A NORMAL APPROXIMATION FOR THE NUMBER OF LOCAL MAXIMA OF A RANDOM FUNCTION ON A GRAPH

BY

P. BALDI, Y. RINOTT, AND C. STEIN

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A Normal Approximation for the Number of Local Maxima of a Random Function on a Graph

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Abstract

Consider a random function on the set of vertices of a regular graph with the values at the vertices independently identically distributed according to a continuous distribution. a local maximum is said to occur at a vertex if the value of the function at that vertex is greater than the value of the function at any adjacent vertex. It is proved that the number of local maxima is approximately normally distributed if and only if its variances is large. An example shows that this result cannot be extended directly to all (non–regular) graphs.

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1. Introduction

We shall study conditions for the approximate normality of the distribution of the number of local maxima of a random function on the set of vertices of a graph when the values of the random function are independently identically distributed with a continuous distribution function. For a regular graph, the distribution of the number of local maxima is approximately normal if and only if its variance is large.

A precise statement of this result is given in Section 3. This result covers a number of interesting special cases that were not covered by an earlier paper of Baldi and Rinott (1988). Section 2 contains a basic lemma on normal approximation for sums of indicator random variables. Section 3 contains the main result. Section 4 contains a number of examples. In particular, Example 4 shows that, without the conditions of regularity, asymptotic normality is not implied by a large variance.

Before writing down expressions for the mean and variance of W, the number of local maxima of the random function, we recall some terminology and introduce some notation. A graph $(\mathcal{V},\mathcal{E})$ consists of a finite set $\mathcal{V}$ of vertices and a set $\mathcal{E}$ of edges, which may be thought of as two-element subsets of $\mathcal{V}$. If $\{v_1,v_2\}\in\mathcal{E}$, the vertices v_1 and v_2 are said to be neighbors. The distance $\delta(v,v')$ is the smallest number n for which there exist $v_0,\ldots,v_n$ with $v_0=v$ and $v_n=v'$, such that each pair $\{v_i,v_{i+1}\}$ belongs to $\mathcal{E}$. The degree $d(v)$ of a vertex v is the number of edges to which it belongs. A graph is regular if all vertices have the same degree, which we denote by d. A triangle is a set of three vertices v_1,v_2,v_3 with $\delta(v_1,v_2)=\delta(v_2,v_3)=\delta(v_3,v_1)=1$.

For a regular graph, the mean and variance of W are given by

$$\lambda = EW = \frac{|\mathcal{V}|}{d+1}, \tag{1.1}$$

where $|\mathcal{V}|$ is the number of elements in the set $\mathcal{V}$ and

$$\sigma^2 = \operatorname{Var} W = \sum_{\substack{u,v\\ \delta(u,v)=2}} s(u,v)(2d+2-s(u,v))^{-1}(d+1)^{-2}, \tag{1.2}$$

where $s(u,v)$ is the number of common neighbors of u and v. Observe that

$$\frac{\mathcal{S}}{2(d+1)^3} \le \sigma^2 \le \frac{\mathcal{S}}{(d+1)^3}, \tag{1.3}$$

where

$$\mathcal{S} = \sum_{\substack{u,v\\ \delta(u,v)=2}} s(u,v) = |\mathcal{V}|d(d-1)-6T \tag{1.4}$$

and T is the number of triangles in the graph. From (1.1), (1.3), and (1.4) it follows that the variance is always less than the mean. More precisely

$$(1.5) \qquad \frac{1}{2}\frac{d(d-1)}{(d+1)^2}\left(1-6\frac{T}{|\mathcal{V}|d(d-1)}\right) \leq \frac{\sigma^2}{\lambda} \leq \frac{d(d-1)}{(d+1)^2}\left(1-6\frac{T}{|\mathcal{V}|d(d-1)}\right).$$

Finally note that by (1.5), the variance σ^2 is large if $\lambda = |\mathcal{V}|/(d+1)$ is large and the ratio of T to the maximal possible number of triangles in a regular graph of degree d, which is of the order of $\frac{1}{6}|\mathcal{V}|d(d-1)$, is suitably bounded away from 1.

2. A Normal approximation theorem for sums of indicator random variables

In this section we develop a normal approximation theorem that will be used, in section 3, to study the distribution of the number of local maxima of a random function on a regular graph. The basic idea of this proof was developed by Stein (1972) and other versions are treated in Stein (1986) and elsewhere. The present version was influenced by a paper by Barbour (1982). It exploits the fact that we are interested in a sum of indicator random variables. The first lemma, related to work of Chen (1975) is an identity applicable to any sum W of indicator random variables. The second lemma gives an expression for the error of the normal approximation to the expectation of any reasonable function of W. With the aid of a third lemma, of a technical nature, which is stated without proof, this leads to the main theorem, which gives a bound for the error of the normal approximation to the distribution of W. We shall write $E^Q Z$ for the conditional expectation of the random variable Z given the random variable Q (not necessarily real valued).

LEMMA 2.1. *Let $\mathcal{V}$ be a finite set and X and X^* random functions on $\mathcal{V}$ taking on only the values 0 and 1. Also let V be a random variable uniformly distributed in $\mathcal{V}$, independent of X. Let the unconditional distribution of X^* be the same as the conditional distribution of X given that $X_V = 1$. Define*

$$(2.1) \qquad W = \sum_{x \in V} X_v = |\mathcal{V}| E^X X_V$$

$$(2.2) \qquad \lambda = EW = |\mathcal{V}| E X_V$$

and

$$(2.3) \qquad W^* = \sum_{v \in \mathcal{V}} X_v^*.$$

Then, for all f: $\mathbf{Z}^+ \to \mathbf{R}$

$$EWf(W) = \lambda Ef(W^*). \tag{2.4}$$

PROOF.

$$\begin{aligned} EWf(W) &= E(|\mathcal{V}|E^X X_V)f(W) \\ &= |\mathcal{V}|EX_V f(W) \\ &= |\mathcal{V}|EX_V E^{X_V} f(W) \\ &= |\mathcal{V}|EX_V E\left[f(W)|X_V = 1\right] = \lambda Ef(W^*). \end{aligned}$$

∎

This lemma is useful if we have a good approximation to the conditional distribution of W^* given W. Starting on p. 90 of Stein (1986) it was used in order to show that, if $E|W+1-W^*|$ is small, then W has approximately a Poisson distribution. The argument is essentially equivalent to an earlier argument of Chen (1975).

