On the Use of Numeraires in Option Pricing

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Abstract

In this paper we discuss the significant computational simplification that occurs when option pricing is approached through the change of numeraire technique. By pricing an asset in terms of another traded asset (the numeraire), this technique reduces the number of sources of risk which need to be accounted for. Originally developed by Geman (1989) and Jamshidian (1989), this technique is useful in pricing complicated derivatives. In this paper we discuss the underlying theory of the numeraire technique, and illustrate it with five pricing problems:

- Pricing savings plans which incorporate a choice of linkage.
- Pricing convertible bonds.
- Pricing employee stock ownership plans
- Pricing options where the strike price is in a currency different from the stock price.
- Pricing options where the strike price is correlated with the short-term interest rate

JEL classification: G12, G13

Contents

1 Introduction

In this paper we explore five applications of the numeraire method in option pricing. While the numeraire method is well-known in the theoretical literature, it appears to be infrequently used in more applied papers, and many practitioners seem to be unaware of how to use it as well as when it is profitable (or not) to use it. In order to illustrate the uses (and possible misuses) of the method we discuss in some detail five concrete applied problems in option pricing:

- Pricing savings plans which incorporate a choice of linkage.
- Pricing convertible bonds.
- Pricing employee stock ownership plans
- Pricing options where the strike price is in a currency different from the stock price.
- Pricing options where the strike price is correlated with the short-term interest rate

The standard Black-Scholes (BS) formula prices a European option on an asset that follows a geometric Brownian motion. The asset's uncertainty is the only risk factor in the model. A more general approach developed by Black-Merton-Scholes leads to a partial differential equation. The most general method developed so far for the pricing of contingent claims is the martingale approach to arbitrage thory developed by Harrison-Kreps (1981), Harrison-Pliska (1981) and others. However, whether one uses the PDE or the standard "risk neutral valuation" formulas of the martingale method, it is in most cases very hard to obtain analytic pricing formulas. Thus, for many important cases, special formulas (typically modifications of the original BS formula), were developed. See Haug (1997) for an extensive set of examples.

One of the most typical cases of several risk factors occurs when an option is to choose among two assets with stochastic prices. In such a case it is often of considerable advantage to use a **change of numeraire** in the pricing of the option. In what follows we demonstrate examples where the numeraire approach leads to significant simplifications but, in order not to oversell the method, also examples where the numeraire change is trivial or where an obvious numeraire change really does not simplify the computations. The main message is still that in many cases the change of numeraire approach leads to a drastic simplification of the computational work.

In section 2 we start with a brief introductory review of the numeraire method, followed by a mathematical summary (which can be skipped on first reading of this paper). In sections 7-4 we then present five different option pricing problems. For each problem we present the possible choices of numeraire, discuss the pros and cons of the various numeraires, and compute the option prices.

2 The change of numeraire approach

The basic idea of the numeraire approach can be described as follows: Suppose that an option's price depends on several (say n) sources of risk. We may then compute the price of the option according to the following scheme:

- Fix a security which embodies one of the sources of risk, and choose this security as the numeraire.
- Express all prices on the market, including that of the option, in terms of the chosen numeraire. In other words, we perform all our computations in a relative price system.
- Since the numeraire asset in the new price system is riskless (by definition) we have decreased the number of risk factors by one from n to $n-1$. If, for example, we started out with two sources of risk, we can now often apply standard one-risk-factor option pricing formulas (such as Black-Scholes).
- We thus derive the option price in terms of the numeraire. A simple translation from the numeraire back to the local currency will then give the price of the option in monetary terms.

These ideas were developed independently by Geman (1989) and Jamshidian (1989).[1] The standard reference in an abstract setting is Geman, et.al. (1995). In the remainder of this section, we consider a Markovian framework which is simpler than that of the last paper, but which is still reasonably general. All details and proofs can be found in Björk (1999).[2]

Assumption 2.1 *The following objects are given a priori.*

- *An empirically observable* $(k+1)$*-dimensional stochastic process*

$$X = (X_1, \dots, X_{k+1}),$$

with the notational convention

$$X_{k+1}(t) = r(t).$$

- *We assume that under a fixed risk neutral martingale measure* Q*, the factor dynamics have the form*

$$dX_i(t) = \mu_i\left(t, X(t)\right) dt + \delta_i\left(t, X(t)\right) dW(t), \quad i = 1, \dots, k+1,$$

where $W = (W_1, \dots, W_d)'$ *is a standard* d*-dimensional* Q*-Wiener process. The superscript* $'$ *denotes transpose.*

[1]The earliest incarnation of a similar idea is to be found in papers by Fischer (1978) and Brenner and Galai (1978).

[2]The remainder of this section can be skipped by readers interested only in the implementation of the numeraire method.

- *A risk free asset (money account) with the dynamics*

$$dB(t) = r(t)B(t)dt.$$

The interpetation of this is that the components of the vector process X are the underlying factors in the economy. We make no a priori market assumptions, so whether or not a particular component is the price process of a traded asset in the market will depend on the particular application. We now also introduce asset prices, driven by the underlying factors, in the economy.

Assumption 2.2

- *We consider a fixed set of price processes $S_0(t), \ldots, S_n(t)$, each of which is assumed to be the arbitrage free price process for some traded asset without dividends.*
- *Under the risk neutral measure Q, the S-dynamics have the form*

$$dS_i(t) = r(t)S_i(t)dt + S_i(t)\sum_{j=1}^{d} \sigma_{ij}(t, X(t))dW_j(t), \tag{1}$$

  for $i = 0, \ldots, n-1$.
- *The nth asset price is always given by*

$$S_n(t) = B(t),$$

  and thus (1) also holds for $i = n$ with $\sigma_{nj} = 0$ for $j = 1, \ldots, d$.

We now fix an arbitary asset as the numeraire, and for notational convenience we assume that it is S_0. We may then express all other asset prices in terms of the numeraire S_0, thus obtaining the **normalized** price vector $Z = (Z_0, Z_1, \ldots, Z_n)$, defined by

$$Z_i(t) = \frac{S_i(t)}{S_0(t)}.$$

We now have two formal economies: the S economy where prices are measured in he local currency (such as dollars), and the Z-economy, where prices are measured in terms of the numeraire S_0.

