3 The main theorems of decision theory

3.1 Admissibility and completeness

Definition 18. *A decision rule δ_1, is said to be as good as a rule δ_2, if $R(\theta, \delta_1) \leq R(\theta, \delta_2)$ for all $\theta \in \Theta$. A rule δ_1, is said to be better than a rule δ_2, if $R(\theta, \delta_1) \leq R(\theta, \delta_2)$ for all $\theta \in \Theta$ and a strict inequality holds for at least one θ. Two rules are said to be equivalent if $R(\theta, \delta_1) = R(\theta, \delta_2)$ for all $\theta \in \Theta$.*

The Bayes and the minimax ordering are consistent with this natural definition of order.

Definition 19. *A rule δ is said to be admissible if there exists no rule better than δ.*

Definition 20. *A class $\mathcal{C} \subset \mathcal{D}^*$, is said to be complete, if, given any rule $\delta \in \mathcal{D}^*$ not in $\mathcal{C}$, there exists a rule $\delta_0 \in \mathcal{C}$ that is better than δ. A class $\mathcal{C}$ is said to be essentially complete if, given any rule $\delta \in \mathcal{D}^*$ not in $\mathcal{C}$, there exists a rule $\delta_0 \in \mathcal{C}$ that is as good as δ.*

Lemma 1. *The class of all admissible rules is a subset of any complete class.*

Lemma 2. *If $\mathcal{C}$ is essentially complete and there exists an admissible $\delta \not{in}\mathcal{C}$, then there exists a $\delta' \in \mathcal{C}$ which is equivalent to δ.*

Definition 21. *A class $\mathcal{C}$ of decision rules is said to be minimal complete if $\mathcal{C}$ is complete and if no proper subset of $\mathcal{C}$ is complete. A class $\mathcal{C}$ of decision rules is said to be minimal essentially complete if $\mathcal{C}$ is essentially complete and if no proper subset of $\mathcal{C}$ is essentially complete.*

Theorem 2. *If a minimal complete class exists, it consists of exactly the admissible rules.*

3.2 Admissibility of Bayes rules

Theorem 3. *If, for a given distribution τ, a Bayes rule with respect to τ is unique up to equivalence then this Bayes rule is admissible.*

Theorem 4. *Assume that the support of τ is Θ, and that Θ is finite. If a Bayes rule against τ exists then it is admissible.*

Example 7. *Let $\Theta = \{\theta_1, \theta_2\}$ and take $S = [1, 2] \times [0, 1]$. The Bayes rules against $\tau = (1, 0)$ are not admissible.*

Theorem 5. *Assume that the support of τ is Θ, and that $\Theta = \mathbb{R}$. Let $R(\theta, \delta)$ be a continuous function of θ for all $\delta \in \mathcal{D}^*$. If a Bayes rule against τ exists and has a finite risk then it is admissible.*

4 The Complete Class Theorem

Definition 22. *A set $S \in \mathbb{R}^k$ is said to be bounded from below if $\min_{1 \le i \le k} y_i \ge -M$ for some $M < \infty$ and for all $y \in S$.*

Definition 23. *The lower quantant of a point $x \in \mathbb{R}^k$ is*

$$Q_x = \{y : y_i \le x_i, 1 \le i \le k\}.$$

Definition 24. *A point x is said to be a lower boundary point of a convex $S \subset \mathbb{R}^k$ if $Q_x \cap \bar{S} = x$. The set of lower boundary points of S is denoted by $\lambda(S)$.*

Definition 25. *A convex set $S \subset \mathbb{R}^k$ is said to be closed from below if $\lambda(S) \subset S$.*

Lemma 3. *If a nonempty convex set is bounded from below, then $\lambda(S)$ is not empty.*

Theorem 6. *Suppose Θ is finite and that the risk set is bounded from below and closed from below. The class of decision rules*

$$\mathcal{D}_0 = \{\delta \in \mathcal{D}^* : (R(\theta_1, \delta), \ldots, R(\theta_k, \delta)) \in \lambda(S)\}$$

is then a minimal complete class.

Corollary 1. *The class $\mathcal{D}_0$ consists exactly of the admissible rules.*

Theorem 7 (The Separating Hyperplane Theorem). *Let S_1 and S_2 be two disjoint convex subsets of $\mathbb{R}^k$. Then there exists a vector $p \neq 0$ such that $p'x \le p'y$ for all $x \in S_1$ and $y \in S_2$.*

Theorem 8. *If δ is admissible and Θ is finite, then δ is Bayes (with respect to some τ.*

Theorem 9 (The Complete Class Theorem). *If, for a given decision problem with finite Θ, the risk set is bounded from below and closed from below, then the class of all Bayes rules is complete and the admissible Bayes rules form a minimal complete class.*