We shall now study the normal approximation to the distribution of W. For this purpose the following notation will be convenient. For h: $\mathbf{R} \to \mathbf{R}$ of bounded variation, let

$$Nh = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(x)e^{-\frac{1}{2}x^2}\,dx \tag{2.6}$$

and

$$(U_N h)(y) = e^{\frac{1}{2}y^2} \int_{-\infty}^{y} \left[h(x) - Nh\right] e^{-\frac{1}{2}x^2}\,dx. \tag{2.7}$$

LEMMA 2.2. *Suppose the hypotheses of Lemma 2.1 are satisfied and let*

$$\sigma^2 = E(W-\lambda)^2. \tag{2.8}$$

Then

$$\sigma^2 = \lambda E(W^* - W), \tag{2.9}$$

and, for any h: $\mathbf{R} \to \mathbf{R}$ *of bounded variation,*

$$\begin{aligned} &Eh(\frac{W-\lambda}{\sigma}) \\ &= Nh - \frac{\lambda}{\sigma^2}E\{[E^W(W^*-W) - E(W^*-W)](U_N h)'(\frac{W-\lambda}{\sigma})\} \\ &\quad - \frac{\lambda}{\sigma^2}E\int_W^{W^*}(W^*-w)d(U_N h)'(\frac{w-\lambda}{\sigma}). \end{aligned} \tag{2.10}$$

For $W^* < W$, it is understood that

$$\int_{W^*}^{W}(W^* - w)d(U_N h)'(\frac{w-\lambda}{\sigma}) = -\int_{W}^{W^*}(W^* - w)d(U_N h)'(\frac{w-\lambda}{\sigma}). \tag{2.11}$$

PROOF. The second expression for σ^2 follows from Lemma 2.1 by setting $f(w) = w - \lambda$. In order to prove (2.10) we first rewrite (2.4), assuming f to be differentiable with derivative of bounded variation, as

$$\begin{aligned} EWf(W) &= \lambda Ef(W^*) \\ &= \lambda E\big[f(W) + (W^* - W)f'(W) + \int_W^{W^*}(W^* - w)df'(w)\big] \\ &= \lambda Ef(W) + \sigma^2 Ef'(W) \\ &\quad + \lambda E\{\big[E^W(W^* - W) - E(W^* - W)\big]\, f'(W)\} \\ &\quad + \lambda E\int_W^{W^*}(W^* - w)df'(w). \end{aligned} \tag{2.12}$$

At the second equality sign we have used Taylor's theorem with remainder and the last equality follows from (2.9). We can rewrite (2.12) as

$$\begin{aligned} &E\big[f'(W) - \frac{W-\lambda}{\sigma^2}f(W)\big] \\ &= -\frac{\lambda}{\sigma^2}E\{[E^W(W^* - W) - E(W^* - W)]f'(W) + \int_W^{W^*}(W^* - w)df'(w)\}. \end{aligned} \tag{2.13}$$

Next, observing that the function $U_N h$, defined by (2.7), satisfies the differential equation

$$(U_N h)'(y) - y(U_N h)(y) = h(y) - Nh, \tag{2.14}$$

we substitute

$$f(w) = \sigma(U_N h)(\frac{w-\lambda}{\sigma}) \tag{2.15}$$

in (2.13) to obtain

$$\begin{aligned} E\big[h(\frac{W-\lambda}{\sigma}) - Nh\big] &= E\big[(U_N h)'(\frac{W-\lambda}{\sigma}) - \frac{W-\lambda}{\sigma}(U_N h)(\frac{W-\lambda}{\sigma})\big] \\ &= -\frac{\lambda}{\sigma^2}E\{[E^W(W^* - W) - E(W^* - W)](U_N h)'(\frac{W-\lambda}{\sigma})\} \\ &\quad - \frac{\lambda}{\sigma^2}E\int_W^{W^*}(W^* - w)d(U_N h)'(\frac{w-\lambda}{\sigma}), \end{aligned} \tag{2.16}$$

which is (2.10). ∎

The following lemma will be used for the purpose of bounding the remainder in (2.10). It is proved as Lemma 3, formulas (46) and (47) on p. 25 of Stein (1986). A trivial improvement has been made in (2.17).

LEMMA 2.3. *If h: $\mathbf{R} \to \mathbf{R}$ is bounded and piecewise continuously differentiable with bounded derivative, then*

$$\sup|(U_N h)'| \leq \sup h - \inf h \tag{2.17}$$

and

$$\sup|(U_N h)''| \leq 2\sup|h'|. \tag{2.18}$$

By combining Lemmas 2.2 and 2.3 we obtain a result that asserts, qualitatively, that, under the hypotheses of Lemma 2.1, W is approximately normally distributed with mean λ and variance σ^2 given by (2.9) if

$$\frac{\sqrt{\operatorname{Var} E^W(W^* - W)}}{E(W^* - W)} \tag{2.19}$$

and

$$\frac{1}{\sigma}\frac{E(W^* - W)^2}{E(W^* - W)} \tag{2.20}$$

are small.

LEMMA 2.4. *Under the hypotheses of Lemma 2.1, for arbitrary piecewise continuously differentiable h: $\mathbf{R} \to \mathbf{R}$,*

$$\begin{aligned} |Eh(\frac{W-\lambda}{\sigma}) - Nh| \leq (\sup h - \inf h)&\frac{\sqrt{\operatorname{Var} E^W(W^* - W)}}{E(W^* - W)} \\ &+ \sup|h'|\frac{E(W^* - W)^2}{\sigma E(W^* - W)}. \end{aligned} \tag{2.21}$$

PROOF. This follows immediately from the identity (2.10) with the aid of (2.9) and Lemma 2.3. ∎

Finally, we obtain a bound for the error in the normal approximation for the distribution of W.

THEOREM 2.1. *Under the hypothesis of Lemma 2.1, we have for all* $w \in \mathbf{R}$

$$|P(W \le w) - \Phi(\frac{w-\lambda}{\sigma})| \le \frac{\sqrt{\operatorname{Var} E^W(W^* - W)}}{E(W^* - W)} + (\frac{2}{\pi})^{1/4}\sqrt{\frac{1}{\sigma}\frac{E(W^*-W)^2}{E(W^*-W)}}. \tag{2.22}$$

PROOF. We apply (2.21) with

$$h(x) = \begin{cases} 1 & \text{if } x \le \frac{w-\lambda}{\sigma}; \\ 1 - \frac{1}{\epsilon}(x - \frac{w-\lambda}{\sigma}) & \text{if } \frac{w-\lambda}{\sigma} \le x \le \frac{w-\lambda}{\sigma} + \epsilon; \\ 0 & \text{otherwise.} \end{cases}$$

to obtain

$$\begin{aligned} P(W \le w) &\le Eh(\frac{W-\lambda}{\sigma}) \\ &\le Nh + \frac{\sqrt{\operatorname{Var} E^W(W^*-W)}}{E(W^*-W)} + \frac{1}{\epsilon}\frac{E(W^*-W)^2}{\sigma E(W^*-W)} \\ &\le \Phi(\frac{w-\lambda}{\sigma}) + \frac{\epsilon}{2\sqrt{2\pi}} + \frac{\sqrt{\operatorname{Var} E^W(W^*-W)}}{E(W^*-W)} \\ &\quad + \frac{1}{\epsilon}\frac{E(W^*-W)^2}{\sigma E(W^*-W)}. \end{aligned} \tag{2.23}$$

