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On the Number of Pure Strategy Nash Equilibria in Random Games

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How many pure Nash equilibria can we expect to have in a finite game chosen at random? Solutions to the above problem have been proposed in some special cases. In this paper we assume independence among the profiles, but we allow either positive or negative dependence among the players' payoffs in a same profile. We provide asymptotic results for the distribution of the number of Nash equilibria when either the number of players or the number of strategies increases. We will show that different dependence assumptions lead to different asymptotic results. *Journal of Economic Literature* Classification Number C72. © 2000 Academic Press

1. INTRODUCTION

The idea of Nash equilibrium is one of the most powerful concepts in game theory. Nash (1950; 1951) proved the existence of mixed strategy equilibria for finite normal games. From a decision theoretic viewpoint the concept of mixed strategy Nash equilibrium is less compelling than the concept of pure strategy Nash equilibrium (PNE). It is therefore interesting to

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study how many PNE's we can expect to have in a finite game, under different conditions. We will try to address this problem by considering finite normal games with random payoffs, and by studying how the distribution of the (random) number of PNE's for these games varies under different assumptions. Zero-sum games or common-payoff games constitute extreme cases of negative or positive dependence among the players' payoffs for a given profile of strategies. Such games, with additional random factors or some "noise" in the payoffs lead to intermediate forms of dependence between these two extremes. The particular intermediate case of independence has drawn most of the attention.

Several authors have dealt with the above problem under different assumptions on the structure of the games and on the distribution of the payoffs.

For a two-person zero-sum game, Goldman (1957) shows that if the payoffs for the first player are i.i.d. for different strategy profiles, coming from any continuous distribution, then there can be at most one PNE with probability one, and the probability that there will be a PNE converges to zero as the number of strategies converges to infinity.

For any number of players and strategies, if the payoffs of different players are assumed independent, the number of PNE's has an expected value of 1, and asymptotically as the number of players and/or strategies increase, it has the Poisson distribution. Partial results in this direction can be found in the pioneering work of Goldberg *et al.* (1968), who compute the probability that a finite two-person random game with independent payoffs has at least one Nash equilibrium. Dresher (1970) generalizes this result to n-person games. See also Papavassilopoulos (1995; 1996).

Powers (1990) follows the line of the above authors and investigates the distribution of the number of PNE's. She assumes that the payoffs of the different players are independent and proves that, as the number of strategies of two or more players approaches infinity, the number of PNE's converges in distribution to a Poisson (1).

Stanford (1995) obtains exact distributional results for the number of PNE's, for any fixed number of strategies. The asymptotic results by Powers are then obtained as a by-product.

In a subsequent paper Stanford (1996) deals with the case of symmetric bimatrix games and proves that, as the number of strategies increases, the number of symmetric PNE's converges in distribution to a Poisson (1), and $\frac{1}{2}$ the number of asymmetric PNE's converges to a Poisson ($\frac{1}{2}$).

An asymptotic result in the number of players can be deduced from Example 1 in Arratia *et al.* (1989), which deals with an application of the Chen–Stein method to a graph problem.

Other models lead to a large number of PNE's. For instance Stanford (1997) treats games with vector payoffs, namely, in his model the payoff of each player is a vector of fixed dimension. Under the assumption of random i.i.d. payoffs Stanford proves that, as the number of strategies increases, the expected number of PNE's diverges to infinity, and the difference between the variance and the expected value (of the number of PNE's) converges to zero. This allows to prove a Poisson approximation, which in turn gives a normal limit theorem. The divergence of the expected number of PNE's is due to the fact that, since vector payoffs can only be partially ordered, it is easier for a strategy profile to be a PNE.

Another way to get a large number of PNE's is given by Stanford (1999), who considers common-payoff two-person games, namely, games where the payoffs corresponding to any profile of strategies are the same for the two players. He computes the expected number of PNE's and proves that it diverges as the number of strategies increases, and that for any k the number of PNE's exceeds k with probability that increases to 1.

Some of the results in Stanford (1999) are contained in Baldi *et al.* (1989) (BRS), and in Baldi and Rinott (1989a; 1989b) who studied the distribution of local maxima under random ranking of graph vertices. Example 1 of BRS can be easily translated into game-theoretic language. BRS compute the expected value and the variance of the number of PNE's, and use a version of Stein's method to obtain asymptotic normality. For the case of two players they also provide the exact distribution of the number of PNE's.

The main goal of this paper is to study the expectation and asymptotic distribution of the number of (pure strategy) Nash equilibria under various assumptions about the relations between the players' payoffs. In our model the payoffs of the players corresponding to a given profile of strategies will be correlated, the extreme cases being zero-sum games (maximal negative correlation) and the common-payoff games (maximal positive correlation) considered by BRS and Stanford (1999).

We focus on a particular dependence (correlation) model, the one of the normal copula. For a recent reference, see e.g., Klaassen and Wellner (1997) and further references therein.

We will study two different kinds of asymptotics, in the number of players, and in the number of strategies.

Qualitatively, our findings can be summarized as follows. When the payoffs exhibit negative dependence, one should expect a small number of PNE's, and asymptotically this number converges to zero as the number of strategies increases. For independent payoffs, the asymptotic distribution is Poisson with expectation equal to 1, so that one should expect very few PNE's. On the other hand, for positively dependent payoffs the number of PNE's is large: its expectation diverges and its (standardized) distribution is asymptotically normal as the number of strategies or players gets large. This means that in "large" games with positively correlated payoffs, one should indeed expect to see a large number of PNE's very often.

2. THE MODEL

We consider a strategic game $\langle \mathscr{P}; A_1, \ldots, A_p; u_1, \ldots, u_p \rangle$, where $\mathscr{P} = \{1, \ldots, p\}$ is the set of players, A_i is the set of actions for player i, $u_i\colon \times_{j=1}^{p} A_j \to \mathbb{R}$ is the payoff of player i. For simplicity we assume $\mathrm{card}(A_i) = s$ for $i \in \mathscr{P}$.

For $\mathbf{a} = (a_1, \ldots, a_p)$, define $Y_{\mathbf{a}}^{(i)} = u_i(a_1, \ldots, a_p)$, and $\mathbf{Y}_{\mathbf{a}} = (Y_{\mathbf{a}}^{(1)}, \ldots, Y_{\mathbf{a}}^{(p)})$. Thus, $\mathbf{Y}_{\mathbf{a}}$ is the (random) vector whose p components are the p players' payoffs when player i chooses strategy a_i, $i \in \mathscr{P}$, resulting in the profile $\mathbf{a}$.

