsilon} E_{\mathcal{P}_s}\left|F_N\left(Y_{j^*}\right) - \frac{k}{N}\right|^r.$$

For $r = 2$ the above j^* can be computed explicitly. However, in general, it may be hard or impossible to compute j^*, and it may depend on r. For example, for $N = 9$, $n = 7$, $k = 2$, direct calculations show that for $r \leq c$ we have $j^* = 2$ whereas $r > c$ implies $j^* = 1$, where $c = log(17/3)/log(2) \approx 2.5$. This means that the minimax estimator $t = Y_{j^*}$ depends on the loss function. This is a natural but somewhat undesirable state of affairs, since statisticians often do not have a precise loss function in mind.

By far, the most widely used quantile is the median, and it is interesting to note that in this case, the minimax rule does not depend on r, and in fact, it is the same rule for any loss function of the form $g(\left|F_N(a) - \frac{k}{N}\right|)$ with g increasing. This follows from the fact (stated for n and N odd for simplicity) proved in this paper, that for the median ($k = (N+1)/2$) we have for $j = 1, \ldots, n$, $\left|F_N(Y_j) - \frac{N+1}{2N}\right| \geq_{st} \left|F_N(Y_{\frac{n+1}{2}}) - \frac{N+1}{2N}\right|$, where $\geq_{st}$ indicates stochastic order. Clearly this implies that $j^* = \frac{n+1}{2}$ for any r, and that the strategy $(\mathcal{P}_s, Y_{j^*})$ is minimax in the sense of the above theorem.

We now turn to the results of this paper, which include the above facts on stochastic ordering of the variables $\left|F_N(Y_j) - \frac{N+1}{2N}\right|$ and generalizations to other orders.

1.2. Main results

We consider two closely related problems.

Problem 1.

Let $Y_1, \ldots, Y_n$ be the order statistics of a simple random sample *without replacement* from a finite population consisting of N distinct values. For simplicity we henceforth assume that n and N are odd. We start with the following problem: find the index j which minimizes $E\left|F_N(Y_j) - \frac{N+1}{2N}\right|$. It is very natural to guess that the minimizing j is such that Y_j is the median, that is,

$$\arg\min_{1 \leq j \leq n} E\left|F_N(Y_j) - \frac{N+1}{2N}\right| = \frac{n+1}{2}. \tag{1}$$

This is indeed true, but to the best of our knowledge has not been proved before, and the proof is less straightforward than expected.

The distribution of $NF_N(Y_1), \ldots, NF_N(Y_n)$ is the same as that of the order statistics of a simple random sample without replacement from $\{1, \ldots, N\}$. Therefore, an equivalent formulation is: let $Y_1, \ldots, Y_n$ be the order statistics of a simple random sample without replacement from $\{1, \ldots, N\}$. Then

$$\arg\min_{1 \leq j \leq n} E\left|Y_j - \frac{N+1}{2}\right| = \frac{n+1}{2}. \tag{2}$$

A stronger result holds: setting $M = \frac{N+1}{2}$ we have $|Y_j - M| \geq_{st} \left|Y_{\frac{n+1}{2}} - M\right|$ for $j = 1, \ldots, n$, where $\geq_{st}$ indicates stochastic order. The latter result can be stated equivalently by saying that $Y_{\frac{n+1}{2}}$ is more peaked about M than Y_j in the sense defined by Birnbaum (1948). Moreover, we can replace stochastic order by the stronger likelihood ratio order, denoted by $\geq_{lr}$, where $S \geq_{lr} T$ means that $f(t)/g(t)$ is nondecreasing in t in the union of the supports of S and T, where f and g are the densities or discrete probability functions of S and T, respectively. See, e.g., Müller and Stoyan (2002), Shaked and Shanthikumar (2006) for further details, implications, and references concerning these orders, and numerous results relating order statistics and stochastic, likelihood ratio, and peakedness orders. It is well known that likelihood ratio order implies stochastic order. Thus we prove

Theorem 1. *Let $Y_1, \ldots, Y_n$ be the order statistics of a simple random sample without replacement from $\{1, \ldots, N\}$, where n and N are odd. Then*

$$\left|Y_j - \frac{N+1}{2}\right| \geq_{lr} \left|Y_{\frac{n+1}{2}} - \frac{N+1}{2}\right| \quad \textit{for } j = 1, \ldots, n. \tag{3}$$