The latter expression is minimized by

$$\epsilon = \left[\frac{2\sqrt{2\pi}}{\sigma}\frac{E(W^*-W)^2}{E(W^*-W)}\right]^{\frac{1}{2}}. \tag{2.24}$$

The resulting upper bound for $P(W \le w)$ and the symmetric lower bound yield (2.22). ∎

REMARK: The classical De Moivre-Laplace CLT can be obtained from Theorem 2.1 as follows. Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli random variables with $P(X_i = 1) = p$, $W = \sum_{i=1}^n X_i$. Let I^* be a uniform random variable on $\{1, \ldots, n\}$ independent of $X_1, \ldots, X_n$ and set $X^*_{I^*} = 1$, $X^*_i = X_i$ for $i \ne I^*$ and $W^* = \sum_{i=1}^n X^*_i$. It is readily seen that X^* satisfies the conditions of Lemma 2.1. Now the relations

$$E(W^* - W)^2 = E(W^* - W) = E(1 - \frac{W}{n}) = 1 - p$$

$$\operatorname{Var} E^W(W^* - W) = \operatorname{Var}(1 - \frac{W}{n}) = p(1-p)/n,$$

combined with Theorem 2.1, yield the asymptotic normality of W.

3. A normal approximation for the distribution of the number of local maxima

Returning to the problem described in Section 1, we consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and independently identically distributed continuous random variables $Y = \{Y_v, v \in \mathcal{V}\}$. We shall see in Theorem 3.1, that the distribution of W, the number of local maxima of the random function Y, is approximately normal with mean λ and variance σ^2 given in section 1, with error bounded by $C\sigma^{-1/2}$, where C is an absolute constant. A more careful argument in Section 2 presumably would yield an error of the form $C\sigma^{-1}$ as in the treatment of a similar problem by Barbour (1982).

Define $X = \{X_v, v \in \mathcal{V}\}$ by

$$X_v = \begin{cases} 1 & \text{if } Y_v > Y_u \text{ for all } u \in N(v); \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

where $N(v) = \{u \in \mathcal{V} : \delta(u,v) = 1\}$, the neighborhood of v. Let $d(v) = |N(v)|$ denote the degree of v. In order to apply Theorem 2.1 we shall construct a random function X^* which will be seen, in Lemma 3.1, to satisfy the condition required of X^* in Section 2. Let V^* be a random variable, independent of Y, taking values in $\mathcal{V}$ and satisfying

$$P(V^* = v) = (d(v)+1)^{-1} / \sum_{u \in \mathcal{V}} (d(u)+1)^{-1}. \tag{3.2}$$

For $v \in \mathcal{V}$ define $Z(v)$ to be the vertex in $N(v) \cup \{v\}$ satisfying

$$Y_{Z(v)} = \max_{w \in N(v) \cup \{v\}} Y_w. \tag{3.3}$$

Note that $Z(v)$ is a random variable, depending on Y. Next define

$$Y_u^* = \begin{cases} Y_{Z(V^*)} & \text{if } u = V^*; \\ Y_{V^*} & \text{if } u = Z(V^*); \\ Y_u & \text{otherwise.} \end{cases} \tag{3.4}$$

In words, if V^* is a vertex at which Y has a local maximum, then $Y^* = Y$. Otherwise, to obtain Y^* from Y, interchange the Y-values of V^* and the vertex $Z(V^*)$ having the largest Y-value in $N(V^*)$, leaving all other values of Y unchanged. Now define X^* from Y^* by analogy to (3.1) and let

$$W = \sum_{v \in \mathcal{V}} X_v, \quad W^* = \sum_{v \in \mathcal{V}} X_v^* \tag{3.5}$$

count the number of local maxima of Y and Y^*, respectively.

LEMMA 3.1. *Let V be a uniform random variable taking values in $\mathcal{V}$, independently of Y, and let Y^* be defined in (3.4). Then, for any measurable set $A \subset \mathbf{R}^{|\mathcal{V}|}$,*

$$P(Y^* \in A) = P(Y \in A | X_V = 1).$$

PROOF. The key observation is that under the present assumptions

$$P(Y^* \in A | V^* = v) = P(Y \in A | X_v = 1). \tag{3.6}$$

Note simply that, conditioned on $V^* = v$, Y_v^* is a local maximum (i.e., $X_v^* = 1$), and the remaining Y_v^* have the conditional distribution given a maximum at v. Similarly, $X_v = 1$ indicates a maximum of Y at v and the remaining Y's as above. From (3.2) and (3.6) we have

$$\begin{aligned} P(Y^* \in A) &= \sum_{v \in \mathcal{V}} P(Y^* \in A | V^* = v) P(V^* = v) \\ &= \sum_{v \in \mathcal{V}} P(Y \in A | X_v = 1)(d(v)+1)^{-1} / \sum_{u \in \mathcal{V}} (d(u)+1)^{-1}. \end{aligned} \tag{3.7}$$

Also

$$\begin{aligned} P(X_V = 1) &= \sum_{u \in \mathcal{V}} P(X_u = 1 | V = u) P(V = u) \\ &= |\mathcal{V}|^{-1} \sum_{u \in \mathcal{V}} (d(u)+1)^{-1}, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} P(Y \in A | X_V = 1) &= \sum_{v \in \mathcal{V}} P(Y \in A, X_v = 1, V = v) / P(X_V = 1) \\ &= \sum_{v \in \mathcal{V}} P(Y \in A | X_v = 1) P(X_v = 1) P(V = v) / P(X_V = 1). \end{aligned}$$

The desired result now follows from $P(V = v) = |\mathcal{V}|^{-1}$, $P(X_v = 1) = (d(v)+1)^{-1}$, (3.7) and (3.8). ∎

Before we state the main Theorem, we recall that a regular graph is one in which all vertices have the same degree, which will be denoted by d. Also, for $u, v \in \mathcal{V}$ let $s(u,v) = |N(u) \cap N(v)|$.