For different $\mathbf{a}$'s the payoff profiles $\mathbf{Y}_{\mathbf{a}}$ are assumed to be i.i.d. random vectors. The components of each random payoff profile will be assumed exchangeable, but not necessarily independent (The assumption of exchangeability is made only for simplicity, and is unnecessary. See Remark 2.2 below.)

Furthermore, we postulate that the random payoff profile $\mathbf{Y}_{\mathbf{a}} = (Y_{\mathbf{a}}^{(1)}, \ldots, Y_{\mathbf{a}}^{(p)})$ is a standard multi-normal and

$$\mathrm{Cov}[Y_{\mathbf{a}}^{(i)}, Y_{\mathbf{a}}^{(j)}] = \rho, \qquad i, j \in \mathscr{P},\ i \neq j.$$

It is well known that, when $\rho \geq 0$, this implies the following representation for the Y's

$$Y_{\mathbf{a}}^{(i)} = \sqrt{\rho} X_{\mathbf{a}} + \sqrt{1-\rho} X_{\mathbf{a}}^{(i)}, \qquad i \in \mathscr{P},$$

where all the random variables X's are i.i.d. standard normal.

We emphasize that we use the assumption of normality only to model the dependence structure of the random payoffs, since any other distribution with continuous marginals and the same dependence structure would give the same result. More precisely, any increasing transformation applied to each of the $Y_{\mathbf{a}}^{(i)}$'s marginally, has no effect on the number of PNE's, and thus our model is really a normal copula model with arbitrary equal continuous marginals. The normal copula is used since in it the dependence can be parametrized in a simple way in terms of the correlation coefficient. However, consider for example, p merchants whose payoffs are the sums of profits from a large number of independent customers. Then a multivariate normal payoff distribution may be a natural approximation. Positive correlations may arise if the merchants tend to cooperate with each other, or if their profits depend primarily on the state of the market in which they all operate. Negative correlations would arise if they mostly compete over a more or less fixed number of customers in a fixed economic environment.

A profile $\mathbf{a} = (a_1, \ldots, a_p)$ is a PNE if for all $i \in \mathscr{P}$, and for all $b_i \in A_i$,

$$u_i(a_1, \ldots, a_i, \ldots, a_p) \geq u_i(a_1, \ldots, b_i, \ldots, a_p),$$

namely,

$$Y_{\mathbf{a}}^{(i)} \geq Y_{\mathbf{a}|b_i}^{(i)},$$

where $\mathbf{a}|b_i = (a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_p)$. Define $V_{\mathbf{a}}$ the indicator of the event $\{\mathbf{a}$ is a PNE$\}$, $Q = P(V_{\mathbf{a}} = 1)$, and N be the number of PNE's, namely,

$$N = \sum_{\mathbf{a}} V_{\mathbf{a}}.$$

Note that $E[N] = s^p Q$.

The probability Q that a given strategy profile $\mathbf{a}$ is a PNE can be written as follows:

$$Q = P\big(Y_{\mathbf{a}}^{(i)} \geq W_{\ell}^{(i)},\ i \in \mathscr{P},\ \ell \in A_i \setminus \{a_i\}\big), \tag{2.1}$$

where the random variables W's are i.i.d. standard normal. If the correlation among the Y's is non-negative, a useful representation of (2.1) can be obtained. If we denote by ϕ the standard normal density and by Φ its distribution function, we have, for $\rho \geq 0$,

$$\begin{aligned} Q &= P\big(\sqrt{\rho} X_{\mathbf{a}} + \sqrt{1-\rho} X_{\mathbf{a}}^{(i)} \geq W_{\ell}^{(i)}, i \in \mathscr{P}, \ell \in A_i \setminus \{a_i\}\big) \\ &= \int \Big[\int \Phi^{s-1}\big(\sqrt{\rho} x + \sqrt{1-\rho} z\big) \phi(z)\, dz\Big]^p \phi(x)\, dx, \end{aligned} \tag{2.2}$$

where the expression is obtained by first conditioning on the X's.

Several results in this article rely on the following well-known inequality, whose proof can be found in the original paper by Slepian (1962) and, for instance, in Tong (1980).

THEOREM 2.1 (Slepian's inequality). *Let $\mathbf{X}$ be an n-dimensional multinormal random vector with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma} = (\sigma_{ij})$, and let $\mathbf{Y}$ be an n-dimensional multinormal random vector with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Gamma} = (\gamma_{ij})$. If $\sigma_{ii} = \gamma_{ii}$ for $i = 1, \dots, n$, and $\sigma_{ij} \geq \gamma_{ij}$ for all $i \neq j$, then*

$$P\left(\bigcap_{i=1}^{n} \{X_i > a_i\}\right) \geq P\left(\bigcap_{i=1}^{n} \{Y_i > a_i\}\right)$$

and

$$P\left(\bigcap_{i=1}^{n} \{X_i < a_i\}\right) \geq P\left(\bigcap_{i=1}^{n} \{Y_i < a_i\}\right).$$

Remark 2.2. It is not hard to express Q also as the probability that a multivariate normal vector with a correlation matrix whose entries are either $1/2$ or $\rho/2$, exceeds zero in every coordinate. By Slepian's inequality, it follows that Q is increasing in ρ. Thus simple lower and upper bounds on Q for $0 < \rho < 1$ are obtained by considering the cases $\rho = 0$ and $\rho = 1$, respectively, and the case of $\rho = 0$ also provides an upper bound to Q for negatively correlated payoffs. Moreover, in the case that the components of the payoff profile $\mathbf{Y_a}$ are multivariate normal but not exchangeable, and the smallest (largest) correlations between them has the value ρ, then Slepian's inequality easily shows that the value of Q computed with that ρ provides a lower (upper) bound for Q. Thus our results can be easily modified for the nonexchangeable case, when the correlations are all positive (or all negative).