The main result is the following theorem, which shows how to price an arbitrary contingent claim in terms of the chosen numeraire. For brevity, a contingent claim with exercise date T will henceforth be referred to as a "T-claim".

Theorem 2.1 (Main theorem) *Let the numeraire S_0 be the price process for a traded assset with $S_0(t) > 0$ for all t. Then there exists a probability measure, denoted by Q^0, with the following properties.*

- *For every T-claim Y, the corresponding arbitrage free price process $\Pi(t;Y)$ in the S-economy is given by*

$$\Pi(t;Y) = S_0(t)\Pi^Z\left(t;\frac{Y}{S_0(T)}\right), \tag{2}$$

where Π^Z denotes the arbitrage free price in the Z-economy.

- *For any T-claim $\tilde{Y}$, its arbitrage free price process Π^Z in the Z economy is given by the formula*

$$\Pi^Z\left(t;\tilde{Y}\right) = E^0_{t,X(t)}\left[\tilde{Y}\right], \tag{3}$$

where E^0 denotes expectation w.r.t. Q^0. In particular, the pricing formula (2) can be written

$$\Pi(t;Y) = S_0(t)E^0_{t,X(t)}\left[\frac{Y}{S_0(T)}\right]. \tag{4}$$

- *The Q^0-dynamics of the Z-processes are given by*

$$dZ_i = Z_i\left[\sigma_i - \sigma_0\right]dW^0, \quad i = 0,\dots,n. \tag{5}$$

- *The Q^0-dynamics of the price processes are given by*

$$dS_i = S_i\left(r + \sigma_i\sigma_0'\right)dt + S_i\sigma_i dW^0, \tag{6}$$

where W^0 is a Q^0-Wiener process.

- *The Q^0-dynamics of the X-processes are given by*

$$dX_i = \left(\mu_i + \delta_i\sigma_0'\right)dt + \delta_i dW^0. \tag{7}$$

- *The measure Q^0 depends upon the choice of numeraire asset S_0, but the same measure is used for all claims, regardless of their exercise dates.*

In passing we note that if we use the money account B as the numeraire, then the pricing formula above reduces to the well known standard risk neutral valuation formula

$$\Pi(t;Y) = B(t)E^0_{t,X(t)}\left[\frac{Y}{B(T)}\right] = E^0_{t,X(t)}\left[e^{-\int_t^T r(s)ds}Y\right] \tag{8}$$

In more pedestrian terms, the main points of the Theorem above are as follows.

- The pricing formula (2) shows that the measure Q^0 "takes care of the stochasticity" related to the numeraire S_0. Note that we do not have to compute the price $S_0(t)$–we simply use the observed market price. We also see that if the claim Y is of the form $Y = Y_0 \cdot S_0(T)$ (where Y_0 is some T-claim) then the change of numeraire is a huge simplification of the standard risk neutral formula (8): Instead of computing the joint distribution of $\int_t^T r(s)ds$ and Y (under Q) we only have to compute the distribution of Y_0 (under Q^0).

- The formula (3) shows that in the Z economy, prices are computed as expected values of the claim. Observe that there is no discounting factor in (3). The reason for this is that in the Z economy, the price process Z_0 has the property that $Z_0(t) = 1$ for all t. Thus, in the Z-economy there is a riskless asset with unit price, i.e. **in the Z economy the short rate equals zero**.
- Formula (5) says that the normalized price processes are martingales (i.e. zero drift) under Q^0, and identifies the relevant volatility.
- Formulas (6)-(7) shows how the dynamics of the asset prices and the underlying a factors change when we move from Q to Q^0. Note that the crucial object is the volatility σ_0 if the numeraire asset.

In the following sections we show examples of the use of the numeraire method which illustrate the considerable conceptual and implementational simplification to which this method leads.

3 Employee stock ownership plans

3.1 Institutional setup

In employee stock ownership plans (ESOP) it is common to include an option of essentially the following form: The holder has the right to buy a stock at the minimum between its price in 6 months and in 1 year minus a rebate (say 15%). The exercise is one year.

3.2 Mathematical model

In a more general setting the ESOP is a contingent claim Y, to be paid out at time T_1, of the form

$$Y = S(T) - \beta \min\left[S(T_1), S(T_0)\right], \tag{9}$$

so in the concrete case above we would have $\beta = 0.85$, $T_0 = 1/2$ and $T_1 = 1$. The problem is to price Y at some time $t \leq T_0$, and to this end we assume a standard Black-Scholes model where, under the usual risk neutral measure Q we have the dynamics

$$\begin{aligned} dS(t) &= rS(t)dt + \sigma S(t)dW(t), & (10) \\ dB(t) &= rB(t)dt, & (11) \end{aligned}$$

with a deterministic and constant short rate r. The price $\Pi(t;Y)$ of the option can obviously be written

$$\Pi(t;Y) = S(t) - \beta\Pi(t;Y_0)$$

where the T_1-claim Y_0 is defined by

$$Y_0 = \min[S(T_1), S(T_0)].$$

In order to compute the price of Y_0 we now basically want to do as follows.

- Perform a suitable change of numeraire.
- Use a standard version of some well known option pricing formula.

The problem with carrying out this small program is that, at the exercise time T_1, the term $S(T_0)$ does not have a natural interpretation as a spot price of a traded asset. In order to overcome this difficulty we therefore introduce a new asset S_0 defined by

$$S_0(t) = \begin{cases} S(t), & 0 \le t \le T_0, \\ S(T_0)e^{r(t-T_0)}, & T_0 \le t \le T_1. \end{cases}$$

In other words, S_0 can be thought of as the value of a self financing portfolio where you at $t = 0$ buy one share of the underlying stock and keep it until $t = T_0$. At $t = T_0$ you then sell the share and put all the money into the bank account.