THEOREM 3.1. *Let $(\mathcal{V}, \mathcal{E})$ be a regular graph and Y a random function on $\mathcal{V}$ whose values are independently distributed with a common continuous distribution and let W be the number of of local maxima of Y. Then the mean and variance of W are given by*

$$\lambda = EW = \frac{|\mathcal{V}|}{d+1}$$

and

$$\sigma^2 = \mathrm{Var}\, W = \sum_{\substack{u,v \\ \delta(u,v)=2}} s(u,v)(2d+2-s(u,v))^{-1}(d+1)^{-2},$$

and, for all $w \in \mathbf{R}^+$,

$$|P(W \le w) - \Phi(\frac{w-\lambda}{\sigma})| \le C\sigma^{-\frac{1}{2}}, \tag{3.9}$$

where C is an absolute constant.

For the proof of Theorem 3.1 we need the following lemma, where we use the notation $a \vee b = \max(a, b)$.

LEMMA 3.2. *Let u_i and S_j denote elements and subsets of $\mathcal{V}$, respectively, and assume that for all i and j, $u_i \notin S_j$, and let $Y_S = \max_{v \in S} Y_v$, for $S \subset \mathcal{V}$. Then*

$$(a) \qquad P(\{Y_{u_2} > Y_{u_1} \vee Y_{S_2}\} \cap \{Y_{u_1} > Y_{S_1}\}) = (|S_1 \cup S_2| + 2)^{-1}(|S_1| + 1)^{-1}.$$

In general

$$(b) \qquad P(\cap_{i=2}^k \{Y_{u_i} > Y_{u_{i-1}} \vee Y_{S_i}\} \cap \{Y_{u_1} > Y_{S_1}\}) = \prod_{i=1}^k (|\cup_{j=1}^i S_i| + i)^{-1}.$$

PROOF. The left hand side of (a) equals

$$P(Y_{u_2} > Y_{u_1} \vee Y_{S_2} \vee Y_{S_1}) P(Y_{u_1} > Y_{S_1} | Y_{u_2} > Y_{u_1} \vee Y_{S_2} \vee Y_{S_1}).$$

Since $|\{u_1\} \cup \{u_2\} \cup S_1 \cup S_2| = |S_1 \cup S_2| + 2$ and the Y-values are i. i. d. (hence exchangeable, in fact Y's exchangeable would suffice) we have

$$P(Y_{u_2} > Y_{u_1} \vee Y_{S_2} \vee Y_{S_1}) = (|S_1 \cup S_2| + 2)^{-1}.$$

Conditionally on the event $Y_{u_2} > Y_{u_1} \vee Y_{S_2} \vee Y_{S_1}$, the Y-values in S_1 and Y_{u_1} are again i.i.d. and exchangeable and Y_{u_1} is the largest among $\{Y_{u_1}, Y_{S_1}\}$ with conditional probability

$|u_1 \cup S_1|^{-1} = (|S_1|+1)^{-1}$ and (a) follows. A similar argument and induction lead to (b). ∎

PROOF OF THEOREM 3.1. First note that

$$E^W(W^* - W) = E^W E^Y(W^* - W),$$

implying

$$\text{(3.10)} \qquad \operatorname{Var} E^W(W^* - W) \leq \operatorname{Var} E^Y(W^* - W).$$

Thus, in order to apply Theorem 2.1 we need only bound the quantities $E(W^* - W)^2/E(W^* - W)$ and $(\operatorname{Var} E^Y(W^* - W))^{1/2}/E(W^* - W)$. In order to write down an expression for $W^* - W$, which is needed for the computation of the bounds, we observe that

$$\text{(3.11)} \qquad X_u^* = \begin{cases} X_u & \text{if } \delta(u, V^*) \geq 3; \\ X_u + A(u, V^*) & \text{if } \delta(u, V^*) = 2; \\ 0 & \text{if } \delta(u, V^*) = 1; \\ 1 & \text{if } u = V^*; \end{cases}$$

where $A(u, v)$ is the event that $\delta(u, v) = 2$ and Y does not have a local maximum at u but, if the values of Y at v and $Z(v)$ of (3.3) are interchanged, then the value at u will be a local maximum for the new function. In the sequel, we identify an event with its indicator function. It follows that

$$\text{(3.12)} \qquad \begin{aligned} W^* - W &= \sum_u (X_u^* - X_u) \\ &= \left[1 - (X_{V^*} + \sum_{u \in N(V^*)} X_u)\right] + \sum_u A(u, V^*). \end{aligned}$$

In order to compute $E^Y(W^* - W)$ we observe that, because the graph is regular and V^* is independent of Y, $P^Y(V^* = v) = P(V^* = v) = |\mathcal{V}|^{-1}$. Thus it follows from (3.12) that

$$\text{(3.13)} \qquad E^Y(W^* - W) = (1 - \frac{d+1}{|\mathcal{V}|} W) + \frac{1}{|\mathcal{V}|} \sum_u \sum_v A(u, v).$$

Next let us verify the formulas for the mean and variance of W. Clearly

$$EW = E \sum_v X_v = \frac{|\mathcal{V}|}{d+1}$$

and thus, by (3.13),

$$\text{(3.14)} \qquad E(W^* - W) = \frac{1}{|\mathcal{V}|} \sum_u \sum_v P(A(u, v)).$$

Let $B(u,v,z)$ be the event that $A(u,v)$ occurs and $Z(v)=z$. Observe that this can only occur with $z \in N(u) \cap N(v)$ and $\delta(u,v)=2$. In order to express the $B(u,v,z)$ explicitely in terms of Y, let

$$C = \{v\} \cup N(u) - \{z\} \tag{3.15}$$

and

$$D = \{v\} \cup N(v) - \{z\}. \tag{3.16}$$

Then

$$B(u,v,z) = \{Y_z > Y_u \vee Y_D\} \cap \{Y_u > Y_C\}. \tag{3.17}$$

Thus, by Lemma 3.2

$$\begin{aligned} P(B(u,v,z)) &= (|C \cup D| + 2)^{-1}(|C|+1)^{-1} \\ &= (2d+2-s(u,v))^{-1}(d+1)^{-1} \end{aligned} \tag{3.18}$$

when $z \in N(u) \cap N(v)$ and $\delta(u,v)=2$, otherwise 0. Finally, it follows from (3.14) that

$$\begin{aligned} &E(W^* - W) \\ &= \frac{1}{|\mathcal{V}|} \sum_{\substack{u,v \\ \delta(u,v)=2}} \sum_{z \in N(u) \cap N(v)} (2d+2-s(u,v))^{-1}(d+1)^{-1} \\ &= \frac{1}{|\mathcal{V}|} \sum_{\substack{u,v \\ \delta(u,v)=2}} s(u,v)(2d+2-s(u,v))^{-1}(d+1)^{-1}. \end{aligned} \tag{3.19}$$

Thus, by (2.9)

$$\sigma^2 = \lambda E(W^* - W) = (d+1)^{-2} \sum_{\substack{u,v \\ \delta(u,v)=2}} s(u,v)(2d+2-s(u,v))^{-1}$$

A straightforward calculation which appears in Baldi and Rinott (1988) shows that, for any graph (not necessarily regular),

$$\begin{aligned} \operatorname{Var} W = \frac{1}{2} &\sum_{\substack{u,v \\ \delta(u,v)=1}} [(d(u)+1)^{-1} - (d(v)+1)^{-1}]^2 \\ + &\sum_{\substack{u,v \\ \delta(u,v)=2}} s(u,v)[d(u)+1]^{-1}[d(v)+1]^{-1}[d(u)+d(v)+2-s(u,v)]^{-1}. \end{aligned} \tag{3.20}$$

The verification of the formula for the variance in our case was included because it is similar to, but simpler than calculations that will be needed later.