It is not difficult to see that when $\rho = 0$, namely when the components of the payoff profile are independent, then

$$Q = \int \left[\int \Phi^{s-1}(z)\phi(z)\, dz \right]^p \phi(x)\, dx = \left(\frac{1}{s} \right)^p. \tag{2.3}$$

If $\rho = 1$, i.e., the components of the profile are identical, then

$$\begin{aligned} Q &= \int \left[\int \Phi^{s-1}(x)\phi(z)\, dz \right]^p \phi(x)\, dx \\ &= \int \Phi^{(s-1)p}(x)\phi(x)\, dx \\ &= \frac{1}{(s-1)p+1}. \end{aligned} \tag{2.4}$$

We need the following

DEFINITION 2.3. For positive functions $g(d)$ and $h(d)$, we say that $g(d) \simeq h(d)$ for large d (or as $d \to \infty$), if there exists a positive constant c not depending on d, such that for all $d > c$,

$$\frac{1}{c} h(d) \le g(d) \le c h(d).$$

We summarize some of our results in the following three propositions. The first says that for independent payoffs $E[N] = 1$ and the limit is Poisson, and the two others treat the case that $\rho < 0$ and N converges to zero, and the case $\rho > 0$, $E[N]$ diverges to infinity, and the standardized N is asymptotically normal.

Note that since both $s, p \ge 2$, the existence (and value) of a limit when "$sp \to \infty$" and when "either $s \to \infty$ or $p \to \infty$ or both" is equivalent.

PROPOSITION 2.4. *If* $\rho = 0$, *then* $E[N] = 1$. *Moreover, N is asymptotically Poisson as* $sp \to \infty$.

PROPOSITION 2.5. (1) *If* $\rho = -1/(p-1)$, *namely if the correlation is minimal, which happens in zero-sum games, then* $E[N] \le s^p 2^{-p(s-1)}$.

Hence $E[N] \to 0$ *as* $sp \to \infty$ (*provided* $s > 2$).

(2) *For constant p and* $-1/(p-1) < \rho < 0$, $E[N] \le s^p R_{p,\rho}(s)$, *where*

$$R_{p,\rho}(s) \simeq (\log s)^{\beta(\rho,p)} s^{-p/[1-|\rho|(p-1)]}, \qquad \text{as } s \to \infty,$$

and

$$\beta(\rho, p) = (p-1)(1-\rho)/2[1+\rho(p-1)].$$

Thus $E[N] \to 0$ *as* $s \to \infty$.

In all these cases where $E[N] \to 0$, $N \to 0$ *in probability.*

PROPOSITION 2.6. (1) *If* $0 < \rho < 1$, *then* $Q < T_\rho(sp)$, *where*

$$T_\rho(sp) \simeq (\log(sp))^{(1-\rho)/2(1+\rho)}(sp)^{-2/(1+\rho)} \qquad \text{as } sp \to \infty.$$

(2) *If* $\rho = 1$, *then* $E[N] = s^p/[(s-1)p+1] \to \infty$ *as* $sp \to \infty$. *Moreover, N is asymptotically normal as* $sp \to \infty$.

(3) *If* $0 < \rho < 1$, *then for s fixed,*

$$E[N] \simeq (\log p)^{(1-\rho)/\rho} s^p p^{-(2-\rho)/\rho} \qquad \text{as } p \to \infty.$$

Hence $E[N] \to \infty$ *as* $p \to \infty$.

(4) *If* $0 < \rho < 1$, *then for all* $s, p \ge 2$,

$$E[N] \ge (1/6)(6e)^{-p}(\log s)^{\alpha(\rho,p)} \cdot s^p s^{-p/[1+\rho(p-1)]},$$

with $\alpha(\rho, p) = [\rho(1-p^2) - 1]/2[1+\rho(p-1)]$. *Hence* (*together with part* (3)) $E[N] \to \infty$ *as* $sp \to \infty$.

Also, for fixed $p \ge 2$,

$$E[N] \le A_{\rho,p}(s) \cdot s^p s^{-p/[1+\rho(p-1)]},$$

where $A_{\rho,p}(s) \simeq (\log s)^{\beta(\rho,p)}$ *as* $s \to \infty$, *with* $\beta(\rho, p)$ *defined in Proposition* 2.5 (2). *Moreover, N is asymptotically normal as either* $p \to \infty$ *or as* $s \to \infty$, *with some conditions on p* (*see Corollary* 5.5) *or as both* $s, p \to \infty$.

Proposition 2.4 follows from (2.3) and the discussion of Section 3. Proposition 2.5 is discussed in Section 4. The first part of Proposition 2.6 is proved by Corollary 6.4 in Section 6. The second part is in Example 1 of BRS. The result on $E[N]$ in the third follows from Lemmas 6.1; the fourth part follows from Lemmas 6.3 and 4.3, and the asymptotic normality is covered in detail in Section 5.

All the above results rely only on the assumption of a normal copula, allowing arbitrary equal continuous marginals. In the cases of independent ($\rho = 0$) and common payoffs ($\rho = 1$) all results on the distribution of PNE's such as (2.3) and (2.4) hold without any assumption of normality. All that is needed is that the components of $\mathbf{Y}_{\mathbf{a}}$ are either independent from a continuous distribution, or that all components are equal.

3. INDEPENDENT PAYOFFS AND POISSON LIMITS

In this section we deal with the case of independent payoffs, as considered by Goldberg *et al.* (1968), Dresher (1970), Powers (1990), Stanford (1995), and show that asymptotic results both in the number of players and in the number of strategies can be obtained by a simple application of the Chen–Stein method, as indicated by Arratia *et al.* (1989) (AGG). Powers uses a similar tool to prove a Poisson approximation as $s \to \infty$, while AGG's Example 1 treats the case of $s = 2$ and $p \to \infty$, without using game theoretic terminology.

Consider a game as described in Section 2 with independent payoffs for different players. Again the assumption of normality is not needed. It is possible to prove that the distribution of the number of PNE's converges in total variation to a standard Poisson.