We then have $S_0(T_1) = S(T_0)e^{r(T_1-T_0)}$ so we can now express Y_0 in terms of $S_0(T_1)$ as

$$Y_0 = \min[S(T_1), K \cdot S_0(T_1)] \tag{12}$$

where

$$K = e^{-r(T_1-T_0)} \tag{13}$$

The point of this is that $S_0(T_1)$ in (12) can formally be treated as the price at T_1 of a traded asset. In fact, from the definition above we have the following trivial Q-dynamics for S_0

$$dS_0(t) = rS_0(t)dt + S_0(t)\sigma_0(t)dW(t)$$

where the deterministic volatility is defined by

$$\sigma_0(t) = \begin{cases} \sigma, & 0 \le t \le T_0, \\ 0, & T_0 \le t \le T_1. \end{cases} \tag{14}$$

It is now time to perform a change of numeraire, and we can choose either S or S_0 as the numeraire. From a logical point of view the choice is irrelevant, but the computations become somewhat easier if we choose S_0. With S_0 as the numeraire we obtain (always with $t < T_0$) the following pricing formula from Theorem 2.1

$$\Pi\left(t; Y_0\right) = S_0(t) E^0_{t,S_0(t)}\left[\min\left\{Z(T_1), K\right\}\right] \tag{15}$$

where

$$Z(t) = \frac{S(t)}{S_0(t)}$$

is the normalized price process. From (5) we furthermore have

$$dZ(t) = Z(t)\left[\sigma - \sigma_0(t)\right] dW^0(t) \tag{16}$$

where W^0 is Q^0-Wiener. Using the simple equality

$$\min\left\{Z(T_1), K\right\} = Z(T_1) - \max\left\{Z(T_1) - K, 0\right\},$$

and noting that for $t \leq T_0$ we have $S_0(t) = S(t)$, we obtain from (15)

$$\Pi\left(t; Y_0\right) = S(t) E^0_{t,S(t)}\left[Z(T_1)\right] - S(t) E^0_{t,S(t)}\left[\max\left\{Z(T_1) - K, 0\right\}\right].$$

Since Z is a Q^0 martingale (zero drift) and $Z(t) = 1$ for $t \leq T_0$ we have

$$S(t) E^0_{t,S(t)}\left[Z(T_1)\right] = S(t) Z(t) = S(t).$$

It now only remains to compute $E^0_{t,S(t)}\left[\max\left\{Z(T_1) - K, 0\right\}\right]$ but this is just the price of a European call with strike price K in a world with, a stock price process Z following GBM as in (16), and zero short rate (see the remarks after Theorem 2.1). ¿From (16), and the definition of σ_0 in (14), the integrated squared volatility for Z over the time interval $[t, T_1]$ is given by

$$\int_t^{T_1} \left[\sigma - \sigma_0(u)\right]^2 du = \sigma^2 \cdot (T_1 - T_0).$$

¿From the Black-Scholes formula with zero short rate and deterministic but time varying volatility we now have

$$E^0_{t,S(t)}\left[\max\left\{Z(T_1) - K, 0\right\}\right] = Z(t) N[d_1] - K N[d_2]$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(Z(t)/K\right) + \frac{1}{2}\sigma^2 (T_1 - T_0)}{\sigma\sqrt{T_1 - T_0}}, \\ d_2 &= d_1 - \sigma\sqrt{T_1 - T_0}. \end{aligned}$$

Using again the trivial fact that, by definition $Z(t) = 1$ for all $t \leq T_0$, and collecting the computations above we finally obtain the price of the ESOP as

$$\Pi\left(t; ESOP\right) = S(t) - \beta S(t) N[d_1] + \beta S(t) K N[d_2], \tag{17}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(1/K\right) + \frac{1}{2}\sigma^2 (T_1 - T_0)}{\sigma\sqrt{T_1 - T_0}}, \\ d_2 &= d_1 - \sigma\sqrt{T_1 - T_0}, \end{aligned}$$

and where K is given by (13).

4 Options with a foreign-currency strike price

In this section we discuss options whose strike price is linked to a non-domestic currency. We illustrate with the example of an option with a US dollar strike price on a stock denominated in UK pounds. Such options might be part of an executive compensation program; such options might be given to motivate managers to maximize the dollar price of their stock. Another example is an option where the strike price is CPI-indexed.

4.1 Institutional setup

For purposes of illustration we assume that the underlying security is traded in the UK in pound sterling and that the option exercise price is in dollars. The institutional setup is as follows.

- The option is initially (i.e. at $t = 0$) an at-the-money option, when the strike price is expressed in pounds.[3]
- This pound strike price is, at $t = 0$, converted into dollars.
- The dollar strike price thus computed is kept constant during the life of the option.
- At the exercise date $t = T$ the holder can pay the fixed dollar strike price in order to obtain the underlying stock.
- The option is fully dividend protected.

Since the stock is traded in pounds, the fixed dollar strike corresponds to a randomly changing strike price when expressed in pounds; thus we have a non-trivial valuation problem. The numeraire approach can be used to simplify the valuation of such an option. The resulting valuation is given in (23).

4.2 Mathematical model

We model the stock price S (in pounds) as a standard geometric Brownian motion under the objective probability measure P, and we assume deterministic short rates r_p and r_d in the UK and the US market respectively. Since we have assumed complete dividend protection we may as well assume (from a formal point of view) that S is without dividends. We thus have the following P-dynamics for the stock price.

$$dS(t) = \alpha S(t)dt + S(t)\delta_S dW^S(t),$$

[3] For tax reasons most executive stock options are initially at-the-money.

We denote the dollar/pound exchange rate by X, and assume a standard Garman-Kohlhagen (1983) model for X. We thus have P-dynamics given by

$$dX(t) = \alpha_X X(t)dt + X(t)\delta_X dW^X(t),$$

Denoting the pound/dollar exchange rate by Y, where $Y = 1/X$, we immediately have the dynamics

$$dY(t) = \alpha_Y Y(t)dt + Y(t)\delta_Y dW^Y(t)$$

where α_Y is of no interest for pricing purposes. Here W^S, W^X and W^Y are scalar Wiener processes and we have the relations

$$\begin{aligned}
\delta_Y &= \delta_X && (18)\\
W^Y &= -W^X, && (19)\\
dW^S(t) \cdot dW^X(t) &= \rho dt, && (20)\\
dW^S(t) \cdot dW^Y(t) &= -\rho dt. && (21)
\end{aligned}$$