We now return to the bounds required for the application of Theorem 2.1 and prove first that

$$E(W^* - W)^2/E(W^* - W) \leq 10. \tag{3.21}$$

By (3.12),

$$E(W^* - W)^2 \leq 2\Big\{E\big[1 - (X_{V^*} + \sum_{u \in N(V^*)} X_u)\big]^2 + E\big[\sum_{\substack{u \\ \delta(u,V^*)=2}} A(u, V^*)\big]^2\Big\}. \tag{3.22}$$

Since $X_{V^*}X_u = 0$ when $u \in N(V^*)$, and $E(X_{V^*} + \sum_{u \in N(V^*)} X_u) = 1$, we have

$$\begin{aligned} &E[1 - (X_{V^*} + \sum_{u \in N(V^*)} X_u)]^2 \\ &= E(X_{V^*} + \sum_{u \in N(V^*)} X_u)^2 - E(X_{V^*} + \sum_{u \in N(V^*)} X_u) \\ &= E \sum_{u_1 \in N(V^*)} \sum_{\substack{u_2 \neq u_1 \\ u_2 \in N(V^*)}} X_{u_1}X_{u_2} = \frac{1}{|\mathcal{V}|} \sum_{v} \sum_{u_1 \in N(v)} \sum_{\substack{u_2 \neq u_1 \\ u_2 \in N(v)}} EX_{u_1}X_{u_2}. \end{aligned} \tag{3.23}$$

If $\delta(u_1, u_2) = 1$, then $X_{u_1}X_{u_2} = 0$. If $\delta(u_1, u_2) = 2$, then, by Lemma 3.2,

$$\begin{aligned} EX_{u_1}X_{u_2} &= P\{Y \text{ has local maxima at } u_1 \text{and } u_2\} \\ &= P(\{Y_{u_1} > Y_{u_2} \vee Y_{N(u_1)}\} \cap \{Y_{u_2} > Y_{N(u_2)}\}) \\ &\quad + P(\{Y_{u_2} > Y_{u_1} \vee Y_{N(u_2)}\} \cap \{Y_{u_1} > Y_{N(u_1)}\}) \\ &= 2[|N(u_1) \cup N(u_2)| + 2]^{-1}(d+1)^{-1} \\ &= 2[2d + 2 - s(u_1, u_2)]^{-1}(d+1)^{-1}. \end{aligned} \tag{3.24}$$

It follows from (3.23) and (3.24) that

$$\begin{aligned} &E[1 - (X_{V^*} + \sum_{u \in N(V^*)} X_u)]^2 \\ &= \frac{1}{|\mathcal{V}|} \sum_{u_1} \sum_{\substack{u_2 \\ \delta(u_1,u_2)=2}} \sum_{v \in N(u_1) \cap N(u_2)} 2[2d + 2 - s(u_1, u_2)]^{-1}(d+1)^{-1} \\ &= 2E(W^* - W). \end{aligned} \tag{3.25}$$

by (3.19).

In order to bound the second term on the right hand side of (3.22) we recall the formula (3.17) for $B(u,v,z)$ and write

$$\begin{aligned}\Delta &= E\Big[\sum_{\delta(u,V^*)=2} A(u,V^*)\Big]^2 - E\sum_{\delta(u,V^*)=2} A(u,V^*) \\ &= \sum_{u_1}\sum_{\substack{u_2\\ u_2\neq u_1}}\sum_{z_1}\sum_{z_2} P(B(u_1,V^*,z_1)B(u_2,V^*,z_2)) \\ &= \frac{1}{|\mathcal{V}|}\sum_{v}\sum_{u_1}\sum_{\substack{u_2\\ u_2\neq u_1}}\sum_{z_1}\sum_{z_2} P(B(u_1,v,z_1)B(u_2,v,z_2)).\end{aligned} \tag{3.26}$$

But for both $B(u_1,v,z_1)$ and $B(u_2,v,z_2)$ to occur we must have $z_1 = z_2 \in N(v)\cap N(u_1)\cap N(u_2)$ and $\delta(u_1,u_2) = 2$. In fact $z_1 = z_2$ because both are the vertices in $N(V^*)$ where Y attains its maximum, $z_1 \in N(u_1)$ because a local maximum is created at u_1 by exchanging the value of Y at V^* and the larger value at z_1 and $\delta(u_1,u_2) \neq 1$ because there cannot be local maxima at two neighboring vertices. With D as defined in (3.16) with $z = z_1 = z_2$ and C_i defined for $i \in \{1,2\}$ by

$$C_i = \{v\} \cup N(u_i) - \{z\},$$

we use (3.17) and Lemma 3.2 to conclude that, under these conditions

$$\begin{aligned}&P(B(u_1,v,z)B(u_2,v,z)) \\ &= P(\{Y_z > Y_{u_1} \vee Y_{u_2} \vee Y_D\} \cap \{Y_{u_1} > Y_{C_1}\} \cap \{Y_{u_2} > Y_{C_2}\}) \\ &= (|C_1 \cup C_2 \cup D| + 3)^{-1}(|C_1 \cup C_2| + 2)^{-1}(|C_1| + 1)^{-1} \\ &\quad + (|C_1 \cup C_2 \cup D| + 3)^{-1}(|C_1 \cup C_2| + 2)^{-1}(|C_2| + 1)^{-1} \\ &\leq 2(2d + 2 - s(u_1,u_2))^{-1}(d+1)^{-2}.\end{aligned} \tag{3.27}$$

It follows from (3.26) and (3.27) and the associated remarks that

$$\begin{aligned}&\Delta \\ &\leq \frac{1}{|\mathcal{V}|}\sum_{u_1}\sum_{\substack{u_2\\ \delta(u_2,u_1)=2}}\sum_{z\in N(u_1)\cap N(u_2)}\sum_{v\in N(z)} 2(2d+2-s(u_1,u_2))^{-1}(d+1)^{-2} \\ &\leq 2\frac{1}{|\mathcal{V}|}\sum_{\substack{u_1,u_2\\ \delta(u_1,u_2)=2}} s(u_1,u_2)(2d+2-s(u_1,u_2))^{-1}(d+1)^{-1} \\ &= 2E(W^* - W)\end{aligned} \tag{3.28}$$

since, after u_1 and u_2 have been chosen, z can be chosen in $s(u_1,u_2)$ ways and then v in at most d ways for each choice of z. Combining (3.22), (3.25), (3.28) and the fact that, by (3.14)

$$E(W^* - W) = E\sum_{\delta(u,V^*)=2} A(u,V^*), \tag{3.29}$$

we obtain (3.21).