Let A be an arbitrary index set, and for $\mathbf{a} \in A$, let $V_{\mathbf{a}}$ be a Bernoulli random variable with $p_{\mathbf{a}} := P(V_{\mathbf{a}} = 1) = 1 - P(V_{\mathbf{a}} = 0) > 0$. Let

$$N = \sum_{\mathbf{a}\in A} V_{\mathbf{a}} \text{ and } \lambda = E[N] = \sum_{\mathbf{a}\in A} p_{\mathbf{a}}.$$

For each $\mathbf{a} \in A$ let $B_{\mathbf{a}}$ (the neighborhood of dependence of $\mathbf{a}$) be a subset of A such that $\mathbf{a} \in B_{\mathbf{a}}$. Define

$$\beta_1 = \sum_{\mathbf{a}\in A} \sum_{\mathbf{b}\in B_{\mathbf{a}}} p_{\mathbf{a}} p_{\mathbf{b}},$$

$$\beta_2 = \sum_{\mathbf{a}\in A} \sum_{\mathbf{a}\neq\mathbf{b}\in B_{\mathbf{a}}} p_{\mathbf{ab}}, \qquad \text{where } p_{\mathbf{ab}} = E[V_{\mathbf{a}} V_{\mathbf{b}}],$$

$$\beta_3 = \sum_{\mathbf{a}\in A} s_{\mathbf{a}},$$

where

$$s_{\mathbf{a}} = E\big|E\big[V_{\mathbf{a}} - p_{\mathbf{a}} \mid \{V_{\mathbf{b}} : \mathbf{b} \in A \setminus B_{\mathbf{a}}\}\big]\big|.$$

Given two probability measures P, P' on a measurable space $(\Omega, \mathscr{F})$ their total variation distance $d_{\mathrm{TV}}(P, P')$ is defined as follows

$$d_{\mathrm{TV}}(P, P') = 2 \sup_{A\in\mathscr{F}} |P(A) - P'(A)|.$$

A sequence of probability measures $\{P_n\}$ converges in total variation to a measure P if $\lim_{n\to\infty} d_{\mathrm{TV}}(P_n, P) = 0$. Convergence in total variation is quite strong and it implies weak convergence (convergence in distribution).

Given a random variable X, we denote its law by $\mathscr{L}(X)$.

THEOREM 3.1 [Arratia *et al.* (1989)]. *Let Z be a Poisson random variable with $E[Z] = E[N] = \lambda$. Then*

$$d_{\mathrm{TV}}(\mathscr{L}(N) - \mathscr{L}(Z)) \leq 2(\beta_1 + \beta_2 + \beta_3).$$

In the case of a game with independent payoffs, $A = \times_{i=1}^{p} A_i$ is the set of s^p possible strategy profiles, $V_{\mathbf{a}}$ is, as before, the indicator of the event that the profile $\mathbf{a}$ is a PNE, $p_{\mathbf{a}} = s^{-p}$, $B_{\mathbf{a}}$ is the set of all indices $\mathbf{b}$ such that $\mathbf{b}$ differs from $\mathbf{a}$ in at most one coordinate. Therefore $\beta_2 = \beta_3 = 0$, and

$$\beta_1 = s^p[(s-1)p + 1]s^{-2p} = \frac{(s-1)p+1}{s^p}.$$

The above quantity converges to zero as either s or p diverge to infinity. Hence Proposition 2.4 holds.

4. ZERO-SUM GAMES

Because of their importance in game theory, we discuss in this section zero-sum games separately, and then games with negatively dependent payoffs.

Goldman (1957) deals with zero-sum games with only two players. Assuming that payoffs for the first player are i.i.d. for different strategy profiles, coming from any continuous distribution, he shows that $Q = [(s-1)!]^2/(2s-1)!$, and that $P(N > 1) = 0$, and hence the probability that there will be a PNE is $[s!]^2/(2s-1)!$, converging to zero as $s \to \infty$.

In a p-person zero-sum game, that is when the sum $Y_{\mathbf{a}}^{(1)} + \cdots + Y_{\mathbf{a}}^{(p)}$ is constant, the correlation between the payoffs is minimal:

$$\rho = \mathrm{Cov}[Y_{\mathbf{a}}^{(i)}, Y_{\mathbf{a}}^{(j)}] = -\frac{1}{p-1}.$$

In our normal copula model, if we call $\Phi_\rho^{(p)}$ the distribution of a p-variate standard exchangeable multinormal with correlation coefficient $\rho \in [-1/(p-1), 1]$, we have

$$\begin{aligned} Q &= P(Y_{\mathbf{a}}^{(i)} \geq W_\ell^{(i)},\ i \in \mathscr{P},\ \ell \in A_i \setminus \{a_i\}) \\ &= \int \cdots \int \prod_{i=1}^{p} \Phi^{s-1}(z_i)\, d\Phi_\rho^{(p)}(z_1, \ldots, z_p). \end{aligned}$$

With the notation $\bar{z} = p^{-1}\sum_{i=1}^{p} z_i$, the fact that the normal distribution function is log-concave, i.e., $\log\Phi$ is a concave function, implies $\prod_{i=1}^{p}\Phi^{s-1}(z_i) \le \Phi^{p(s-1)}(\bar{z})$, and we obtain

$$Q \le \int\cdots\int \Phi^{p(s-1)}(\bar{z})\, d\Phi_{\rho}^{(p)}(z_1,\ldots,z_p). \tag{4.1}$$

For the case that $\rho = -1/(p-1)$, the zero-sum game, clearly $\bar{z} = 0$ with probability one relative to the measure $d\Phi_{\rho}^{(p)}(z_1,\ldots,z_p)$ and since $\Phi(0) = 1/2$, we obtain

LEMMA 4.1. *For $\rho = -1/(p-1)$, $Q \le 2^{-p(s-1)}$ and hence $E[N] \le s^p 2^{-p(s-1)}$.*

Therefore, for $s > 2$,

$$\lim_{sp\to\infty} E[N] = 0. \tag{4.2}$$

Since N is a non-negative random variable, (4.2) implies that N converges to zero in probability as $sp \to \infty$.

Lemma 4.1 proves the first part of Proposition 2.5 concerning zero-sum games. Note that for such games the first part shows that Q and $E[N]$ converge to zero exponentially fast in the number of strategies s and the number of players p. The calculations below are valid for $-1/(p-1) < \rho < 1$. Lemma 4.3 taken for $-1/(p-1) < \rho < 0$, proves the second part of Proposition 2.5, which concerns games with negatively dependent payoffs. It shows that negatively correlated payoffs have qualitatively similar results to those of zero-sum games: the number of PNE's N converges to zero as $s \to \infty$. However, the convergence is much slower when the correlations are not minimal, being polynomial rather than exponential. Even small "noise" added to a zero-sum game may change N quantitatively by slowing its convergence to zero.