Corollary 1. *Let $Y_1, \ldots, Y_n$ be the order statistics of a simple random sample without replacement from a finite population consisting of N distinct values, where n and N are odd. Then*

$$\left|F_N(Y_j) - \frac{N+1}{2N}\right| \geq_{lr} \left|F_N(Y_{\frac{n+1}{2}}) - \frac{N+1}{2N}\right| \quad \textit{for } j = 1, \ldots, n. \tag{4}$$

Problem 2. Let $Y_1, \ldots, Y_n$ be the order statistics of iid observations having a distribution function F, and suppose F is symmetric around some values M, that is, for $X \sim F$ the variables $X - M$ and $M - X$ are identically distributed. Equivalently, F satisfies $1 - F(M - x) = F(M + x)$ whenever $M \pm x$ are points of continuity of F. For simplicity we henceforth assume that n is odd. Then it is very natural to conjecture that $\arg\min_{1 \leq j \leq n} E\,|Y_j - M| = \frac{n+1}{2}$, provided the expectation exists. If F is a uniform distribution then this follows from Theorem 1 by letting $N \to \infty$, and was proved in Zieliński (1999), with a generalization to other quantiles.

Here we prove: $|Y_j - M| \geq_{st} \left|Y_{\frac{n+1}{2}} - M\right|$. Moreover we prove $|Y_{j+1} - M| \geq_{st} |Y_j - M|$ for all $j \geq \frac{n+1}{2}$. Furthermore, we can again replace stochastic order by the stronger likelihood ratio order when the latter is defined, and thus we prove

Theorem 2. *Let $Y_1, \dots, Y_n$ be the order statistics of iid observations having a distribution function F that is symmetric around its median M. Assume that F is discrete or absolutely continuous. Then for $j \geq \frac{n+1}{2}$, we have $|Y_{j+1} - M| \geq_{lr} |Y_j - M|$.*

Without assuming that F is absolutely continuous or discrete the likelihood ratio order is not defined. However, we obtain

Corollary 2. *Let $Y_1, \dots, Y_n$ be the order statistics of iid observations having a distribution function F that is symmetric around its median M. Then for $j \geq \frac{n+1}{2}$, we have $|Y_{j+1} - M| \geq_{st} |Y_j - M|$.*

Corollary 2 can be restated as follows: for $j \geq \frac{n+1}{2}$, the random variable Y_j is more peaked about M than Y_{j+1}.

2. Proofs

We start with the simpler Theorem 2.
Proof of Theorem 2. We prove the theorem assuming that $X_1, \dots, X_n$ is a sample from an absolutely continuous and symmetric distribution F. For a discrete (symmetric) distribution F, the result will then follow by taking the convolution of F with a Normal$(0, \sigma^2)$ or Uniform$(-\sigma, \sigma)$ distribution, and letting $\sigma \to 0$. Note that such an approximate identity convolution is absolutely continuous and, of course, it is symmetric. We assume WLOG that the median $M = 0$. Then, using the fact that for a continuous F, symmetric around $M = 0$, $\quad F(t) + F(-t) = 1$, we have for any integrable function ψ

$$E\psi(|Y_j|) = \frac{n!}{(j-1)!(n-j)!} \int_{-\infty}^{\infty} \psi(|t|)[F(t)]^{j-1}[1 - F(t)]^{n-j} f(t) dt$$
$$= \frac{n!}{(j-1)!(n-j)!} \int_0^{\infty} \psi(t) f(t) \left\{ [F(t)]^{j-1}[1 - F(t)]^{n-j} + [F(t)]^{n-j}[1 - F(t)]^{j-1} \right\} dt$$
$$= \int_0^{\infty} \psi(t) g_j(t) dt,$$

and so $g_j(t) = \frac{n!}{(j-1)!(n-j)!} f(t) \left\{ [F(t)]^{j-1}[1 - F(t)]^{n-j} + [F(t)]^{n-j}[1 - F(t)]^{j-1} \right\}$ is the density of $|Y_j|$.