Next we bound $\operatorname{Var} E^Y(W^* - W)$. In view of (3.13), we start with a bound for $\operatorname{Var}((d+1)W/|\mathcal{V}|)$. We obtain

$$\begin{aligned}
\text{(3.30)} \qquad \operatorname{Var}\Big(\frac{d+1}{|\mathcal{V}|}W\Big) &= \frac{(d+1)^2}{|\mathcal{V}|^2}\sigma^2 \\
&= |\mathcal{V}|^{-2} \sum_{\substack{u,v \\ \delta(u,v)=2}} s(u,v)(2d+2-s(u,v))^{-1}.
\end{aligned}$$

Next we deal with

$$\begin{aligned}
\text{(3.31)} \qquad \operatorname{Var}\Big(|\mathcal{V}|^{-1}\sum_{u,v} A(u,v)\Big) &= |\mathcal{V}|^{-2} \sum_{\substack{u,v \\ \delta(u,v)=2}} \operatorname{Var} A(u,v) \\
&+ |\mathcal{V}|^{-2} \sum_{u_1,v_1} \sum_{u_2,v_2} \operatorname{Cov}(A(u_1,v_1), A(u_2,v_2)),
\end{aligned}$$

where in the last double sum u_1, v_1, u_2, v_2 satisfy $\delta(u_1,v_1) = \delta(u_2,v_2) = 2$ and $(u_1,v_1) \neq (u_2,v_2)$. Recalling the fact that

$$\text{(3.32)} \qquad A(u,v) = \sum_{z \in N(u) \cap N(v)} B(u,v,z)$$

and henceforth suppressing the condition $z \in N(u) \cap N(v)$ in the summations, we have

$$\begin{aligned}
\text{(3.33)} \qquad \operatorname{Var} A(u,v) &= \operatorname{Var} \sum_z B(u,v,z) \\
&\leq \sum_z \operatorname{Var} B(u,v,z)
\end{aligned}$$

the inequality following from $E(B(u,v,z_1)B(u,v,z_2)) = 0$ for $z_1 \neq z_2$. Clearly $\operatorname{Var} B(u,v,z) \leq P(B(u,v,z))$, and with (3.18) and (3.33) we obtain that the first term on the right hand side of (3.31) is bounded by

$$\text{(3.34)} \qquad |\mathcal{V}|^{-2} \sum_{\substack{u,v \\ \delta(u,v)=2}} s(u,v)(2d+2-s(u,v))^{-1}(d+1)^{-1}.$$

We now consider the sum of the covariances on the right hand side of (3.31), which by (3.32) equals

$$\text{(3.35)} \qquad |\mathcal{V}|^{-2} \sum_{u_1,v_1,z_1} \sum_{u_2,v_2,z_2} \operatorname{Cov}(B(u_1,v_1,z_1), B(u_2,v_2,z_2)),$$

where again u_1, v_1, u_2, v_2 satisfy $\delta(u_1, v_1) = \delta(u_2, v_2) = 2$ and $(u_1, v_1) \neq (u_2, v_2)$. In order to obtain a bound for this expression, we shall consider different cases depending on the distances $\delta(u_1, u_2)$, $\delta(v_1, v_2)$, etc. For each case we calculate a bound for $\mathrm{Cov}(B(u_1, v_1, z_1), B(u_2, v_2, z_2))$ and a bound for the number of (u_1, v_1, z_1), (u_2, v_2, z_2) of this case, thus obtaining a bound for the sum of all covariances. We consider the following cases, with subcases where $z_1 = z_2$ and $z_1 \neq z_2$.

Case 1: all $\delta(u_1, u_2), \delta(u_1, v_2), \delta(v_1, u_2), \delta(v_1, v_2) \geq 3$

Case 2: all the above distances are ≥ 2 and at least one is equal to 2.

Case 3: all the above distances are ≥ 1 and at least one is equal to 1.

Case 4: at least one of them is equal to 0.

In case 1, the corresponding covariances of (3.35) vanish because in this case the indicated events $B(u_1, v_1, z_1)$, $B(u_2, v_2, z_2)$ are independent.

Next we consider case 2, with $z_1 \neq z_2$. In order to compute a bound on $\mathrm{Cov}(B(u_1, v_1, z_1), B(u_2, v_2, z_2))$, define $C_i = \{v_i\} \cup N(u_i) - \{z_i\}$, $D_i = \{v_i\} \cup N(v_i) - \{z_i\}$ and for brevity write B_i for $B(u_i, v_i, z_i)$, $i = 1, 2$. Then we can write the probability of $B_1 \cap B_2$ as a sum of six terms, each of which can be computed with the aid of Lemma 3.2. We have

$$\begin{aligned}
P_1 &= P(\{Y_{z_1} > Y_{z_2} > Y_{u_2} > Y_{u_1}\} \cap B_1 \cap B_2) \\
&= (|C_1 \cup C_2 \cup D_1 \cup D_2| + 4)^{-1}(|C_1 \cup C_2 \cup D_2| + 3)^{-1} \\
&\qquad (|C_1 \cup C_2| + 2)^{-1}(|C_1| + 1)^{-1}
\end{aligned}$$

$$\begin{aligned}
P_2 &= P(\{Y_{z_1} > Y_{z_2} > Y_{u_1} > Y_{u_2}\} \cap B_1 \cap B_2) \\
&= (|C_1 \cup C_2 \cup D_1 \cup D_2| + 4)^{-1}(|C_1 \cup C_2 \cup D_2| + 3)^{-1} \\
&\qquad (|C_1 \cup C_2| + 2)^{-1}(|C_2| + 1)^{-1}
\end{aligned}$$

$$\begin{aligned}
P_3 &= P(\{Y_{z_1} > Y_{u_1} > Y_{z_2} > Y_{u_2}\} \cap B_1 \cap B_2) \\
&= (|C_1 \cup C_2 \cup D_1 \cup C_2| + 4)^{-1}(|C_1 \cup C_2 \cup D_2| + 3)^{-1} \\
&\qquad (|C_2 \cup D_2| + 2)^{-1}(|C_2| + 1)^{-1}
\end{aligned}$$