Returning to (4.1) for any $-1/(p-1) < \rho < 1$, a straightforward calculation shows that relative to the measure $d\Phi_{\rho}^{(p)}(z_1,\ldots,z_p)$, $\bar{z}$ is a $N(0,a^2)$ variable with a given in (4.3) below. Therefore,

$$Q \le \int \Phi^{p(s-1)}(ax)\phi(x)\,dx, \qquad \text{where } a = \sqrt{[1+\rho(p-1)]/p}. \tag{4.3}$$

In order to evaluate Q we need the following lemma which is equivalent to Proposition 1 in Rinott and Rotar (1999). The relation "$\sim$" between two quantities below indicates that their ratio converges to 1 as $d \to \infty$.

LEMMA 4.2. *For any fixed $a > 0$,*

$$\int \Phi^{d}(ax)\phi(x)\,dx \simeq (\log d)^{(1-a^2)/2a^2}\left(\frac{1}{d}\right)^{1/a^2} \qquad \textit{as } d \to \infty. \tag{4.4}$$

More precisely,

$$\int \Phi^d(ax)\phi(x)\,dx \sim \frac{1}{a}\Gamma(1/a^2)(2\sqrt{\pi})^{(1-a^2)/a^2}(\log d)^{(1-a^2)/2a^2}\left(\frac{1}{d}\right)^{1/a^2} \quad \text{as } d \to \infty. \tag{4.5}$$

Since a of (4.3) depends on p, we can apply Lemma 4.2 to (4.3) only as $s \to \infty$. As p is fixed, we can simplify the asymptotic expression obtained by relegating it into the constants.

LEMMA 4.3. *For constant* p *and* $-1/(p-1) < \rho < 1$,

$$Q \leq R_{p,\rho}(s), \qquad \text{where } R_{p,\rho}(s) \simeq (\log s)^{\beta(\rho,p)} s^{-p/[1+\rho(p-1)]} \quad \text{as } s \to \infty,$$

with $\beta(\rho, p) = (p-1)(1-\rho)/2[1+\rho(p-1)]$.

As mentioned above, Lemma 4.3 taken for negative ρ implies the second part of Proposition 2.5.

The fact that the probability of finding at least one Nash equilibrium converges to zero as s gets large, whenever payoffs are negatively correlated, magnifies the importance of the Nash existence theorem in mixed strategies.

5. POSITIVELY DEPENDENT PAYOFFS AND APPROXIMATE NORMALITY OF N

In this section we consider the number of PNE's for the case $\rho > 0$. In this case, by Proposition 2.6, $E[N] \to \infty$ as $sp \to \infty$, and one may hope for a normal approximation to the distribution of the standardized N, which we now discuss.

Note again that in the case of $\rho = 1$, i.e., common payoffs, the assumption of normality of the payoffs plays no role. This case is completely covered by BRS Example 1, where it is shown that $E[N] = s^p/[(s-1)p+1]$, $\mathrm{Var}[N] = s^p(p-1)(s-1)/2[(s-1)p+1]^2$, and N is asymptotically normal when either $p \to \infty$ or $s \to \infty$ (or both). Therefore we now focus on the case $0 < \rho < 1$. It is unclear whether the method of BRS extends to the case of $\rho < 1$, and therefore we will use another approach based on Stein's method, see, e.g., Stein (1986), which was used for the case of $\rho = 1$ in Baldi and Rinott (1989a; 1989b). We make use of the following variant, which appears with appropriate references in Dembo and Rinott (1996).

THEOREM 5.1. *Let $V_1, \dots, V_n$ be random variables satisfying $|V_i - E(V_i)| \le B$ a.s., $i = 1, \dots, n$, $E[\sum_{i=1}^n V_i] = \lambda$, $\mathrm{Var}[\sum_{i=1}^n V_i] = \sigma^2 > 0$ and $n^{-1}E[\sum_{i=1}^n |V_i - E(V_i)|] = \mu$. Let $S_i \subset \{1, \dots, n\}$ be such that $j \in S_i$ if and only if $i \in S_j$, and (V_i, V_j) is independent of $\{V_k\}_{k \notin S_i \cup S_j}$ for $i, j = 1, \dots, n$. Set $D = \max_{1 \le i \le n}\{\mathrm{card}(S_i)\}$. Then for all t,*

$$\left| P\left(\frac{\sum_{i=1}^n V_i - \lambda}{\sigma} \le t \right) - \Phi(t) \right| \le 7 \frac{n\mu}{\sigma^3} (DB)^2. \tag{5.1}$$

We can now prove the following

THEOREM 5.2. *If $0 < \rho < 1$, then there exists a constant c depending only on ρ, such that for all t,*

$$\left| P\left(\frac{N - \lambda}{\sigma} \le t \right) - \Phi(t) \right| \le c \frac{s^4 p^4}{s^{p/2} Q^{1/2}},$$

where $\lambda = E[N] = s^p Q$ and $\sigma^2 = \mathrm{Var}[N]$.

Proof. In order to apply Theorem 5.1 to $\sum_{\mathbf{a}} V_{\mathbf{a}}$, we first look at the dependence structure and determine the value of D. Note that, if $V_{\mathbf{a}} = 1$, then $V_{\mathbf{b}} = 0$ for any $\mathbf{b}$ differing from $\mathbf{a}$ at exactly one coordinate and hence $V_{\mathbf{a}}$ and $V_{\mathbf{b}}$ are negatively dependent. On the other hand, with some reflection, it is not hard to see that if $\mathbf{a}$ and $\mathbf{b}$ differ by exactly two coordinates, then $V_{\mathbf{a}}$ and $V_{\mathbf{b}}$ are positively correlated, whereas if $\mathbf{a}$ and $\mathbf{b}$ differ by three coordinates or more, then $V_{\mathbf{a}}$ and $V_{\mathbf{b}}$ are independent (in order to prove the last assertions one must invoke the independence of the $Y_{\mathbf{a}}$'s. This is explained in more detail towards the end of the proof).