It is required to prove that for $j \geq \frac{n+1}{2}$, the ratio $g_{j+1}(t)/g_j(t)$ is nondecreasing in t in the support of F which contained in the set $\{t : f(t) > 0\}$. This follows directly from

Lemma 1. *For n odd,*

$$\zeta(a) = \frac{a^j(1-a)^{n-j-1} + a^{n-j-1}(1-a)^j}{a^{j-1}(1-a)^{n-j} + a^{n-j}(1-a)^{j-1}} \text{ is an increasing function of } a$$

$$\text{for all } a \geq \frac{1}{2} \text{ and for all } j \geq \frac{n+1}{2}.$$

Proof. Note that

$$a^{j-1}(1-a)^{n-j} + a^{n-j}(1-a)^{j-1} = \frac{(1-a)^n}{a}\left[\left(\frac{a}{1-a}\right)^j + \left(\frac{a}{1-a}\right)^{n-j+1}\right].$$

Therefore,

$$\zeta(a) = \frac{\left(\frac{a}{1-a}\right)^{j+1} + \left(\frac{a}{1-a}\right)^{n-j}}{\left(\frac{a}{1-a}\right)^j + \left(\frac{a}{1-a}\right)^{n-j+1}} = \frac{b^{j+1} + b^{n-j}}{b^j + b^{n-j+1}} = \frac{b^{s+1}+1}{b^s+b} \equiv \kappa(b),$$

where $s = 2j - n > 0,\ b = a/(1-a) \geq 1.$

$$\frac{d\kappa(b)}{db} = \frac{(s+1)b^s(b^s+b) - (b^{s+1}+1)(sb^{s-1}+1)}{(b^s+b)^2}$$

$$= \frac{s(b^{2s} + b^{s+1} - b^{2s} - b^{s-1}) + (b^{2s} + b^{s+1} - b^{s+1} - 1)}{(b^s+b)^2} = \frac{s(b^{s+1} - b^{s-1}) + (b^{2s}-1)}{(b^s+b)^2} > 0,$$

for all $b > 1$ and for all $\frac{n+1}{2} \leq j \leq n-1$. This proves Lemma 1, completing the proof of Theorem 2. □

Corollary 2 follows easily from Theorem 2 by the smoothing approximation described in the beginning of the proof of the theorem.

Proof of Theorem 1. The distribution of Y_j is

$$P\left(Y_j = m\right) = \frac{\binom{m-1}{j-1}\binom{N-m}{n-j}}{\binom{N}{n}}; \quad m = j, \ldots, N-n+j \text{ and } j = 1, \ldots, n, \tag{5}$$

see, e.g., Wilks(1962, p.243), Arnold, Balakrishnan and Nagaraja (1992, p.54) David and Nagaraja (2003, p.23). We express (3) in the form

$$\left|Y_{\frac{n+1}{2}+r} - \frac{N+1}{2}\right| \geq_{lr} \left|Y_{\frac{n+1}{2}} - \frac{N+1}{2}\right|$$

for any integer r satisfying $-\frac{n-1}{2} \leq r \leq \frac{n-1}{2}$, and by symmetry, and as the case $r = 0$ is trivial, it suffices to consider $r = 1, \ldots, \frac{n-1}{2}$. The random variable $Y_{\frac{n+1}{2}+r}$ takes the values $\frac{n+1}{2} + r, \frac{n+1}{2} + r + 1, \ldots, N - n + \frac{n+1}{2} + r$, and its probability function is

$$P\left(Y_{\frac{n+1}{2}+r} = m\right) = \frac{\binom{m-1}{\frac{n-1}{2}+r}\binom{N-m}{\frac{n-1}{2}-r}}{\binom{N}{n}} \tag{6}$$

$$m = \frac{n+1}{2} + r, \ldots, N - n + \frac{n+1}{2} + r, \text{ and } r = 0, \ldots, \frac{n-1}{2}.$$

Hence, $Y_{\frac{n+1}{2}+r} - \frac{N+1}{2}$ takes the values $-\frac{N-n}{2} + r, \ldots, 0, \ldots, \frac{N-n}{2} + r$ $(r = 0, \ldots, \frac{n-1}{2})$ and the random variable $Y_{\frac{n+1}{2}} - \frac{N+1}{2}$ takes the values $-\frac{N-n}{2}, \ldots, 0, \ldots, \frac{N-n}{2}$.