and we define P_4, P_5, P_6 similarly with the indices 1 and 2 interchanged. It follows that

$$\begin{aligned}
P_1 + P_2 &= (|C_1 \cup C_2 \cup D_1 \cup D_2| + 4)^{-1}(|C_1 \cup C_2 \cup D_2| + 3)^{-1} \\
&\qquad (|C_1| + 1)^{-1}(|C_2| + 1)^{-1}(1 + \frac{|C_1 \cap C_2|}{|C_1 \cup C_2| + 2})
\end{aligned}$$

and consequently

$$P_1 + P_2 + P_3 = (|C_1 \cup C_2 \cup D_1 \cup D_2| + 4)^{-1}(|C_2 \cup D_2| + 2)^{-1}(|C_1| + 1)^{-1}$$

$$(|C_2|+1)^{-1}\Big[1+\frac{|C_1\cap C_2|}{|C_1\cup C_2|+2}\frac{|C_2\cup D_2|+2}{|C_1\cup C_2\cup D_2|+3}+\frac{|C_1\cap(C_2\cup D_2)|}{|C_1\cup C_2\cup D_2|+3}\Big]$$

adding this to the quantity obtained by interchanging the indices and using (3.18) we obtain

$$
\begin{aligned}
(3.36)\qquad \mathrm{Cov}(B_1,B_2) &= P(B_1\cap B_2)-P(B_1)P(B_2)\\
&=\sum_{i=1}^{6}P_i-(|C_1\cup D_1|+2)^{-1}(|C_1|+1)^{-1}(|C_2\cup D_2|+2)^{-1}(|C_2|+1)^{-1}\\
&=(|C_1\cup C_2\cup D_1\cup D_2|+4)^{-1}(|C_1|+1)^{-1}(|C_2|+1)^{-1}\\
&\Big[(|C_1\cap C_2|)(|C_1\cup C_2|+2)^{-1}((|C_1\cup C_2\cup D_2|+3)^{-1}+(|C_1\cup C_2\cup D_1|+3)^{-1})\\
&+(|C_1\cap(C_2\cup D_2)|)(|C_2\cup D_2|+2)^{-1}(|C_1\cup C_2\cup D_2|+3)^{-1}\\
&+(|C_2\cap(C_1\cup D_1)|)(|C_1\cup D_1|+2)^{-1}(|C_1\cup C_2\cup D_1|+3)^{-1}\\
&+(|(C_1\cup D_1)\cap(C_2\cup D_2)|)(|C_1\cup D_1|+2)^{-1}(|C_2\cup D_2|+2)^{-1}\Big]\\
&\le\frac{1}{(d+1)^4}\Big[\frac{5s(u_1,u_2)}{2d+2-s(u_1,u_2)}+\frac{2s(u_1,v_2)}{2d+2-s(u_1,v_2)}\\
&\quad+\frac{2s(u_2,v_1)}{2d+2-s(u_2,v_1)}+\frac{s(v_1,v_2)}{2d+2-s(v_1,v_2)}\Big].
\end{aligned}
$$

We now sum over all (u_1,v_1,z_1) and (u_2,v_2,z_2) of case 2. The first term on the right hand side of (3.36) gives

$$\frac{5}{(d+1)^4}\sum_{u_1}\sum_{u_2}s(u_1,u_2)(2d+2-s(u_1,u_2))^{-1}[d(d-1)]^2$$

where the term $[d(d-1)]^2$ arises from the fact that given u_1,u_2 there are at most $[d(d-1)]^2$ possible choices of z_1,z_2,v_1,v_2. With a similar argument for the other terms it follows that the sum of covariances in (3.35) of case 2 and $z_1\neq z_2$ is bounded by

$$(3.37)\qquad 10|\mathcal{V}|^{-2}\sum\sum s(u_1,u_2)(2d+2-s(u_1,u_2))^{-1}.$$

Next, consider $z_1=z_2=z$. Here and in all remaining cases it suffices to use the bound

$$(3.38)\qquad \mathrm{Cov}(B_1,B_2)\le E(B_1B_2)$$

and the calculations become considerably simpler. In the present case, Lemma 3.2 yields

$$
\begin{aligned}
(3.39)\qquad E(B_1B_2) &= 2(|C_1\cup C_2\cup D_1\cup D_2|+4)^{-1}\\
&\quad(|C_1\cup C_2|+2)^{-1}(d+1)^{-1}\\
&\le(2d+2-s(u_1,u_2))^{-1}(d+1)^{-1}.
\end{aligned}
$$

To obtain a bound for the sum of covariances in (3.35), we sum the right hand side of (3.39) over $u_1, v_1, z_1, u_2, v_2, z_2$ of this case. Observe that after choosing u_1, v_1, the vertex $z = z_1 = z_2$ can be chosen in $s(u_1, v_1)$ ways and given z, $u_2, v_2 \in N(z)$ can be chosen in $(d-2)(d-3)$ ways. Combining (3.38) and (3.39) we conclude that an upper bound on the sum of covariances in case 2 when $z_1 = z_2$ is

$$2|\mathcal{V}|^{-2} \sum \sum s(u_1, u_2)(2d + 2 - s(u_1, v_1))^{-1}. \tag{3.40}$$

The remaining cases are treated similarly. For example in the case that $\delta(u_1, u_2) = 2$, $\delta(v_1, v_2) = 1$ and $z_1 \neq z_2$ we obtain

$$E(B_1 B_2) \le 6(2d + 2 - s(u_1, u_2))^{-1}(d+1)^{-3} \tag{3.41}$$

(with the factor 6 corresponding to the six arrangements described in case 2). The number of terms in this case is counted as follows. Given u_1 and v_1, z_1 can be chosen in $s(u_1, v_1)$ ways, then v_2 in $d-1$ ways since $\delta(v_1, v_2) = 1$, and z_2, u_2 in $d(d-1)$ ways. This shows that an upper bound on covariances in this case is given by

$$6|\mathcal{V}|^{-2} \sum_{\substack{u_1, u_2 \\ \delta(u_1, u_2) = 2}} s(u_1, u_2)(2d + 2 - s(u_1, u_2))^{-1}. \tag{3.42}$$

In other cases, the bound on $E(B_1, B_2)$ may be of order $(2d + 2 - s(u_1, u_2))^{-1}$

$(d+1)^{-i}$ $i = 1, 2$ or 3 but in each case the count of the associated terms leads to a bound similar to (3.37), (3.40), (3.42). In conclusion of this discussion, we obtain

$$\begin{aligned} &\operatorname{Var} E^Y(W^* - W) \\ &\le \alpha|\mathcal{V}|^{-2} \sum_{\substack{u_1, u_2 \\ \delta(u_1, u_2) = 2}} s(u_1, u_2)(2d + 2 - s(u_1, u_2))^{-1}. \end{aligned} \tag{3.43}$$

where α is an absolute constant, independent of the particular graph. Recalling also (3.19) and (1.2) we have

$$\begin{aligned} &\frac{\operatorname{Var} E^Y(W^* - W)}{[E(W^* - W)]^2} \\ &\le \alpha(d+1)^2 \Big[\sum_{u_1} \sum_{u_2} s(u_1, u_2)(2d + 2 - s(u_1, u_2))^{-1}\Big]^{-1} \\ &\le \alpha\sigma^{-2}. \end{aligned} \tag{3.44}$$

Theorem 3.1 now follows from Theorem 2.1, with (3.21) and (3.44). ∎

4. Examples

The emphasis in this section is on the asymptotics; the bounds of section 3 will be computed only to the extent needed to verify asymptotic normality.