We can define $S_{\mathbf{a}}$ to be the set of strategy profile $\mathbf{b}$ which differ from $\mathbf{a}$ by at most two coordinates. We then have $D \le \binom{p}{2} s^2 \le p^2 s^2$. Note that in the case at hand the number of summands is $n = s^p$ and clearly, we can take $B = 1$ and $\mu \le 2Q$.

We now turn to calculations concerning $\sigma^2 = \mathrm{Var}[N]$. Expressing the variance as a sum of all covariances, the discussion above implies that most of the covariances vanish, and

$$\sigma^2 = s^p Q(1 - Q) - s^p p(s - 1)Q^2 + s^p \binom{p}{2} (s - 1)^2 \,\mathrm{Cov}(V_{\mathbf{a}}, V_{\mathbf{b}}), \tag{5.2}$$

where $\mathbf{a}$ and $\mathbf{b}$ are any two profiles which differ in exactly two coordinates. The first term on the r.h.s. is the sum of the variances, the second corresponds to the sum of covariances between $V_{\mathbf{a}}$ and $V_{\mathbf{c}}$ with $\mathbf{a}$ and $\mathbf{c}$ differing in exactly one coordinate (and hence $E[V_{\mathbf{a}} V_{\mathbf{c}}] = 0$, since in this case at most one of $\mathbf{a}$ or $\mathbf{c}$ can be a Nash point). The third term corresponds to the sum of covariances between indicators of profiles which differ in exactly two coordinates. In the normal model assumed throughout this paper, a straightforward calculation using Slepian's inequality, shows that the latter covariances are positive. Indeed, consider the events $F_{\mathbf{a}} := \{V_{\mathbf{a}} = 1\}$ and $F_{\mathbf{c}} := \{V_{\mathbf{c}} = 1\}$. They can be expressed as

$$F_{\mathbf{a}} = \left\{Y_{\mathbf{a}}^{(i)} - Y_{\mathbf{a}|b_i}^{(i)} > 0,\ \forall i \in \mathcal{P},\ \forall b_i \in A_i\right\},$$

$$F_{\mathbf{c}} = \left\{Y_{\mathbf{c}}^{(i)} - Y_{\mathbf{c}|d_i}^{(i)} > 0,\ \forall i \in \mathcal{P},\ \forall d_i \in A_i\right\}.$$

It is easy to see that if the profiles **a** and **c** differ in three coordinates, the variables appearing in $F_{\mathbf{a}}$ are independent of those in $F_{\mathbf{c}}$ and the events are independent.

Suppose now that **a** and **c** agree on all coordinates, except say, the first two. Then $\mathbf{a}|c_2 = \mathbf{c}|a_1$ and $\mathbf{a}|c_1 = \mathbf{c}|a_2$. Thus $Y_{\mathbf{a}|c_2}^{(2)}$ appears in $F_{\mathbf{a}}$ while $Y_{\mathbf{c}|a_1}^{(1)}$ appears in $F_{\mathbf{c}}$, and there is another such pair. These two variables have a correlation ρ and it is now easy to see that Slepian's inequality implies that $P(F_{\mathbf{a}} \cap F_{\mathbf{c}})$ is increasing in ρ. If $\rho = 0$, then $F_{\mathbf{a}}$ and $F_{\mathbf{c}}$ are independent. It follows that, in the case $\rho > 0$, $V_{\mathbf{a}}$ and $V_{\mathbf{c}}$ are positively correlated.

By the first part of Proposition 2.6, we have $p(s-1)Q \to 0$ as $sp \to \infty$, and it is easy to see that $\sigma^2 \geq cs^pQ$ provided sp is large.

Thus, the r.h.s. bound in (5.1) can be bounded above by a constant times $s^pQp^4s^4/[s^pQ]^{3/2}$, and the result follows. ■

Remark 5.3. By Eq. (5.2) we can obtain also a useful upper bound for $\mathrm{Var}[N]$. In fact $\mathrm{Cov}(V_{\mathbf{a}}, V_{\mathbf{b}}) \leq Q$, therefore $\mathrm{Var}[N] \leq Qs^pp^2s^2 = E[N]s^2p^2$. It follows that

$$\sqrt{\mathrm{Var}[N]}/E[N] \leq sp/[s^{p/2}Q^{1/2}].$$

The latter quantity is obviously smaller than the bound of Theorem 5.2. Therefore whenever we can assert the convergence of N to normality by showing that the bound of Theorem 5.2 coverges to zero (see below), it follows that the standard deviation σ is of a smaller order than $E[N]$. Therefore, as $E[N] \to \infty$, in all cases for which we can prove normal convergence of N in distribution, we are guaranteed that, with probability converging to 1, N will be large. For instance we can obtain for all positive ρ results similar to the ones that Stanford (1999) has for the particular case of $\rho = 1$, namely, for any positive integer k, $P(N > k) \to 1$.

Because our evaluations of Q as a function of p and s are done when one is held constant and the other increases to infinity (see Proposition 2.6), our results distinguish between asymptotics in p and in s, and we begin with the limit in p. Applying the evaluation of Q from Lemma 6.1 to Theorem 5.2 we obtain

COROLLARY 5.4. *With the same terminology used in Theorem* 5.2, *for all* t

$$\left|P\left(\frac{N-\lambda}{\sigma} \leq t\right) - \Phi(t)\right| \leq K(p),$$

where

$$K(p) \simeq (\log p)^{-(1-\rho)/2\rho} \frac{p^{3.5+1/\rho}}{s^{p/2}} \qquad \text{as } p \to \infty.$$

It follows that $(N - \lambda)/\sigma \to N(0, 1)$ *in distribution as* $p \to \infty$ *for any fixed* s.

The next result shows asymptotic normality when both s and $p \to \infty$, and also when only $s \to \infty$ and p is fixed. For the latter limit in s, the present results require a restriction on p. This problem (for the case $\rho = 1$) arose also in Baldi and Rinott (1989a; 1989b) and the method introduced in BRS was designed to avoid this restriction. However, as mentioned above, we are unable to extend the method of BRS to the case of $\rho < 1$, and we do not know if asymptotic normality as $s \to \infty$ holds for all p, or whether for $\rho < 1$ some restrictions on p are necessary. Applying the lower bound of Q from Lemma 6.3 to Theorem 5.2 we obtain

COROLLARY 5.5. *For all* $s, p \geq 2$ *and all* t,

$$\left| P\left(\frac{N-\lambda}{\sigma} \leq t\right) - \Phi(t) \right| \leq cJ(s),$$

where

$$J(s) = \sqrt{6}(6e)^{p/2}(\log s)^{-[\rho(1-p^2)-1]/4[1+\rho(p-1)]}\left(\frac{1}{s}\right)^{[\rho p(p-1)]/2[1+\rho(p-1)]-4}.$$

It follows that $(N - \lambda)/\sigma \to N(0, 1)$ *in distribution as both* s *and* $p \to \infty$, *and also if only* $s \to \infty$ *for any* p *large enough such that the exponent of* $1/s$ *in* $J(s)$ *is positive.*

For example, for $\rho = 1/2$, we have asymptotic normality as $s \to \infty$ provided $p > 10$.