For computational purposes it is sometimes convenient to express the dynamics in terms of a two dimensional Wiener process W with independent components intead of using the two correlated processes W^X and W^S. Logically the two approaches are equivalent, and in the new W-formalism we then have the P-dynamics

$$\begin{aligned}
dS(t) &= \alpha S(t)dt + S(t)\sigma_S dW(t),\\
dX(t) &= \alpha_X X(t)dt + X(t)\sigma_X dW(t),\\
dY(t) &= \alpha_Y Y(t)dt + Y(t)\sigma_Y dW(t).
\end{aligned}$$

The volatilities σ_S, σ_X and σ_Y are two-dimensional row vectors with the properties that

$$\begin{aligned}
\sigma_Y &= -\sigma_X\\
\|\sigma_X\|^2 &= \delta_X^2,\\
\|\sigma_Y\|^2 &= \delta_Y^2,\\
\|\sigma_S\|^2 &= \delta_S^2,\\
\sigma_X \sigma_S' &= \rho\delta_X\delta_S\\
\sigma_Y \sigma_S' &= -\rho\delta_Y\delta_S
\end{aligned}$$

where $'$ denotes transpose and $\| \ \|$ denotes the Euclidian norm in R^2.

The initial strike price expressed in pounds is by definition given by

$$K_p(0) = S(0),$$

and the corresponding dollar strike price is thus

$$K_d = K_p(0) \cdot X(0) = S(0)X(0).$$

The dollar strike price is kept constant until the exercise date. However, expressed in pounds the strike price evolves dynamically as a result of the varying exchange rate, so the pound strike at maturity is given by

$$K_p(T) = K_d \cdot X(T)^{-1} = S(0) \cdot X(0) \cdot X(T)^{-1}. \tag{22}$$

There are now two natural ways to value this option: we can work in dollars or in pounds, and initially it is not obvious which way is the easier. We will in fact perform the calculations in both alternatives and compare the computational effort. As will be seen below it turms out to be slightly easier to work in dollars than in pounds.

4.3 Pricing the option in dollars

In this approach we transfer all data into dollars. The stock price, expressed in dollars, is given by

$$S_d(t) = S(t) \cdot X(t),$$

so in dollar terms the payout Φ_d of the option at maturity is given by the expression

$$\Phi_d = \max\left[S(T)X(T) - K_d, 0\right]$$

Since the dollar strike K_d is constant we can use the Black-Scholes formula applied to the dollar price process $S_d(t)$. The Itô formula applied to $S_d(t) = S(t)X(t)$ immediately gives us the P-dynamics of $S_d(t)$ as

$$dS_d(t) = S_d(t)\left(\alpha + \alpha_X + \sigma_S\sigma_X'\right)dt + S_d(t)\left(\sigma_S + \sigma_X\right)dW(t)$$

We can write this as

$$dS_d(t) = S_d(t)\left(\alpha + \alpha_X + \sigma_S\sigma_X'\right)dt + S_d(t)\delta_{S,d}dV(t)$$

where V is a scalar Wiener process and where

$$\delta_{S,d} = \|\sigma_S + \sigma_X\| = \sqrt{\delta_S^2 + \delta_X^2 + 2\rho\delta_S\delta_X}$$

is the dollar volatility of the stock price.

The dollar price (expressed in dollar data) at t of the option is now obtained directly from the Black-Scholes formula as

$$\begin{aligned} C_d(t) &= S_d(t)N[d_1] - e^{-r_d(T-t)}K_dN[d_2], \\ d_1 &= \frac{\ln\left(S_d(t)/K_d\right) + \left(r_d + \frac{1}{2}\delta_{S,d}^2\right)(T-t)}{\delta_{S,d}\sqrt{T-t}}, \\ d_2 &= d_1 - \delta_{S,d}\sqrt{T-t}. \end{aligned} \tag{23}$$

The corresponding price in pound terms is finally obtained as

$$C_p(t) = C_d(t) \cdot \frac{1}{X(t)},$$

so we have the final pricing formula

$$\begin{aligned}
C_p(t) &= S(t)N[d_1] - e^{-r_d(T-t)}\frac{S(0)X(0)}{X(t)}N[d_2], \qquad (24)\\
d_1 &= \frac{\ln\left(\frac{S(t)X(t)}{S(0)X(0)}\right) + \left(r_d + \frac{1}{2}\delta_{S,d}^2\right)(T-t)}{\delta_{S,d}\sqrt{T-t}},\\
d_2 &= d_1 - \delta_{S,d}\sqrt{T-t},\\
\delta_{S,d} &= \sqrt{\delta_S^2 + \delta_X^2 + 2\rho\delta_S\delta_X}
\end{aligned}$$

4.4 Pricing the option directly in pounds

Although this is not immediately obvious, pricing the option directly in pounds is a bit more complicated than pricing the option in dollars. The pricing problem, expressed in pound terms, is that of pricing the T-claim Φ_p defined by

$$\Phi_p = \max\left[S(T) - K_p(T), 0\right].$$

Using (22) and denoting the pound/dollar exchange rate by Y (where of course $Y = 1/X$) we obtain

$$\Phi_p = \max\left[S(T) - S(0)\frac{Y(T)}{Y(0)}, 0\right].$$

It is now tempting to use the punds/dollar exchange rate Y as the numeraire but this is not allowed. The reason is that although Y is the price of a traded asset (dollar bills) it is not the price of a traded asset *without dividends*, the obvious reason being that dollars are put into an American (or perhaps Eurodollar) account where they will command the interest rate r_d. Thus the role of Y is rather that of the price of an asset with the continuous dividend yield r_d. In order to convert the present situation into the standard case covered by Theorem 2.1 we therefore do as follows.

- We denote the dollar bank account by B_d with dynamics

$$dB_d(t) = r_d B_d(t)dt.$$

- The value in pounds, of the dollar bank account, is then given by the process $\hat{Y}_d$, defined by

$$\hat{Y}_d(t) = B_d(t) \cdot Y(t) = Y(t)e^{r_d t}.$$

- The process $\hat{Y}_d$ can now be interpreted as the price process (denoted in pounds) of a traded asset without dividends.
- We may thus use $\hat{Y}_d$ as a numeraire.

