

Introduction to vine copulas

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Lehrstuhl für
Mathematische Statistik

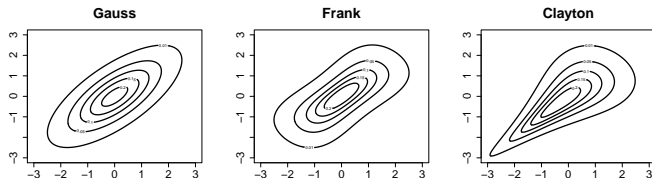


NIPS Workshop, Granada, December 18, 2011

- 1 Motivation and background
- 2 Pair-copula construction (PCC) of vine distribution
- 3 Model selection and estimation
- 4 Applications and extensions
- 5 Summary and Outlook

Motivation

- Copulas model marginal and common dependencies separately.
- There is a wide range of parametric copula families:



- **But:** Standard multivariate copulas
 - can become inflexible in high dimensions.
 - do not allow for different dependency structures between pairs of variables.

⇒ **Vine copulas** for higher-dimensional data.

Overview Vines

Vine pair-copulas

- **Bivariate copulas** are building blocks for higher-dimensional distributions.
 - The dependency structure is determined by the bivariate copulas and a **nested set of trees**.
- Vine approach is more flexible, as we can select bivariate copulas from a wide range of (parametric) families.

Model estimation

- 1 **graph theory** to determine the dependency structure of the data
- 2 **statistical inference** (maximum-likelihood, Bayesian approach ...) to fit bivariate copulas.

Background - Bivariate Copulas

Bivariate Copula

A bivariate copula function

$$C : [0, 1]^2 \rightarrow \mathbb{R}$$

is a distribution on $[0, 1]^2$ with uniform marginals.

Let F be a bivariate distribution with marginal distributions F_1, F_2 .

Sklar's Theorem (1959)

There exists a two dimensional copula $C(\cdot, \cdot)$, such that

$$\forall (x_1, x_2) \in \mathbb{R}^2 : F(x_1, x_2) = C(F_1(x_1), F_2(x_2)).$$

If F_1 and F_2 are continuous, the copula C is unique.

Copula densities

Copula density (2-dimensional)

$$c_{12}(u_1, u_2) = \frac{\partial^2 C_{12}(u_1, u_2)}{\partial u_1 \partial u_2}$$

This implies

- joint density

$$f(x_1, x_2) = c_{12}(F_1(x_1), F_2(x_2)) \cdot f_1(x_1) \cdot f_2(x_2)$$

- conditional density

$$f(x_2|x_1) = c_{12}(F_1(x_1), F_2(x_2)) \cdot f_2(x_2)$$

Important: pair-copula constructions

We can represent a density $f(x_1, \dots, x_d)$ as a product of **pair** copula densities and marginal densities!

Example: $d = 3$ dimensions. One possible decomposition of $f(x_1, x_2, x_3)$ is:

$$f(x_1, x_2, x_3) = f_{3|12}(x_3|x_1, x_2) f_{2|1}(x_2|x_1) f_1(x_1)$$

$$f_{2|1}(x_2|x_1) = c_{12}(F_1(x_1), F_2(x_2)) f_2(x_2)$$

$$f_{3|12}(x_3|x_1, x_2) = c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) f_{3|2}(x_3|x_2)$$

$$f_{3|2}(x_3|x_2) = c_{23}(F_2(x_2), F_3(x_3)) f_3(x_3)$$

$$\begin{aligned} f(x_1, x_2, x_3) &= f_3(x_3) f_2(x_2) f_1(x_1) \text{ (marginals)} \\ &\times c_{12}(F_1(x_1), F_2(x_2)) \cdot c_{23}(F_2(x_2), F_3(x_3)) \text{ (unconditional pairs)} \\ &\times c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) \text{ (conditional pair)} \end{aligned}$$

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Pair-copula construction (PCC) in d dimensions