By direct calculations and (6) we can write

$$E\left|Y_{\frac{n+1}{2}} - \frac{N+1}{2}\right| = \sum_{i=0}^{\frac{N-n}{2}} ip_i, \text{ and } E\left|Y_{\frac{n+1}{2}+r} - \frac{N+1}{2}\right| = \sum_{i=max\left(0,-\frac{N-n}{2}+r\right)}^{\frac{N-n}{2}+r} iq_i,$$

where

$$p_i = P\left(Y_{\frac{n+1}{2}} - \frac{N+1}{2} = i\right) + P\left(Y_{\frac{n+1}{2}} - \frac{N+1}{2} = -i\right)\mathbb{I}(i > 0)$$
$$= 2^{\mathbb{I}(i>0)}\frac{\binom{\frac{N-1}{2}+i}{\frac{n-1}{2}}\binom{\frac{N-1}{2}-i}{\frac{n-1}{2}}}{\binom{N}{n}},$$

$$i = 0, \ldots, \frac{N-n}{2}, \text{ and } \mathbb{I} \text{ denotes the indicator function, and}$$

$$q_i = P\left(Y_{\frac{n+1}{2}+r} - \frac{N+1}{2} = i\right) + P\left(Y_{\frac{n+1}{2}+r} - \frac{N+1}{2} = -i\right)\mathbb{I}\left(0 < i \le \frac{N-n}{2} - r\right)$$
$$= \frac{\binom{\frac{N-1}{2}+i}{\frac{n-1}{2}+r}\binom{\frac{N-1}{2}-i}{\frac{n-1}{2}-r}}{\binom{N}{n}} + \frac{\binom{\frac{N-1}{2}-i}{\frac{n-1}{2}+r}\binom{\frac{N-1}{2}+i}{\frac{n-1}{2}-r}}{\binom{N}{n}}\mathbb{I}\left(0 < i \le \frac{N-n}{2} - r\right);$$
$$i = \max\left\{0, -\frac{N-n}{2} + r\right\}, \dots, \frac{N-n}{2} + r.$$

Recalling that we have to prove the result in the range $1 \le r \le \frac{n-1}{2}$, we consider three cases within this range:
Case 1. If $-\frac{N-n}{2} + r > \frac{N-n}{2}$, that is, $r > N - n$, then Theorem 1 follows immediately, since the range of $\left|Y_{\frac{n+1}{2}+r} - \frac{N+1}{2}\right|$ is completely to the right of the range of $\left|Y_{\frac{n+1}{2}} - \frac{N+1}{2}\right|$.
Case 2. If $\frac{N-n}{2} < r \le \min\left\{N - n, \frac{n-1}{2}\right\}$, then for $-\frac{N-n}{2} + r \le i \le \frac{N-n}{2}$ we have

$$\frac{q_i}{p_i} = \frac{\binom{\frac{N-1}{2}+i}{\frac{n-1}{2}+r}\binom{\frac{N-1}{2}-i}{\frac{n-1}{2}-r}}{2\binom{\frac{N-1}{2}+i}{\frac{n-1}{2}}\binom{\frac{N-1}{2}-i}{\frac{n-1}{2}}} = K\frac{\prod_{j=1}^{r}\left(\frac{N-n}{2} + i - (j-1)\right)}{\prod_{j=1}^{r}\left(\frac{N-n}{2} - i + j\right)}, \qquad (7)$$

where $K = \dfrac{([(n-1)/2]!)^2}{2[(n-1)/2 + r]![(n-1)/2 - r]!}$. It is clear that in this case $\dfrac{q_i}{p_i}$ is increasing in i in the above range. For other values of i the ratio is either 0 or ∞ in a way that $\dfrac{q_i}{p_i}$ is nondecreasing for all i, and Theorem 1 follows. In the calculations below we consider only the range of i where both $p_i,\ q_i > 0$, and the argument for other i's remains the same as above.
Case 3. If $1 \le r \le \min\left\{\frac{N-n}{2}, \frac{n-1}{2}\right\}$ we have

(a) For $0 \le i \le \frac{N-n}{2} - r$

$$\frac{q_i}{p_i} = \frac{\binom{\frac{N-1}{2}+i}{\frac{n-1}{2}+r}\binom{\frac{N-1}{2}-i}{\frac{n-1}{2}-r} + \binom{\frac{N-1}{2}-i}{\frac{n-1}{2}+r}\binom{\frac{N-1}{2}+i}{\frac{n-1}{2}-r}}{2\binom{\frac{N-1}{2}+i}{\frac{n-1}{2}}\binom{\frac{N-1}{2}-i}{\frac{n-1}{2}}}$$
$$= K\frac{\prod_{j=1}^{r}\left(\frac{N-n-2(j-1)}{2}+i\right)\left(\frac{N-n+2j}{2}+i\right) + \prod_{j=1}^{r}\left(\frac{N-n-2(j-1)}{2}-i\right)\left(\frac{N-n+2j}{2}-i\right)}{\prod_{j=1}^{r}\left(\left(\frac{N-n+2j}{2}\right)^2 - i^2\right)},$$