Since we have $Y(T) = \hat{Y}_d(T)e^{-r_dT}$ we can write

$$\Phi_p = \max\left[S(T) - \hat{Y}_d(T)e^{-r_dT}\frac{S(0)}{Y(0)}, 0\right].$$

Using $\hat{Y}_d$ as the numeraire we immediately obtain from Theorem 2.1

$$\Pi(t; \Phi_p) = \hat{Y}_d(t)E_t^{\hat{Q}}\left[\max\{Z(T) - K, 0\}\right], \tag{25}$$

where $\hat{Q}$ denotes the martingale measure with $\hat{Y}_d$ as the numeraire, where Z is defined by

$$Z(t) = \frac{S(t)}{\hat{Y}_d(t)},$$

and where K is given by

$$K = e^{-r_dT}\frac{S(0)}{Y(0)}. \tag{26}$$

¿From Theorem 2.1 we know that Z has zero drift under $\hat{Q}$, and a simple calculation shows that the $\hat{Q}$ dynamics of Z are given by

$$dZ(t) = Z(t)\left(\sigma_S - \sigma_Y\right)d\hat{W}(t),$$

where $\hat{W}$ is $\hat{Q}$-Wiener. Thus the expectation in (25) is given by the Black-Scholes formula for a call, with strike price K, written on an asset with (scalar) volatility

$$\delta_Z = \|\sigma_S - \sigma_Y\| = \sqrt{\|\sigma_S\|^2 + \|\sigma_Y\|^2 - 2\sigma_S\sigma_Y^\star} = \sqrt{\delta_S^2 + \delta_Y^2 + 2\rho\delta_S\delta_Y}$$

in a world with zero interst rate. We thus obtain the pricing formula

$$\begin{aligned}
\Pi(t; \Phi_p) &= \hat{Y}_d(t)\left\{Z(t)N[d_1] - KN[d_2]\right\} \\
d_1 &= \frac{1}{\delta_Z\sqrt{T-t}}\left\{\ln\left(\frac{Z(t)}{K}\right) + \frac{1}{2}\delta_Z^2(T-t)\right\}, \\
d_2 &= d_1 - \delta_Z\sqrt{T-t}
\end{aligned}$$

After simplification this reduces to the pricing formula, which of course coincides with (24).

$$C_p(t) = \Pi(t; \Phi_p) = S(t)N[d_1] - Y(t)e^{-r_d(T-t)}\frac{S(0)}{Y(0)}N[d_2], \tag{27}$$

where

$$\begin{aligned}
d_1 &= \frac{1}{\delta_Z\sqrt{T-t}}\left\{\ln\left(\frac{S(t)Y(0)}{Y(t)S(0)}\right) + \left\{r_d + \frac{1}{2}\delta_Z^2\right\}(T-t)\right\}, \\
d_2 &= d_1 - \delta_Z\sqrt{T-t}, \\
\delta_Z &= \sqrt{\delta_S^2 + \delta_Y^2 + 2\rho\delta_S\delta_Y}.
\end{aligned}$$

In this case we have thus seen that there were two distinct (but logically equivalent) ways of pricing the option. From the computations above it is also clear (ex post) that the easiest way was by using the dollar bank account as the numeraire, rather than using the pound value of the same account.

5 Pricing convertible bonds

Standard pricing models of convertible bonds concentrate on pricing the bond and its conversion option at date $t = 0$ (see, for example, Brennan-Schwartz 1977, Bardhan et.al. 1993). A somewhat less-standard problem is the pricing of the bond at some date $0 < t < T$, where T is the maturity date of the bond. We consider this problem in this section; again we see that the numeraire approach gives a relatively simple solution to this problem; the "trick" is to use the stock price as the numeraire. This gives a relatively simple pricing formula for the bond (equation (31) below), which we now derive.

5.1 Institutional setup

A convertible bond involves two underlying objects: a discount bond and a stock. The more precise assumptions are as follows.

- The bond is a zero coupon bond with face value 1.
- The bond matures at a fixed date T_1.
- The underlying stock pays no dividends
- At a fixed date T_0, with $T_0 < T_1$, the bond can be converted to one share of the stock.

The problem is of course that of pricing, at time $t < T_0$, the convertible bond.

5.2 Mathematical model

We introduce the following notation

$$\begin{aligned} S(t) &= \text{the price, at time } t\text{, of the stock} \\ p(t,T) &= \text{the price, at time } t\text{, of a zero-coupon bond of the same risk class.} \end{aligned}$$

We now view the convertible bond as a contingent claim Y with exercise date T_0. Given the setup above, the claim Y is thus given by the expression

$$Y = \max\left[S(T_0), p(T_0, T_1)\right].$$

In order to price this claim we have two obvious possibilities: we can use either the stock or the zero-coupon bond maturing at T_1 as the numeraire. Assuming that the T_1 bond actually **is** traded we immediately obtain the price as

$$\Pi\left(t; Y\right) = p(t, T_1) E_t^1\left[\max\left\{Z(T_0), 1\right\}\right],$$

where E^1 denotes expectation under the "forward neutral" martingale measure Q^1 with the T_1 bond as numeraire. The process Z is defined by

$$Z(t) = \frac{S(t)}{p(t,T_1)}.$$

We can now simplify and write

$$\max\{Z(T_0),1\} = \max\{Z(T_0)-1,0\}+1,$$

giving us

$$\Pi(t;Y) = p(t,T_1)E_t^1\left[\max\{Z(T_0)-1,0\}\right] + p(t,T_1) \tag{28}$$

In more verbal terms this just says that the price of the convertible bond equals the price of a conversion option plus the price of the underlying zero coupon bond. Since we assumed that the T_1 bond is traded, we do not have to compute the price $p(t,T_1)$ in the formula above, but instead we simply observe the price on the market. It thus only remains to compute the expectation above, and this is obviously the price, at time t, of a European call with strike price 1 on the price process Z in a world where the short rate equals zero. Thus the numeraire approach gives a big simplification of the computational problem.

In order to obtain more explicit results, we now make more specific assumptions about the stock and bond price dynamics.

Assumption 5.1 *Define, as usual, the forward rates by $f(t,T) = -\frac{\partial}{\partial T}\ln p(t,T)$, We now make the following assumptions, all under the risk neutral martingale measure Q.*

- *The bond market can be described by an HJM model for the forward rates of the form*

$$df(t,T) = \left(\sigma_f(t,T)\int_t^T \sigma_f'(t,u)du\right)dt + \sigma_f(t,T)dW(t)$$

  where the volatility structure $\sigma_f(t,T)$ is assumed to be deterministic. W is a (possibly multidimensional) Q-Wiener process.