Joe (1996), Bedford and Cooke (2001), Aas et al. (2009), Czado (2010)

$$f(x_1, \dots, x_d) = \underbrace{\prod_{j=1}^{d-1} \prod_{i=1}^{d-j} c_{i,(i+j)|(i+1), \dots, (i+j-1)}}_{\text{pair copula densities}} \cdot \underbrace{\prod_{k=1}^d f_k(x_k)}_{\text{marginal densities}}$$

with

$$c_{i,j|i_1, \dots, i_k} := c_{i,j|i_1, \dots, i_k}(F(x_i|x_{i_1}, \dots, x_{i_k}), (F(x_j|x_{i_1}, \dots, x_{i_k})))$$

for i, j, i_1, \dots, i_k with $i < j$ and $i_1 < \dots < i_k$.

Remarks:

- The decomposition is not unique.
- Bedford and Cooke (2001) introduced a **graphical structure** called **regular vine structure** to help organize them.

Important: regular vine structure

Example: $d = 3$ dimensions

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1

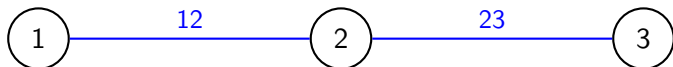
2

3

Important: regular vine structure

Example: $d = 3$ dimensions

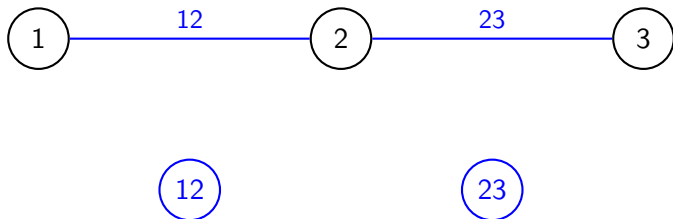
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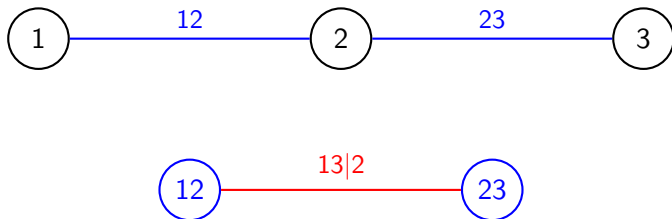
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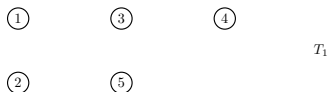
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R-vine structure ($d = 5$)

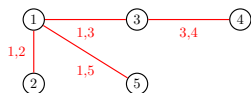
▶ formal definition



Pair-copulas:

- 1 $C_{12}, C_{13}, C_{34}, C_{35}, C_{15}$
- 2 **proximity condition** If two nodes are joined by an edge in tree $j + 1$, the corresponding edges in tree j share a node.
- 3 $C_{23|1}, C_{14|3}, C_{35|1}$
- 4 $C_{24|13}, C_{45|13}$
- 5 $C_{25|134}$

R-vine structure ($d = 5$) ▶ formal definition



T_1

Pair-copulas:

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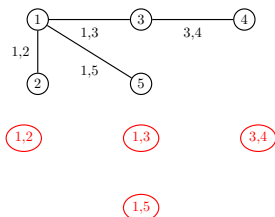
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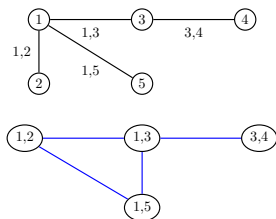
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T_2

R-vine structure ($d = 5$) ▶ formal definition



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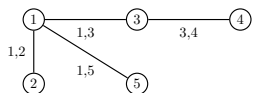
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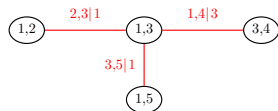
5 $C_{25|134}$

T_2

R-vine structure ($d = 5$) ▶ formal definition



T_1



T_2

(2,3|1)

(1,4|3)

(3,5|1)

T_3

Pair-copulas:

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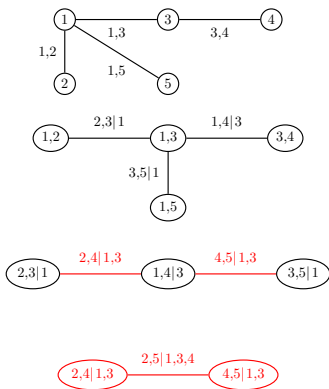
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T_3

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T_4

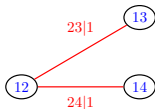
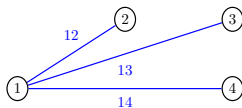
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C-anonical vines

Each tree has a **unique node** that is connected to all other nodes.