where $K = \frac{1}{2}\prod_{j=1}^{r}\frac{n-1-2(j-1)}{n-1+2j}$. If the numerator of the latter expression expanded, some of its terms will cancel, and others will appear twice with a positive coefficient, and it is easily seen to be a polynomial of degree $2r$ in $i \ge 0$, having positive coefficients. Hence the numerator is increasing in i. It is clear that the denominator is decreasing in $i \ge 0$, therefore the whole expression is an increasing function of i.

(b) For $\frac{N-n}{2} - r + 1 \le i \le \frac{N-n}{2}$ we obtain the same ratio as in (7), and hence monotonicity in this range follows.

In the present case, $1 \le r \le \min\left\{\frac{N-n}{2}, \frac{n-1}{2}\right\}$, we have already shown that for $0 \le i \le \frac{N-n}{2} - r$ (case (a)) and for $\frac{N-n}{2} - r + 1 \le i \le \frac{N-n}{2}$ (case (b)), the ratio q_i/p_i is an increasing function of i. In order to show that q_i/p_i is an increasing function of i throughout the whole range of i, $0 \le i \le \frac{N-n}{2}$, it remains to show that it is increasing when i increases from $\frac{N-n}{2} - r$ to $\frac{N-n}{2} - r + 1$. For this purpose we compute:

$$2\frac{q_{\frac{N-n}{2}-r+1}}{p_{\frac{N-n}{2}-r+1}} - 2\frac{q_{\frac{N-n}{2}-r}}{p_{\frac{N-n}{2}-r}}$$
$$= \frac{\binom{\frac{2N-n-1}{2}-r+1}{\frac{n-1}{2}+r}\binom{\frac{n-1}{2}+r-1}{\frac{n-1}{2}-r}}{\binom{\frac{2N-n-1}{2}-r+1}{\frac{n-1}{2}}\binom{\frac{n-1}{2}+r-1}{\frac{n-1}{2}}} - \frac{\binom{\frac{2N-n-1}{2}-r}{\frac{n-1}{2}+r}\binom{\frac{n-1}{2}+r}{\frac{n-1}{2}-r} + \binom{\frac{2N-n-1}{2}-r}{\frac{n-1}{2}-r}}{\binom{\frac{2N-n-1}{2}-r}{\frac{n-1}{2}}\binom{\frac{n-1}{2}+r}{\frac{n-1}{2}}}$$
$$= \frac{\left[\left(\frac{n-1}{2}\right)!\right]^2}{\left(\frac{n-1}{2}-r\right)!\left(\frac{n-1}{2}+r\right)!}\left[\frac{(N-n-r+1)!\,(r-1)!}{(N-n-2r+1)!(2r-1)!} - \frac{(N-n-r)!}{(N-n-2r)!}\frac{r!}{(2r)!}\right.$$
$$\left. - \frac{(N-n-r)!\,r!}{(N-n)!}\right].$$

Straightforward calculations show that the expression in the last square brackets is positive and the result follows. This completes the proof of Theorem 1. $\square$

3. A Counterexample

Here we provide an example of a nonsymmetric distribution, and show that the conclusion of Theorem 2 does not hold.

Example 1. *Let X_1, X_2, X_3 be iid from a nonsymmetric discrete distribution:*

$$X_i = \begin{cases} 10 & \textit{with probability } \frac{1}{3} \\ 19 & \textit{with probability } \frac{1}{3} \\ 20 & \textit{with probability } \frac{1}{3}, \end{cases}$$

and let Y_1, Y_2, Y_3 be their order statistics. The median of this distribution is $M = 19$, and $E|Y_3 - M| = \frac{28}{27}, E|Y_2 - M| = \frac{70}{27}$, and $E|Y_1 - M| = \frac{172}{27}$, showing that Theorem 2 does not hold in this nonsymmetric case. A similar continuous example can be constructed from the above example using the smoothing approximation described in the beginning of the proof of Theorem 2.

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