Solution to HW 4

Question 2.1 (i) Show that the expectation of complex valued random variables is linear, i.e.,

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y) \ ,$$

where $a, b \in \mathbb{C}$ and X and Y are integrable.
(ii) Show that

$$\mathbb{C}ov(X, Y) = \mathbb{E}(X\bar{Y}) - \mathbb{E}(X)\mathbb{E}(\bar{Y}) \ ,$$

for square integrable X and Y.
Solution: (i) By definition, $\mathbb{E}(X) = \mathbb{E}(X_{(1)}) + i\,\mathbb{E}(X_{(2)})$. Observe that

$$\begin{aligned} aX &= (a_1 + ia_2)(X_{(1)} + iX_{(2)}) = (a_1X_{(1)}) - a_2X_{(2)}) + i(a_1X_{(2)} + a_2X_{(1)}) \\ bY &= (b_1 + ib_2)(Y_{(1)} + iY_{(2)}) = (b_1Y_{(1)}) - b_2Y_{(2)}) + i(b_1Y_{(2)} + b_2Y_{(1)}) \end{aligned}$$

Apply the linearity of the expectation, which is given for real valued random variables, to the real and to the imaginary part produces:

$$\begin{aligned} \mathbb{E}(aX) &= (a_1\mathbb{E}(X_{(1)}) - a_2\mathbb{E}(X_{(2)}))) + i(a_1\mathbb{E}(X_{(2)}) + a_2\mathbb{E}(X_{(1)})) = a\mathbb{E}(X) \\ \mathbb{E}(bY) &= (b_1\mathbb{E}((Y_{(1)}) - b_2\mathbb{E}(Y_{(2)})) + i(b_1\mathbb{E}(Y_{(2)}) + b_2\mathbb{E}(Y_{(1)})) = b\mathbb{E}(Y) \end{aligned}$$

The simpler argument shows that the expectation of the sum of random variables is the sum of expectations. Hence

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y) \ .$$

(ii) Observe that

$$\begin{aligned} (X - \mathbb{E}(X))(\overline{Y - \mathbb{E}(Y)}) &= (X - \mathbb{E}(X))(\overline{Y} - \overline{\mathbb{E}(Y)}) \\ &= X\overline{Y} - X\overline{\mathbb{E}(Y)} - \mathbb{E}(X)\overline{Y} + \mathbb{E}(X)\overline{\mathbb{E}(Y)} \end{aligned}$$

The expectations are numbers. Apply (i) to get

$$\begin{aligned} \mathbb{E}[(X - \mathbb{E}(X))(\overline{Y - \mathbb{E}(Y)})] &= \mathbb{E}[X\overline{Y}] - \mathbb{E}(X)\overline{\mathbb{E}(Y)} - \mathbb{E}(X)\overline{\mathbb{E}(Y)} + \mathbb{E}(X)\overline{\mathbb{E}(Y)} \\ &= \mathbb{E}[X\overline{Y}] - \mathbb{E}(X)\overline{\mathbb{E}(Y)} \ . \end{aligned}$$

Question 2.3 Give an example of a stochastic process (Y_t) such that for arbitrary $t_1, t_2 \in \mathbb{Z}$ and $k \neq 0$

$$\mathbb{E}(Y_{t1}) \neq \mathbb{E}(Y_{t1+k}) \quad \text{but} \quad \mathbb{C}ov(Y_{t_1}, Y_{t_2}) = \mathbb{C}ov(Y_{t_1+k}; Y_{t_2+k}) \ .$$

Solution: Take process of independent random variables Y_t with variance 1 and mean $\mathbb{E}(Y_t) = t$.

Question 2.4 (i) Let (X_t), (Y_t) be stationary processes such that $\mathbb{C}ov(X_t, Y_s) = 0$ for $t, s \in \mathbb{Z}$. Show that for arbitrary $a, b \in \mathbb{C}$ the linear combinations $(aX_t + bY_t)$ yield a stationary process.
(ii) Suppose that the decomposition $Z_t = X_t + Y_t$, $t \in \mathbb{Z}$ holds. Show that stationarity of (Z_t) does not necessarily imply stationarity of (X_t).

Solution: (i) Let $Z_t = aX_t + bY_t$. Observe that $\mathbb{E}(Z_t) = a\mathbb{E}(X_t) + b\mathbb{E}(Y_t)$ is equal to a constant since the expectations of the to originating processes are constant. The covariance satisfies

$$\mathbb{C}ov(Z_t, Z_s) = a\bar{a}\mathbb{C}ov(X_t, X_s) + b\bar{b}\mathbb{C}ov(Y_t, Y_s)$$

since $\mathbb{C}ov(X_t, Y_s) = \mathbb{C}ov(Y_t, X_s) = 0$ by the assumption of lack of correlation. The statement that the covariance is a function of the difference $t - s$ results from the fact that this is the case for the covariance of the two processes that produce the sum.
(ii) Take $Y_t = -X_t$. The sum process is stationary (it is always equal to 0) but (X) can be taken to be a non-stationary process.

Question 2.6 Let Z_1, Z_2 be independent and normal $N(\mu_i; \sigma_i^2)$, $i = 1, 2$, distributed random variables and choose $\lambda \in \mathbb{R}$. For which means $\mu_1, \mu_2 \in \mathbb{R}$ and variances $\sigma_1^2, \sigma_2^2 > 0$ is the cosinoid process $Y_t = Z_1 \cos(2\pi\gamma t) + Z_2 \sin(2\pi\gamma t)$, $t \in \mathbb{Z}$ stationary?

Solution: The mean of the process is

$$\mathbb{E}(Y_t) = \mu_1 \cos(2\pi\gamma t) + \mu_2 \sin(2\pi\gamma t).$$

In order for this function to be fix we must have that if $\gamma \neq k/2$, for $k \in \mathbb{Z}$, then $\mu_2 = 0$. For any value of γ we must have that $\mu_1 = 0$. The covariance of the process is given by

$$\mathbb{C}ov(Y_t, Y_s) = \sigma_1^2 \cos(2\pi\gamma t)\cos(2\pi\gamma s) + \sigma_2^2 \sin(2\pi\gamma t)\sin(2\pi\gamma s).$$

From the trigonometric identity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ we will get that for $\sigma_1^2 = \sigma_2^2 = \sigma^2$ we get that the covariance is equal to $\sigma^2\cos(2\pi\gamma(t - s))$, which is stationary for any γ.