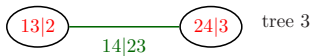
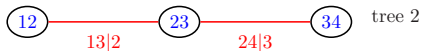
$$f_{1234} = \underbrace{f_1 \cdot f_2 \cdot f_3 \cdot f_4}_{\text{nodes in } T_1} \cdot \underbrace{c_{12} \cdot c_{13} \cdot c_{14}}_{\substack{\text{edges in } T_1 \\ \text{nodes in } T_2}} \cdot \underbrace{c_{23|1} \cdot c_{24|1}}_{\substack{\text{edges in } T_2 \\ \text{nodes in } T_3}} \cdot \underbrace{c_{34|12}}_{\text{edge in } T_3}$$



D_{-rawable} vines

Each tree is a **path**.

$$f_{1234} = \underbrace{f_1 \cdot f_2 \cdot f_3 \cdot f_4}_{\text{nodes in } T_1} \cdot \underbrace{c_{12} \cdot c_{23} \cdot c_{34}}_{\substack{\text{edges in } T_1 \\ \text{nodes in } T_2}} \cdot \underbrace{c_{13|2} \cdot c_{24|3}}_{\substack{\text{edges in } T_2 \\ \text{nodes in } T_3}} \cdot \underbrace{c_{14|23}}_{\text{edge in } T_3}$$



Preliminary summary: pair-copula decomposition

So far

Given a d -dimensional density, we can

- decompose it into products of marginal densities and bivariate copula densities.
- represent this decomposition with nested set of trees that fulfill a proximity condition.

Question

Given data, how can we estimate a pair-copula decomposition?

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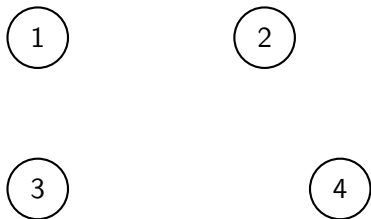
Model selection and parameter estimation

Model = structure (trees) + copula families + copula parameters

Use our software package **CDVine!**
(Brechmann and Schepsmeier (2011))

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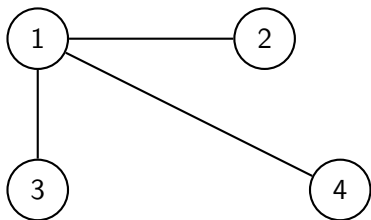


Data

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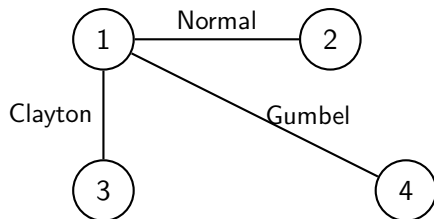
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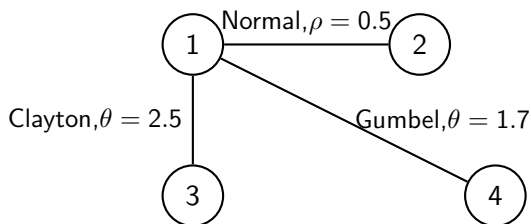
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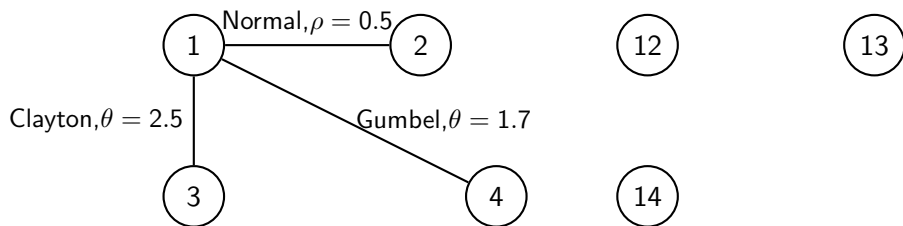
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Pseudo observations