EXAMPLE 1. The graph $\{0,\ldots,m-1\}^n$. Set $\mathcal{V}=\{0,\ldots,m-1\}^n$ and for $u=(u_1,\ldots,u_n)$, $v=(v_1,\ldots,v_n)$ in $\mathcal{V}$ define $\delta(u,v)=|\{i\in\{1,\ldots,n\}: u_i\neq v_i\}|$. Thus two vertices are connected by an edge if and only if they differ in exactly one coordinate. Here $|\mathcal{V}|=m^n$, $d=n(m-1)$, $s(u,v)=2$ for all $u,v\in\mathcal{V}$ satisfying $\delta(u,v)=2$, so that

$$\sigma^2=|\mathcal{V}|\binom{n}{2}(m-1)^2(d+1)^{-2}d^{-1}. \tag{4.1}$$

It is readily seen that σ^2 is of the order of m^{n-1}/n, diverging to ∞ as $m\to\infty$ for any $n\geq 2$, or as $n\to\infty$ for any $m\geq 2$, and asymptotic normality of W follows from Theorem 3.1.

REMARK. In the case $n=2$, the exact distribution of W can be identified as the hypergeometric distribution. We claim

$$P(W=k)=\frac{\binom{m}{k}\binom{m-1}{m-k}}{\binom{2m-1}{m}}\quad k=1,2,\ldots,m. \tag{4.2}$$

To see this, consider the graph $\{0,\ldots,m-1\}^2$ as a two dimensional array of m rows and m columns. Each row (or column) may contain at most one local maximum. Let $I_j=I\{$ there exists a maximum in row $j\}$. Then $W=\sum_{j=1}^m I_j$. The I_j's are exchangeable and by Lemma 3.2 accounting also for the $m(m-1)\ldots(m-k+1)$ possible locations of the maxima at rows $1,\ldots,k$ we obtain

$$\begin{aligned} E(I_1\ldots I_k)&=\frac{k!\,m(m-1)\ldots(m-k+1)}{\prod_{l=1}^k(ld+l-2\frac{l(l-1)}{2})}\\ &=\frac{k!\,m(m-1)\ldots(m-k+1)}{\prod_{l=1}^k l(2m-l)}=\frac{\binom{m}{k}}{\binom{2m-1}{k}}. \end{aligned} \tag{4.3}$$

This determines (4.2). Thus the distribution of the number of local maxima is the same as that of the number of white balls obtained when m balls are drawn at random without replacement from an urn containing m white balls and $m-1$ black balls.

EXAMPLE 2. Again the graph $\{0,\ldots,m-1\}^n$ but with distance δ defined for $u=(u_1,\ldots,u_n)$, $v=(v_1,\ldots,v_n)\in\mathcal{V}$ by

$$\delta(u,v)=\begin{cases}0 & \text{if } u=v;\\ 1 & \text{if } u\neq v \text{ and for some } i,\ u_i=v_i;\\ 2 & \text{otherwise.}\end{cases}$$

Thus the neighborhood of a vertex consists of the union of all $n-1$-dimensional hyperplanes containing it. It is not hard to see that in this case

$$d = m^n - (m-1)^n - 1 \approx nm^{n-1},$$

$$s = s(u,v) = m^n - 2(m-1)^n + (m-2)^n \approx n(n-1)m^{n-2},$$

and

$$\mathcal{S} = [m(m-1)]^n s \approx n(n-1)m^{3n-2}$$

where the asymptotic expressions are valid if $m \to \infty$, $n = o(m)$ (with $n \to \infty$ allowed). Therefore $\mathcal{S}(d+1)^{-3}$ is of the order of m/n. We obtain asymptotic normality if $m/n \to \infty$. If m/n is bounded, then $\sigma^2 = \operatorname{Var} W$ is bounded and aymptotic normality is impossible.

EXAMPLE 3. The complete bipartite graph $K_{d,d}$. In this graph $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ with $|\mathcal{V}_i| = d$ and $\delta(v_1, v_2) = 1$ if and only if $v_1 \in \mathcal{V}_1$ and $v_2 \in \mathcal{V}_2$ or the reverse. This graph is regular with degree d. Also

$$s(u,v) = d, \quad \mathcal{S} = 4\binom{d}{2}d, \quad |\mathcal{V}| = 2d.$$

In this case $\mathcal{S}d^{-3} \approx 2$ as $d \to \infty$ and by (1.3) asymptotic normality is obviously ruled out. In fact for $k = 1, 2, \ldots, d$

$$P(W = k) = 2\frac{d(d-1)\ldots(d-k+1)d}{2d(2d-1)\ldots(2d-k)}. \tag{4.4}$$

Note that as $d \to \infty$, $P(W = k) \to 2^{-k}$ and $W \xrightarrow{\mathcal{D}}$ *Geometric* $(\frac{1}{2})$.

EXAMPLE 4. The complete bipartite graph $K_{1,d}$ (star). Here $|\mathcal{V}_1| = 1$, $|\mathcal{V}_2| = d$, with notation as in Example 3. This graph is not regular. From (3.20) one can readily show that $\sigma^2 = \operatorname{Var} W \to \infty$ as $d \to \infty$. However W is not asymptotically normal. In fact

$$P(W = k) = \begin{cases} 2/(d+1), & \text{if } k = 1; \\ 1/(d+1), & \text{if } k = 2, \ldots, d. \end{cases} \tag{4.5}$$

So the distribution is nearly uniform and $\frac{W-\lambda}{\sigma} \xrightarrow{\mathcal{D}}$ *Uniform* $(-\sqrt{3}, \sqrt{3})$. This example indicates that without some regularity conditions on the graph (perhaps local regularity) the problem is much more complicated and obviously $\operatorname{Var} W \to \infty$ is not a sufficient condition for asymptotic normality.

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