6. EVALUATION OF Q

In order to study Q as given in (2.2), that is, the probability that a given profile is a PNE, we need an approximation of the integral $\int [\Phi(ax)]^d \phi(x)\, dx$ for large values of d. This is given in Lemma 4.2 which is proved in Rinott and Rotar (1999). The latter paper contains references to surveys which lead to numerous further references, however, we have not found the required asymptotic result in the literature. It is easy to see that for fixed a, the above integral expresses the quadrant probability $P(Y_1 \leq 0, \ldots, Y_d \leq 0)$, where the Y_i's are exchangeable normals with correlation $\rho = a^2/(1 + a^2)$.

We are unable to provide precise asymptotic results which cover together all cases where either one of s and p or both s and $p \to \infty$, or equivalently (since $s, p \geq 2$) as $sp \to \infty$, though we have useful bounds on Q which are valid as $sp \to \infty$, see Lemma 6.3 and Corollary 6.4. In Lemmas 6.1 and 6.3 below it may be possible to obtain more precise results by extending the calculations in Rinott and Rotar (1999), see Remark 6.2. However, this would require a great deal more technicalities which we prefer to avoid.

The following lemma will provide the asymptotic behavior of Q in p. It proves the third part of Proposition 2.6.

LEMMA 6.1. *For a constant s and $0 < \rho < 1$,*

$$Q = \int \Big[\int \Phi^{s-1}\Big(\sqrt{\rho}x + \sqrt{1-\rho}\, z\Big)\phi(z)\, dz \Big]^p \phi(x)\, dx$$
$$\simeq (\log p)^{(1-\rho)/\rho} p^{-(2-\rho)/\rho} \qquad \text{as } p \to \infty.$$

Proof. An upper bound to Q is obtained by replacing $\Phi^{s-1}(\cdots)$ with $\Phi(\cdots)$. Now the relation

$$\int \Phi\Big(\sqrt{\rho}x + \sqrt{1-\rho}\, z\Big)\phi(z)\, dz = \Phi\Big(\sqrt{\rho/(2-\rho)}x\Big) \tag{6.1}$$

leads to an upper bound of the required order for constant s and large p by Lemma 4.2 with $a = \sqrt{\rho/(2-\rho)}$, and $d = p$.

For the lower bound, again we use (6.1). By the inequality $E[U^{s-1}] \geq (E[U])^{s-1}$ for $s \geq 2$, we have

$$\int \Big[\int \Phi^{s-1}\Big(\sqrt{\rho}x + \sqrt{1-\rho}\, z\Big)\phi(z)\, dz \Big]^p \phi(x)\, dx$$
$$\geq \int \Big[\int \Phi\Big(\sqrt{\rho}x + \sqrt{1-\rho}\, z\Big)\phi(z)\, dz \Big]^{(s-1)p} \phi(x)\, dx$$
$$= \int \Big[\Phi\Big(\sqrt{\rho/(2-\rho)}x\Big) \Big]^{(s-1)p} \phi(x)\, dx.$$

The result now follows (actually giving the precise asymptotics if all constants are computed, see Remark 6.2 below) from Lemma 4.2 with $a = \sqrt{\rho/(2-\rho)}$, and $d = (s-1)p$, noting that a fixed s can be relegated into the constants. ∎

Remark 6.2. Using calculations as in Rinott and Rotar (1999), we can obtain the following more precise result than Lemma 6.1 at the expense of many technicalities that we omit.

For constant s and $0 < \rho < 1$,

$$Q \sim \sqrt{(2-\rho)/\rho}\,\Gamma((2-\rho)/\rho)(4\pi)^{(1-\rho)/\rho}(\log sp)^{(1-\rho)/\rho}\left(\frac{1}{(s-1)p}\right)^{(2-\rho)/\rho} \qquad \text{as } p \to \infty.$$

For asymptotics as $s \to \infty$ we need another evaluation of Q. An upper bound for Q, valid for large s was given in Lemma 4.3. Below we present a lower bound which holds for all $s, p \geq 2$. The upper bound of Lemma 4.3 and the lower bound below exhibit different powers of $\log s$. However, the main terms (considered for large s) in the upper and lower bounds do coincide. Together they prove the fourth part of Proposition 2.6. In particular, the following lower bound is used in concluding that $E[N] \to \infty$ as both $s, p \to \infty$, in Proposition 2.6.

LEMMA 6.3. *For any constant* $0 < \rho < 1$ *and for all* $s, p \geq 2$,

$$Q = \int \Big[\int \Phi^{s-1}\Big(\sqrt{\rho}x + \sqrt{1-\rho}\ z\Big)\phi(z)\,dz\Big]^p \phi(x)\,dx$$
$$\geq (1/6)(6e)^{-p}(\log s)^{\alpha(\rho,\,p)}\left(\frac{1}{s}\right)^{p/[1+\rho(p-1)]}, \tag{6.2}$$

where $\alpha(\rho, p) = [\rho(1-p^2) - 1]/2[1+\rho(p-1)]$.

Proof. First, we use the following Mills' ratio inequality, Gordon (1941), for $x > 0$,

$$\frac{x}{x^2+1}\phi(x) < 1 - \Phi(x) < \frac{1}{x}\phi(x), \tag{6.3}$$

to prove that for $0 < r < 1$ and $y \geq 1$,

$$1 - \Phi(ry) > (1/4)\frac{1}{y}\phi(ry). \tag{6.4}$$

To prove (6.4) note that if $ry > 1$ then by (6.3),

$$1 - \Phi(ry) > \frac{(ry)^2}{(ry)^2+1}\frac{1}{ry}\phi(ry) > (1/2)\frac{1}{y}\phi(ry).$$

On the other hand, if $ry \leq 1$ then

$$1 - \Phi(ry) \geq 1 - \Phi(1) > [1-\Phi(1)]\sqrt{2\pi}\frac{1}{y}\phi(ry) > (1/4)\frac{1}{y}\phi(ry),$$

and (6.4) is proved.