- *The stock price follows a geometric Brownian motion, i.e.*

$$dS(t) = r(t)S(t)dt + S(t)\sigma_S dW(t),$$

  where $r_t = f(t,t)$ is the short rate. The row vector σ_S is assumed to be constant and deterministic.

In essence we have thus assumed a standard Black-Scholes model for the stock price S, and a Gaussian forward rate model. The point of this is that it will lead to (see below) a lognormal distribution for Z, thus allowing us to use a standard Black-Scholes formula. From the forward rate dynamics above if now follows (Björk (1999), prop. 15.5) that we have bond price dynamics given by

$$dp(t,T) = r(t)p(t,T)dt - p(t,T)\Sigma_p(t,T)dW(t),$$

where the bond price volatility is given by

$$\Sigma_p(t,T) = \int_t^T \sigma_f(t,u)du.$$

We may now attack the expectation in (28), and to this end we compute the Z-dynamics under Q^{T_1}. It follows directly from the Itô formula that the Q-dynamics of Z are given

$$dZ(t) = Z(t)\alpha_Z(t)dt + Z_t\left\{\sigma_S + \Sigma_p(t,T_1)\right\}dW(t)$$

where for the moment we do not bother about the drift process α_Z. Furthermore we know from the general theory (see Theorem 2.1) that the following hold

- The Z process is a Q^1 martingale (i.e. zero drift term).
- The volatility does not change when we change measure from Q to Q^1.

The Q^1 dynamics of Z are thus given by

$$dZ(t) = Z(t)\sigma_Z(t)dW^1(t) \tag{29}$$

where

$$\sigma_Z(t) = \sigma_S + \Sigma_p(t,T_1), \tag{30}$$

and where W^1 is Q^1-Wiener.

Under the assumptions above the volatility σ_Z is deterministic, thus guaranteeing that Z has a lognormal distribution. We can in fact write

$$dZ(t) = Z(t)\left\|\sigma_Z(t)\right\| dV^1(t),$$

where V^1 is a scalar Q^1 Wiener process. We may thus use a small variation of the Black-Scholes formula to obtain the final pricing result

Proposition 5.1 *The price, at t, of the convertible bond is given by the formula*

$$\Pi\left(t;Y\right) = S(t)N[d_1] - p(t,T_1)N[d_2] + p(t,T_1),$$

where

$$\begin{aligned}
d_1 &= \frac{1}{\sqrt{\sigma^2(t,T_0)}}\left\{\ln\left(\frac{S(t)}{p(t,T_1)}\right) + \frac{1}{2}\sigma^2(t,T_0)\right\}, \\
d_2 &= d_1 - \sqrt{\sigma^2(t,T_0)}, \\
\sigma^2(t,T_0) &= \int_t^{T_0} \left\|\sigma_Z(u)\right\|^2 du, \\
\sigma_Z(t) &= \sigma_S + \int_t^{T_1} \sigma_f(t,s)ds
\end{aligned}$$

6 Pricing savings plans with choice of linkage

These plans are common. Typically they give savers an ex-post choice of interest rates to be paid on their account. With the inception of capital requirements, many financial institutions have to recognize these options and price them.

6.1 Institutional setup

We use the example of a common bank account from the Israeli context; this account gives savers the ex-post choice of indexing their savings to an Israeli-shekel interest rate or a US dollar rate.

- The saver deposits NIS 100 ("NIS"= Israeli shekels) today in a shekel/dollar savings account with a maturity of 1 year.
- In one year, the account pays the maximum of:
  - The sum of NIS 100 + real shekel interest, the whole amount indexed to the inflation rate.
  - Today's dollar equivalent of NIS 100 + dollar interest, the whole amount indexed to the dollar exchange rate.

The savings plan is thus an option to exchange the Israeli interest rate for the US interest rate, while at the same time taking on the exchange rate risk. Since the choice is made *ex-post*, it is clear that both the shekel and the dollar interest rates offered on such an account must be below their respective market rates.

6.2 Mathematical model

In this section we derive the value of the exchange option described above; the result is given in equation (35) below.

We consider two economies, one domestic and one foreign, and we introduce the following notation.

$$
\begin{aligned}
r_d &= \text{domestic short rate} \\
r_f &= \text{foreign short rate} \\
I(t) &= \text{domestic inflation process} \\
X(t) &= \text{the exchange rate in terms of domestic currency/foreign currency.} \\
Y(t) &= X(t)^{-1} = \text{the exchange rate in terms of foreign currency/ domestic currency.} \\
T &= \text{the maturity of the savings plan.}
\end{aligned}
$$

The value of the option is linear in the initial shekel amount invested in the savings plan; without loss in generality, we assume that this amount is 1 shekel.

In the domestic currency the contingent T-claim Φ_d to be priced, is thus given by

$$\Phi_d = \max\left[e^{r_d T} I(T),\ X(0)^{-1} e^{r_f T} X(T)\right]$$

In the foreign currency the claim Φ_f is given by

$$\Phi_f = \max\left[e^{r_d T} I(T) Y(T),\ Y(0) e^{r_f T}\right]$$

It turns out that it is easier to work with Φ_f than with Φ_d, and we have

$$\Phi_f = \max\left[e^{r_d T} I(T) Y(T) - Y(0) e^{r_f T}, 0\right] + Y(0) e^{r_f T}.$$

The price (in the foreign currency) at $t = 0$ of this claim is now given by

$$\begin{aligned} \Pi\left(0; \Phi_f\right) &= e^{-r_f T} E^{Q_f}\left[\max\left\{e^{r_d T} I(T) Y(T) - e^{r_f T} Y(0), 0\right\} - e^{r_f T} Y(0)\right] \\ &= E^{Q_f}\left[\max\left\{e^{(r_d - r_f)T} I(T) Y(T) - Y(0), 0\right\}\right] + Y(0), \qquad (31) \end{aligned}$$

where Q_f denotes the risk neutral martingale measure for the foreign market.