Use our software package **CDVine!**
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Model selection and parameter estimation

Model = structure (trees) + copula families + copula parameters

Problem:

- Huge number of possible vines → structure selection
- $\frac{d(d-1)}{2}$ pair-copulas → copula selection
- parameter estimation

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Structure selection

Possible edge weights

- Kendall's τ
- Spearman's ρ
- p-values of Goodness-of-Fit tests
- distances

Model selection

is done tree by tree via

- optimal C-vines structure selection (Czado et al. (2011))
- Traveling Salesman Problem for D-vines
- Maximum Spanning Tree for R-vines (Dissmann et al. (2011))
- Bayesian approaches (Reversible Jump MCMC)

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Copula selection

Copula selection

can be done via

- Goodness-of-fit tests
- Independence test
- AIC/BIC
- graphical tools like contour plots, λ -function, ...

Possible copula families

- Elliptical copulas (Gauss, t-)
- one-parametric Archimedean copulas (Clayton, Gumbel, Frank, Joe,...)
- two-parametric Archimedean copulas (BB1, BB7,...)
- rotated versions of the Archimedean for neg. dependencies
- ...

Parameter estimation

Estimation approaches:

■ Maximum likelihood estimation

■ Sequential estimation:

- Parameters are **estimated sequentially** starting from the top tree.
- Parameter estimates can be used to define pseudo observations for the next tree
- Parameter estimation via $\theta = f(\tau)$ or bivariate MLE
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■ Bayesian estimation

Applications

Dimensionality of applications

- **Gaussian** vines in **arbitrary** dimensions (Kurowicka and Cooke 2006)
- First **non Gaussian D-vine** models using **joint** maximum likelihood in **4** dimensions
- **Bayesian D-vines** with credible intervals in **7** and **12** dimensions
- **Joint** maximum likelihood now feasible in **50** dimensions for **R-vines**
- **Sequential** estimation of R-vines in **100** dimensions
- Sequential estimation for $d \gg 100$ dimensions with **truncation** (i.e. higher order trees only contain independent copulas)
- Heinen and Valdesogo (2009) sequentially fit a **C-vine autoregressive model** in **100** dimensions

Application areas:

- finance
- insurance
- genetics
- health
- images
- ...

Extensions (Projects of our research group)

Special vine models:

- vine copulas with **time varying** parameters
- **regime switching vine** models
- **non parametric** vine pair copulas
- **Non Gaussian** directed acyclic graphical (**DAG**) models based on PCC's
- **discrete vine** copulas
- **truncated and simplified** R-vines
- **spatial** vines
- copula **discriminant analysis**

Summary and outlook

- PCC's such as C-, D- and R-vines allow for very flexible class of multivariate distributions
- Efficient parameter estimation methods are available for dimensions up to 50
- Model selection of vine tree structures and pair copula types for regular vines still needs further work
- Efficient distance measures between vine distributions would be useful

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Reading material, software and current projects:

<http://www-m4.ma.tum.de/en/research/vine-copula-models>

Workshop: Copulae in Mathematical and Quantitative Finance
July 10-11, 2012, Kraków - Przegorzaly

<http://worcotha.mimuw.edu.pl>

→ Short Course: “Vine copulae; Simulation; Estimation methods; Model selection; Truncation; Vine sector models”.

Regular vine distribution

An d -dimensional **regular vine** is a sequence of $d-1$ trees

1 tree 1

- d nodes: X_1, \dots, X_d
- $d - 1$ edges: pair-copula densities between nodes X_1, \dots, X_d

2 tree j

- $d + 1 - j$ nodes: edges of tree $j - 1$
- $d - j$ edges: conditional pair-copula densities

- **Proximity condition:** If two nodes in tree $j + 1$ are joined by an edge, the corresponding edges in tree j share a node.

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