Solution to HW 8

Question 2.34 Compute the autocovariance function of an ARMA(1,2)-process.

Solution: See Homework 7.

Question 2.35 Derive the least squares normal equations for an AR(p)-process and compare them with the Yule-Walker equations.

Solution: We use the approach described in page 106 in order to obtain the least square equations for the AR(p)-process. Observe that

$$\begin{aligned}
\hat{\epsilon}_1 &= y_1 \\
\hat{\epsilon}_2 &= y_2 - a_1 y_1 \\
&\vdots \\
\hat{\epsilon}_{p+1} &= y_{p+1} - a_1 y_p \cdots - a_p y_1 \\
\hat{\epsilon}_{p+2} &= y_{p+2} - a_1 y_{p+1} \cdots - a_p y_2 \\
&\vdots
\end{aligned}$$

It follows that the sum of squares $\sum_{i=1}^{n} \hat{\epsilon}_i^2$ is equal to

$$y_1^2 + (y_2 - a_1 y_1)^2 + \cdots + (y_{p+1} - a_1 y_p \cdots - a_p y_1)^2 + \cdots + (y_n - a_1 y_{n-1} \cdots - a_p y_{n-p})^2$$

Taking a derivative with respect to a_1, equating to 0, and dividing by -2 produces the equation:

$$y_1(y_2 - a_1 y_1) + \cdots + y_p(y_{p+1} - a_1 y_p \cdots - a_p y_1) + \cdots + y_{n-1}(y_n - a_1 y_{n-1} \cdots - a_p y_{n-p}) = 0$$

For a_2 we get the equation

$$y_1(y_3 - a_1 y_2 - a_2 y_1) + \cdots + y_{p-1}(y_{p+1} - a_1 y_p \cdots - a_p y_1) + \cdots + y_{n-2}(y_n - a_1 y_{n-1} \cdots - a_p y_{n-p}) = 0$$

Similar equations are obtained for $a_3, \ldots, a_{p-1}$. Finally, for a_p we get

$$y_1(y_{p+1} - a_1 y_p \cdots - a_p y_1) + \cdots + y_{n-p}(y_n - a_1 y_{n-1} \cdots - a_p y_{n-p}) = 0$$

These p equations can be written in a matrix form as

$$\tilde{\mathbf{R}}\mathbf{a} = \tilde{\mathbf{r}} \ ,$$

The components of the matrix and the vector are:

$$\tilde{\mathbf{R}}_{ij} = \sum_{k=j}^{n-i} y_k y_{k+i-j} \ , \quad \tilde{\mathbf{r}}_i = \sum_{k=1}^{n-i} y_k y_{k+i} \ .$$

Applying the Yule-Walker equations one may obtain estimates of the coefficients $\mathbf{a}$ via the solution of the linear system

$$\mathbf{Ra} = \mathbf{r} \ ,$$

where the components of the matrix $\mathbf{R}$ and the vector $\mathbf{r}$ are the appropriate sample autocorrelations. The solution to the previous system and the solution to the current system are essentially the same. The main difference is that in the computation of the sample autocorrelation one centers the computation by deleting the sample average $\bar{y}$ from each of the observations. As is, the least-squared approach uses a hidden assumption that the expectation of zero mean for the residuals.

Question 2.43 (Zurich Data) The daily value of the Zurich stock index was recorded between January 1st, 1988 and December 31st, 1988. Use a difference filter of first order to remove a possible trend. Plot the (trend-adjusted) data, their squares, the pertaining partial autocorrelation function and parameter estimates. Can the squared process be considered as an AR(1)-process?

Solution: See the attached code. The empirical autocorrelation of the squared difference is consistent with an AR(1)-process.