Let $B(s, x) = \{z: \sqrt{\rho}x + \sqrt{1-\rho}\ z \geq \sqrt{2\log s - \log\log s}\}$. For $z \in B(s, x)$ we have $\Phi^{s-1}(\sqrt{\rho}x + \sqrt{1-\rho}\ z) \geq \Phi^{s-1}(\sqrt{2\log s - \log\log s})$. By (6.3) and straightforward calculations we have for $s \geq 2$,

$$\Phi^{s-1}(\sqrt{2\log s - \log\log s}) \geq (1 - 1/s)^{s-1} > e^{-1}.$$

Thus a lower bound to the quantity on the l.h.s. of (6.2) is

$$\begin{aligned} &\int [e^{-1}P(B(s, x))]^p \phi(x)\, dx \\ &\qquad = \int e^{-p}\left[1 - \Phi\left(\frac{\sqrt{2\log s - \log\log s} - \sqrt{\rho}x}{\sqrt{1-\rho}}\right)\right]^p \phi(x)\, dx. \end{aligned} \tag{6.5}$$

We obtain a lower bound to the integral in (6.5) by integrating only over $x \geq \sqrt{2\log s - \log\log s}(1-r)/\sqrt{\rho}$ for some function $r = r(s, p, \rho)$ taking values in $(0, 1)$ to be determined later (it will actually not depend on s). For such an x we obtain from (6.4) with $y = \sqrt{2\log s - \log\log s} \geq 1$ for $s \geq 2$ and simple calculations,

$$\begin{aligned} 1 &- \Phi\left(\frac{\sqrt{2\log s - \log\log s} - \sqrt{\rho}x}{\sqrt{1-\rho}}\right) \\ &\geq 1 - \Phi\left(\frac{r\sqrt{2\log s - \log\log s}}{\sqrt{1-\rho}}\right) \\ &\geq [1/(4\sqrt{2})](\log s)^{[r^2/2(1-\rho)]-1/2} e^{-(r^2\log s)/(1-\rho)}. \end{aligned} \tag{6.6}$$

Therefore, repeating the argument, the quantity in (6.5) is bounded below by

$$\begin{aligned} &(6e)^{-p}(\log s)^{[pr^2/2(1-\rho)]-p/2} e^{-p(r^2\log s)/(1-\rho)} \\ &\quad \times \left[1 - \Phi\left(\sqrt{2\log s - \log\log s}(1-r)/\sqrt{\rho}\right)\right] \\ &\qquad \geq [1/(4\sqrt{2})](6e)^{-p}(\log s)^{[pr^2/2(1-\rho)+(1-r)^2/2\rho]-(p+1)/2} \\ &\qquad\quad \times \left(\frac{1}{s}\right)^{pr^2/(1-\rho)+(1-r)^2/\rho}. \end{aligned} \tag{6.7}$$

We are free to choose r. We choose it to minimize the exponent of $1/s$ in (6.7) and get $r = (1-\rho)/(p\rho + 1 - \rho)$, and the exponent's minimal value for that r is $p/[1 + \rho(p-1)]$. (Recall that $p \geq 2$.) Some straightforward calculations now lead to (6.2). ■

The following upper bound to Q applies as $sp \to \infty$, and can be used to prove the first part of Proposition 2.6.

COROLLARY 6.4. *For a constant* $0 < \rho < 1$,

$$Q \leq T_\rho(sp), \qquad \text{where } T_\rho(sp) \simeq (\log(sp))^{(1-\rho)/2(1+\rho)} \left(\frac{1}{sp}\right)^{2/(1+\rho)} \quad \text{as } sp \to \infty.$$

Proof. The quantity a of (4.3) is decreasing as a function of p, and its maximal value is therefore $a = \sqrt{[1+\rho]/2}$, attained for $p = 2$. The integral in (4.3) is increasing in a, and therefore we obtain from (4.3) a further upper bound

$$Q \leq \int \Phi^{p(s-1)}(\sqrt{[1+\rho]/2}x)\phi(x)\, dx.$$

We can now apply Lemma 4.2 for large sp and the result follows. ■

7. POSSIBLE EXTENSIONS

Expression (2.2) can be adapted to the case where the card$(A_i) = s_i$ and the s_i are not all equal. In this case

$$\begin{aligned} Q &= P(\sqrt{\rho}X_{\mathbf{a}} + \sqrt{1-\rho}X_{\mathbf{a}}^{(i)} \geq W_\ell^{(i)},\ i \in \mathcal{P},\ \ell \in A_i \setminus \{a_i\}) \\ &= \int \prod_{i\in\mathcal{P}} \Big[\int \Phi^{s_i-1}\Big(\sqrt{\rho}x + \sqrt{1-\rho}z\Big)\phi(z)\, dz\Big]\phi(x)\, dx. \end{aligned} \tag{7.1}$$

The extensions of (2.3) and (2.4) are then easy. If all the s_i are of the same order, the asymptotic results go through with little change.

Note that for any $i \in \mathcal{P}$ we have $N \leq \prod_{j\neq i} s_j$. Therefore it is clear that if only s_i, say, diverges, then N is bounded, and it cannot be asymptotically Poisson, even in the independent case. On the other hand, if two of the s_i diverge, we obtain the Poisson limit by the method indicated in Section 3.

Also for the case $\rho < 0$ it can be shown that if only one s_i diverges, then N does not converge to zero in probability, but it does if two of the s_i diverge. Results of the same flavor can be obtained also for $\rho > 0$.

In the whole paper we assumed a uniform correlation within each random payoff profile. As pointed out in Remark 2.2, it is possible to use Slepian's inequality in order to get bounds, even in the case of different correlation coefficients.

A different and much more complicated problem would arise by assuming that different payoffs are not independent. This is natural when considering, for instance, symmetric games. The whole analysis of the problem changes from the start. We hope to deal with this issue in the future.

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