At this point we have to make some probabilistic assumptions, and in fact we assume that we have a Garman-Kohlhagen model for Y. Standard theory then gives us the Q_f dynamics of Y as

$$dY(t) = Y(t)(r_f - r_d)dt + Y(t)\sigma_Y dW(t). \qquad (32)$$

For simplicity we assume that also the inflation follows a geometric Brownian motion, with Q_f -dynamics given by

$$dI(t) = I(t)\alpha_I dt + I(t)\sigma_I dW(t). \qquad (33)$$

Note that W is assumed to be two-dimensional, thus allowing for correlation between Y and I. Also note that economic theory does not say anything about the mean inflation rate α_I under Q_f.

When computing the expectation in (31) we cannot use a standard change of numeraire technique, the reason being that none of the processes Y, I or $Y \cdot I$ are price processes of traded assets without dividends. Instead we have to attack the expectation directly.

To that end we define the process Z as $Z(t) = Y(t) \cdot I(t)$ and obtain the following Q_f-dynamics.

$$dZ(t) = Z(t)\left(r_f - r_d + \alpha_I + \sigma_Y \sigma_I'\right) dt + Z(t)\left(\sigma_Y + \sigma_I\right) dW(t).$$

¿From this it is easy to see that if we define $S(t)$ by

$$S(t) = e^{-(r_f - r_d + \alpha_I + \sigma_Y \sigma_I')t} Z(t),$$

then we will have the Q_f-dynamics

$$dS(t) = S(t)\left(\sigma_Y + \sigma_I\right) dW(t),$$

the point being that we can interperet $S(t)$ as a stock price in a Black-Scholes world with zero short rate and Q_f as the risk neutral measure. With this notation we obtain easily

$$\Pi(0;\Phi_f) = e^{cT} E^{Q_f}\left[\max\left[S(T) - e^{-cT}Y(0), 0\right]\right] + Y(0),$$

where

$$c = \alpha_I + \sigma_Y \sigma_I'.$$

The expectation above can now be expressed by the Black-Scholes formula for a call option with strike price $e^{-cT}Y(0)$, zero short rate and a volatility given by

$$\sigma = \sqrt{\|\sigma_Y\|^2 + \|\sigma_I\|^2 + 2\sigma_Y\sigma_I'}$$

The price, at $t = 0$ of the claim, expressed in the foreign currency is thus given by the formula

$$\Pi(0;\Phi_f) = e^{cT} I(0)Y(0)N[d_1] - Y(0)N[d_2] + Y(0), \tag{34}$$

$$\begin{aligned} d_1 &= \frac{\ln(I(0)) + \left(c + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned}$$

Finally, the price at $t = 0$ in domestic terms is given by

$$\Pi(0;\Phi_d) = X(0)\Pi(0;\Phi_f) = e^{cT} I_0 N[d_1] - N[d_2] + 1. \tag{35}$$

Remark 6.1 For practical purposes it may be more convenient to model Y and I as

$$\begin{aligned} dY(t) &= Y(t)(r_f - r_d)dt + Y(t)\sigma_Y dW^Y(t), \\ dI(t) &= I(t)\alpha_I dt + I(t)\sigma_I dW^I(t), \end{aligned}$$

where now σ_Y and σ_Y are constant scalars, whereas W^Y and W^I are scalar Wiener processes with local correlation given by $dW^Y(t)dW^I(t) = \rho dt$.

In this model (which of course is logically equivalent to the one above) we have the pricing formulas (34)-(35), but now with the notation

$$\begin{aligned} c &= \alpha_I + \rho\sigma_Y\sigma_I, \\ \sigma &= \sqrt{\sigma_Y^2 + \sigma_I^2 + 2\rho\sigma_Y\sigma_I} \end{aligned}$$

7 Endowment warrants

Endowment options, which are primarily traded in Australia and New Zealand, are very long-term call options on equity. These options were recently discussed in this journal by Hoang-Powell-Shi (1999, henceforth HPS). Endowment warrants have two unusual features: Their dividend protection consists of adjustments to the strike price, and the strike price behaves like a money market fund (i.e., increases over time at the short-term interest rate). HPS assume that the dividend adjustment to the strike price is equivalent to the usual dividend adjustment to the stock price; this assumption is now known to be mistaken.[4] Under this assumption they prove an arbitrage-free warrant price for the case where the short-rate is deterministic and provide an approximation of the option price for the stochastic interest rate.

In this section we discuss a "pseudo-endowment option." This pseudo-endowment option is like the Australian option except that its dividend protection is the usual adjustment to the stock price (i.e., the stock price is raised by the dividends). The "pseudo-endowment option" thus depends on two sources of uncertainty: the (dividend-adjusted) stock price and the short-term interest rate. With a numeraire approach we can eliminate one of these sources of risk. Choosing an interest-rate related instrument (i.e., a money-market account) as a numeraire results in a pricing formula for the "pseudo-endowment option" which is similar to the standard Black-Scholes formula.[5]

7.1 Institutional setup

A pseudo-endowment option is a very long term call option. Typically we have the following setup:

- At issue, the **initial strike price** $K(0)$ is set to approximatively 50% of the current stock price, so the option is initially deep in the money.
- The endowment options are European.
- The time to exercise is typically 10+ years.
- The options are interest rate and dividend protected. The protection is performed by the following two adjustments:
  - The strike price is not fixed over time. Instead it is increased by the short-term interest rate.
  - The stock price is increased by the size of the dividend each time a dividend is paid.
- The payoff at the exercise date T is that of a standard call option, but with the adjusted (as above) strike price $K(T)$.

[4] See "Dividend Protection at a Price: Lessons from Endowment Warrants," Christine Brown and Kevin Davis, mimeo, March 2001, forthcoming in this journal?

[5] This formula was derived by HPS as a solution for the deterministic interest rate.

7.2 Mathematical model

We model the underlying stock price process $S(t)$ in a standard Black-Scholes setting. In other words, under the objective probability measure P, the price process $S(t)$ follows Geometrical Brownian Motion (between dividends) as:

$$dS(t) = \alpha S(t)dt + S(t)\sigma W^p(t),$$

where α and σ are deterministic constants and W^p is a P Wiener process. We allow the short rate r to be an arbitrary random process, thus giving us the following P-dynamics of the money-market account:

$$\begin{aligned} dB(t) &= r(t)B(t)dt, && (36) \\ B(0) &= 1. && (37) \end{aligned}$$

In order to analyze this option we have to formalize the protection features of the option. This is done in the following way.

- We assume that the strike price process $K(t)$ is changed at the continuously compounded instantaneous interest rate. The formal model is thus as follows
$$dK(t) = r(t)K(t)dt. \tag{38}$$
- For simplicity we assume (see Remark 7.1 below) that the dividend protection is perfect. More precisely we assume that the dividend protection is done by reinvesting the dividends into the stock itself. Under this assumption we can view the stock price as the theoretical price of a mutual fund which includes all dividends invested in the stock. Formally this implies that we can treat the stock price process $S(t)$ defined above as the price process of a stock without dividends.

The value of the option at the exercise date T is given by the contingent claim Y, defined by

$$Y = \max[S(T) - K(T), 0]$$

Clearly there are two sources of risk in endowment options: The stock price risk and the risk of the short-term interest rate. In order to analyze this option, we observe that from (36)-(38) it follows that

$$K(T) = K(0)B(T).$$

Thus we can express the claim Y as

$$Y = \max[S(T) - K(0)B(T), 0]$$

and from this expression we see that the natural numeraire process is now obviously the money account $B(t)$. The martingale measure for this numeraire is the standard risk neutral martingale measure Q under which we have the stock price dynamics

$$dS(t) = r(t)S(t)dt + S(t)\sigma dW(t), \tag{39}$$

where W is a Q-Wiener process.

A direct application of Theorem 2.1 gives us the pricing formula

$$\Pi(0;Y) = B(0)E^Q\left[\frac{1}{B(T)}\max\left[S(T) - K(0)B(T), 0\right]\right].$$

After a simple algebraic manipulation, and using the fact that $B(0) = 1$, we thus obtain

$$\Pi(0;Y) = E^Q\left[\max\left[Z(T) - K(0), 0\right]\right] \tag{40}$$

where $Z(t) = S(t)/B(t)$ is the normalized stock price process. It follows immediately from (36), (39), and the Itô formula that under Q we have Z-dynamics given by

$$dZ(t) = Z(t)\sigma dW(t). \tag{41}$$

and from (40)-(41) we now see that our original pricing problem has been reduced to that of computing the price of a standard European call, with strike price $K(0)$, on an underlying stock with volatility σ in a world where the short rate is zero. Thus the Black-Scholes formula gives the endowment warrant price at $t = 0$ directly as

$$C_{EW} = \Pi(0;Y) = S_0N(d_1) - K_0N(d_2) \tag{42}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S(0)/K(0)) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned}$$

Using the numeraire approach price of the endowment option in (42) is given by a standard Black-Scholes formula for the case where $r = 0$. The result does not in any way depend upon assumptions made about the stochastic short rate process $r(t)$.

The pricing formula (42) was in fact earlier derived in HPS, but only for the case of a deterministic short rate. The case of a stochastic short rate is not treated in detail in HPS. Instead the authors of HPS attempt to include the effect of a stochastic interest rate by introducing the following scheme:

- They assume that the short rate r is deterministic and constant.
- The strike price process is assumed to have dynamics of the form

  $$dK(t) = rK(t)dt + \gamma dV(t)$$

  where V is a new Wiener process (possibly correlated with W).

- They then go on to value the claim $Y = \max[S(T) - K(T), 0]$ by using the Margrabe (1978) result about exchange options.

The claim made in HPS is that this setup is an approximation to the case of a stochastic interest rate. Whether it is a good approximation or not is never clarified in HPS, and from our analysis above we see that the entire scheme is in fact unnecessary, since the pricing formula (42) is invariant under the introduction of a stochastic short rate.

Remark 7.1 We note that the result above relies upon our simplifying assumption about perfect dividend protection. A more realistic modeling of the dividend protection would lead to severe computational problems. To see this assume that the stock pays a constant dividend yield rate δ. This would change our model in two ways: The Q-dynamics of the stock price would be different, and the dynamics of the strike process $K(t)$ would have to be changed.

As for the Q-dynamics of the stock price, standard theory immediately gives us

$$dS(t) = (r(t) - \delta) S(t)dt + S(t)\sigma dW(t).$$

Furthermore, from the institutional description above we see that in real life (as opposed to in our simplified model), the dividend protection is done by decreasing the strike price process with the dividend amount at every dividend payment. In terms of our model this means that over an infinitesimal interval $[t, t + dt]$, the strike price should decrease with the amount $\delta S(t)dt$. Thus the K-dynamics are given by

$$dK(t) = [r(t)K(t) - \delta S(t)]\, dt.$$

This equation can be solved as

$$K(T) = e^{\int_0^T r(t)dt} K(0) - \delta \int_0^T e^{\int_t^T r(u)du} S(t)dt$$

The moral is that in the expression of the contingent claim

$$Y = \max[S(T) - K(T), 0]$$

we now have the unpleasant integral expression

$$\int_0^T e^{\int_t^T r(u)du} S(t)dt.$$

Even in the simple case of a deterministic short rate this integral is quite problematic. It is then basically a sum of lognormally distributed random variables, and thus we have the same hard computational problems as in the case of an Asian option.

8 Conclusion

Numeraire methods have been in the derivative pricing literature since papers by Geman (1989) and Jamshidian (1989). These methods afford a considerable simplification in the pricing of many complex options; however, they appear not to be well-known. In this paper we have considered five problems whose computation is vastly aided by the use of numeraire methods. The first of these is the pricing of endowment options (discussed in the *Journal of Derivatives* in a recent paper by Hoang, Powell, and Shi (1999). We also discuss the pricing of options where the strike price is denominated in a currency different from that of the underlying stock, the pricing of savings plans where the choice of interest paid is ex-post chosen by the saver, the pricing of convertible bonds and the pricing of employee stock ownership plans.

Numeraire methods are not a panacea for complex option pricing. However, when there are several risk factors which impact an option's price, choosing one of the factors as a numeraire reduces the dimensionality of the computational problem by one. A clever choice of the numeraire can, in addition, lead to a significant computational simplification in the option's pricing.

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